Bundle-like metrics on a tangent bundle

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Abstract

Let $(M, g, \mathcal{F})$ be a semi-Riemannian manifold with metric g and non-degenerated foliation $\mathcal{F}$. Let $TM = \mathcal{D} \oplus \mathcal{D}^{\perp}$ and $D^{\perp}$ be the intrinsic connection on $\mathcal{D}^{\perp}$. A metric g on M is said to be bundle-like for the non-degenerated foliation $\mathcal{F}$ if the induced semi-Riemannian metric on $\mathcal{D}^{\perp}$ is parallel with respect to the intrinsic connection $D^{\perp}$.

Supposing that a Riemannian metric g on M is bundle-like we shall give a necessary and sufficient condition for a g-natural metric G on TM to be bundle-like. We shall also indicate another bundle-like metrics on TM.

1 Preliminaries

a) Intrinsic connection

Let g be a semi-Riemannian metric on $n-$distribution $\mathcal{D}$ on M, that is $(\mathcal{D}, g)$ is a semi-Riemannian distribution. Let $\mathcal{D}'$ be a distribution complementary to $\mathcal{D}$ in TM :

$$TM = \mathcal{D} \oplus \mathcal{D}'.$$

Denote by Q the projection

$$Q : TM \longrightarrow \mathcal{D}.$$

There exists a unique linear connection D on $\mathcal{D}$ satisfying for all $X, Y, Z \in \Gamma(TM)$ the following two conditions:

$$D_X QY - D_{QY} QX - Q\,[X, QY] = 0$$

(i.e. D on $\mathcal{D}$ is $\mathcal{D}'-$ torsion free) and

$$(D_{QX} g)\,(QY, QZ) = \\ QX\,(g(QY, QZ)) - g\,(D_{QX} QY, QZ) - g\,(QY, D_{QX} QZ) = 0$$

(i.e. g is parallel along $\mathcal{D}$ or g is $\mathcal{D}$ - parallel).

Connection D is given as a mapping

$$D : \Gamma(TM) \times \Gamma(\mathcal{D}) \longrightarrow \Gamma(\mathcal{D})$$

by

$$\begin{aligned} 2g\left(D_{QX}QY, QZ\right) = & \\ & QX(g(QY,QZ)) + QY(g(QZ,QX)) - QZ(g(QX,QY)) + \\ & \quad g(Q[QX,QY],QZ) - g(Q[QY,QZ],QX) + g(Q[QZ,QX],QY) \end{aligned}$$

and

$$D_{Q'X}QY = Q[Q'X, QY].$$

(If $\mathcal{D}' = \{0\}$, then D is the Levi-Civita connection on M).

Now consider $n+p$ - dimensional manifold (M,g) and suppose that $(\mathcal{D},g)$ is a semi-Riemannian $n-$distribution on M. Then $(\mathcal{D}^{\perp},g)$ is a semi-Riemannian $p-$distribution on M. Thus we have

$$TM = \mathcal{D} \oplus \mathcal{D}^{\perp}.$$

On $\mathcal{D}$ and $\mathcal{D}^{\perp}$ there exist uniquely determined connections D and $D^{\perp}$ that we call the intrinsic connections on $\mathcal{D}$ and $\mathcal{D}^{\perp}$ respectively.

Theorem 1 *([BF], Theorem 1.5.3, p. 26) The adapted linear connection on (M,g) determined by the pair $(D,D^{\perp})$ of intrinsic connections is the Vrănceanu connection ∇^{*} defined by the Levi-Civita connection $\widetilde{\nabla}$ on (M,g) :*

$$(D, D^{\perp}) = \nabla^{*}.$$

b) Semi-holonomic frame field

Let (M,g) be an $(n+p)-$dimensional semi-Riemannian manifold with metric g and $\mathcal{F}$ be an $n-$foliation of M. Denote by $\mathcal{D}$ the distribution tangent to $\mathcal{F}$. Let $\mathcal{D}'$ be a fixed distribution on M so that

$$TM = \mathcal{D} \oplus \mathcal{D}^{'}. \tag{1}$$

If $\left\{U,\ (x^{j},x^{\alpha})\right\}$, $j=1,...,n,\ \alpha=n+1,...,p$ is a foliated chart on $(M,\mathcal{F})$ then $\mathcal{D}$ is locally represented on U by the canonical vectors fields $\left\{\frac{\partial}{\partial x^{1}},...\frac{\partial}{\partial x^{n}}\right\} = \{\partial_1,...,\partial_n\}$. Suppose that $\mathcal{D}'$ on U is locally represented by $\{E_{n+1},...E_{n+p}\}$. Then $\{\partial_j, E_\alpha\}$ is a non-holonomic frame field on U with respect to the decomposition (1). With respect to this frame field one can write

$$\partial_\alpha = A^{r}_{\alpha}\partial_r + A^{\beta}_{\alpha}E_\beta, \tag{2}$$

where $\partial_\alpha = \frac{\partial}{\partial x^{\alpha}}$. Since the transition matrix from the non-holonomic frame field to the canonical one is of the form

$$\begin{bmatrix} \delta^i_j & A^i_\alpha \\ 0 & A^\beta_\alpha \end{bmatrix},$$

it is non-singular, so that the vector fields $\delta_\alpha = \frac{\delta}{\delta x^\alpha} = A^\beta_\alpha E_\beta$ also represent locally $\mathcal{D}'$ on U. Thus we have

$$\delta_\alpha = \partial_\alpha - A^r_\alpha \partial_r. \tag{3}$$

It is possible to choose A^r_α so that $g(\partial_j, \delta_\beta) = 0$ ([R1]). The frame $\{\partial_j, \delta_\alpha\}$ is called the semi-holonomic frame field on U ([BF]).

Taking into accoun the semi-Riemannian metric g on M, we define its local components with respect to the semi-holonomic frame field $\{\partial_j, \delta_\alpha\}$:

$$g_{ij} = g(\partial_i, \partial_j), \qquad g_{\alpha\beta} = g(\delta_\alpha, \delta_\beta)$$

and denote by $\left[g^{\alpha\beta}\right]$ the inverse matrix of $[g_{\alpha\beta}]$.

Theorem 2 *([BF], Proposition 3.1.2, p. 99). The local coefficiets of the intrinsic connections D and $D^\perp$ with respect to the semi-holonomic frame field $\{\partial_j, \delta_\alpha\}$ are given by*

$$C^k_{ij} = \frac{1}{2} g^{kh} \left(\partial_i g_{hj} + \partial_j g_{hi} - \partial_h g_{ij}\right), \quad D^k_{j\alpha} = \partial_j A^k_\alpha, \tag{4}$$

and

$$L^\beta_{\alpha j} = 0, \quad \Gamma^\omega_{\alpha\beta} = \frac{1}{2} g^{\omega\mu} \left(\delta_\alpha g_{\mu\beta} + \delta_\beta g_{\mu\alpha} - \delta_\mu g_{\alpha\beta}\right). \tag{5}$$

Since the Vrăncianu connection ∇^* is determined by the pair $(D,\, D^\perp)$, the local components of ∇^* with respect to the frame field $\{\partial_j, \delta_\alpha\}$ are given by (4) and (5) ([BF], Corollary 3.3.1, p. 99).

Let $(M, g, \mathcal{F})$ be a semi-Riemannian manifold with metric g and non-degenerated foliation $\mathcal{F}$ which means that the distribution $\mathcal{D}$ tangent to $\mathcal{F}$ together with the induced metric is a semi-Riemannian manifold. Let

$$TM = \mathcal{D} \oplus \mathcal{D}^\perp$$

and $D^\perp$ be the intrinsic connection on $\mathcal{D}^\perp$. A metric g on M is said to be bundle-like for the non-degenerated foliation $\mathcal{F}$ if the induced semi-Riemannian metric on $\mathcal{D}^\perp$ is parallel with respect to the intrinsic connection $D^\perp$ ([BF]).

Theorem 3 *([BF], Theorem 3.3.2, p. 111). The semi-Riemannian metric g on M is bundle-like if and only if the transversal components $g_{\alpha\beta}$ of g satisfy*

$$\frac{\partial g_{\alpha\beta}}{\partial x^j}$$

for all $j = 1, ..., n$ and all $\alpha,\, \beta = n+1, ..., n+p$.

In other words, with respect to the semi-holonomic frame field $\{\partial_j, \delta_\alpha\}$, the local form of the metric tensor g is

$$g_{ij}\left(x^k, x^\mu\right)\delta x^i \delta x^j + g_{\alpha\beta}(x^\mu)dx^\alpha dx^\beta$$

where $\{\delta x^j = dx^j + A^s_\mu dx^\mu,\, dx^\alpha\}$ is the dual semi-holonomic frame field.

c) TM as a manifold

Let x be a point of a manifold (M,g) and $(x,u) \in TM$, $\dim M = n$. For any vector $X \in T_xM$ there exist the unique vectors: X^h given by $\pi_*(X^h) = X$, where $\pi : TM \longrightarrow M$, and X^v given by $X^v(df) = Xf$ for any function f on M. X^h and X^v are called the horizontal and the vertical lifts of X to the point $(x,u) \in TM$.

The space $T_{(x,u)}TM$ tangent to TM at (x,u) splits into direct sum

$$T_{(x,u)}TM = H_{(x,u)}TM \oplus V_{(x,u)}TM$$

and we have isomorphisms

$$H_{(x,u)}TM \sim T_xM \sim V_{(x,u)}TM.$$

If $((x^j),\ (u^j))$, $i = 1,...,n$, is a local coordinate system around the point $(x,u) \in TM$ and $X = X^j \frac{\partial}{\partial x^j}$, then

$$X^h = X^j \frac{\partial}{\partial x^j} - u^r X^s \Gamma^j_{rs} \frac{\partial}{\partial u^j}, \quad X^v = X^j \frac{\partial}{\partial u^j},$$

where Γ^j_{rs} are Christoffel symbols of the Levi-Civita connection ∇ on (M,g).

d) g-natural metrics

Every metric g on M defines a family of metrics on TM. Between them the class of so called $g-$ natural metrics is of special interest. The well-known Cheeger-Gromoll and Sasaki metrics are the special cases of the $g-$natural metrics ([KS]).

Lemma 4 *([AS], [AS1]) Let (M,g) be a Riemannian manifold and $\widetilde{g}$ be a $g-$natural metric on TM. There exist functions $\alpha_j,\ \beta_j :< 0,\infty) \longrightarrow R$, $j = 1,2,3,$ such that for every $X,\ Y,\ u \in T_xM$*

$$\begin{aligned}
\widetilde{g}_{(x,u)}(X^h, Y^h) &= (\alpha_1 + \alpha_3)(r^2)g_x(X,Y) + (\beta_1 + \beta_3)(r^2)g_x(X,u)g_x(Y,u),\\
\widetilde{g}_{(x,u)}(X^h, Y^v) &= \alpha_2(r^2)g_x(X,Y) + \beta_2(r^2)g_x(X,u)g_x(Y,u),\\
\widetilde{g}_{(x,u)}(X^v, Y^h) &= \alpha_2(r^2)g_x(X,Y) + \beta_2(r^2)g_x(X,u)g_x(Y,u),\\
\widetilde{g}_{(x,u)}(X^v, Y^v) &= \alpha_1(r^2)g_x(X,Y) + \beta_1(r^2)g_x(X,u)g_x(Y,u),
\end{aligned}$$

where $r^2 = g_x(u,u)$. For $\dim M = 1$ the same holds with $\beta_j = 0$, $j = 1,2,3$.

Setting $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = \beta_j = 0$ we obtain the Sasaki metric, while setting $\alpha_1 = \frac{1}{1+r^2}$, $\alpha_2 = \beta_2 = 0 = 0, \alpha_3 = 1 - \alpha_1, \beta_1 = 1$, $\beta_1 + \beta_3 = 0$ we get the Cheeger-Gromoll one.

Following ([AS]) we put

1. $\alpha(t) = \alpha_1(t)\left(\alpha_1(t) + \alpha_3(t)\right) - \alpha_2^2(t)$,
2. $\Phi_j(t) = \alpha_j(t) + t\beta_j(t)$,
3. $\Phi(t) = \Phi_1(t)\left[\Phi_1(t) + \Phi_3(t)\right] - \Phi_2^2(t)$
   for all $t \in < 0, \infty)$.

Lemma 5 *([AS], Proposition 2.7) The necessary and sufficient conditions for a $g-$ natural metric $\widetilde{g}$ on the tangent bundle of a Riemannian manifold (M, g) to be non-degenerate are $\alpha(t) \neq 0$ and $\Phi(t) \neq 0$ for all $t \in < 0, \infty)$. If $\dim M = 1$ this is equivalent to $\alpha(t) \neq 0$ for all $t \in < 0, \infty)$.*

2 Bundle-like $g-$natural metrics generated by a bundle-like metric g

a) Semi-holonomic frame field on TM

Let (M, g) be an $(n+p)-$dimensional semi-Riemannian manifold with metric g and $\mathcal{F}$ be an $n-$foliation of M and $\{\partial_j, \delta_\alpha\}$ be a semi-holonomic frame field on $U \subset M$. We shall construct a semi-holonomic frame field

$$\left\{\partial_j, \frac{\partial}{\partial u^r}, \mu_\alpha, \omega_\beta\right\}$$

on $V \subset TM = \mathcal{D} \oplus \mathcal{D}^{'}$ with $(\mathcal{D}, g)$ being a semi-Riemannian distribution. Let $(V, (x^j, x^\alpha, u^j, v^a))$ be a local chart on TM, where $\left\{U, \ (x^j, x^\alpha)\right\}$, $j, r = 1, ..., n$, $\alpha, \beta = n+1, ..., p$, is a foliated chart on $(M, \mathcal{F})$. With respect to (3)

$$u^j\partial_j + v^\alpha\partial_\alpha = (u^j + v^\beta A^j_\beta)\partial_j + v^\beta\delta_\beta = U^j\partial_j + V^\beta\delta_\beta. \tag{6}$$

The vertical and horizontal lifts of ∂_j to $T\mathcal{D}$ are the vector fields on $T\mathcal{D}$ locally represented respectively by

$$(\partial_j)^v = \frac{\partial}{\partial u^j}, \qquad (\partial_j)^h = \partial_j - \Gamma^r_j \frac{\partial}{\partial u^r}, \tag{7}$$

where $\Gamma^r_j = \Gamma^r_{js}u^s$, Γ^r_{js} being the Christoffel symbols of $(\mathcal{D}, g)$. From (7) we get

$$\partial_j = (\partial_j)^h + \Gamma^r_j (\partial_r)^v. \tag{8}$$

Let $\left\{\partial_j, \frac{\partial}{\partial u^r}, \partial_\alpha, \frac{\partial}{\partial v^\alpha}\right\}$ be a canonical frame on $V \subset TM$. Then

$$\left\{\partial_j, \frac{\partial}{\partial u^r}, (\delta_\alpha)^v, (\delta_\alpha)^h\right\} \tag{9}$$

is a non-holonomic frame field on V. From (3)

$$(\delta_\alpha)^v = (\partial_\alpha)^v - A^r_\alpha\,(\partial_r)^v\,, \tag{10}$$

whence

$$\frac{\partial}{\partial v^\alpha} = (\delta_\alpha)^v + A^r_\alpha\,(\partial_r)^v\,. \tag{11}$$

Since

$$(\partial_\alpha)^h = \partial_\alpha - \Gamma^r_\alpha \frac{\partial}{\partial u^r} - \Gamma^\beta_\alpha \frac{\partial}{\partial v^\beta}, \tag{12}$$

from (3), by the use of (12), (11) and (7) we obtain

$$(\delta_\alpha)^h = \partial_\alpha - \Gamma^\beta_\alpha\,(\delta_\beta)^v - \left(\Gamma^r_\alpha + \Gamma^\beta_\alpha A^r_\beta\right)(\partial_r)^v - A^r_\alpha\,(\partial_r)^h\,. \tag{13}$$

On the other hand we can express the canonical vector fields $\partial_\alpha,\ \frac{\partial}{\partial v^\alpha}$ on $V \subset TM$ in terms of the non-holonomic frame (9):

$$\begin{aligned}
\partial_\alpha &= P^r_\alpha \partial_r + Q^r_\alpha \frac{\partial}{\partial u^r} + P^\beta_\alpha\,(\delta_\beta)^v + Q^\beta_\alpha\,(\delta_\beta)^h\,, &(14)\\
\frac{\partial}{\partial v^\alpha} &= \overline{P}^r_\alpha \partial_r + \overline{Q}^r_\alpha \frac{\partial}{\partial u^r} + \overline{P}^\beta_\alpha\,(\delta_\beta)^v + \overline{Q}^\beta_\alpha\,(\delta_\beta)^h\,. &(15)
\end{aligned}$$

Setting

$$\mu_\alpha = P^\beta_\alpha\,(\delta_\beta)^v + Q^\beta_\alpha\,(\delta_\beta)^h\,, \qquad \omega_\alpha = \overline{P}^\beta_\alpha\,(\delta_\beta)^v + \overline{Q}^\beta_\alpha\,(\delta_\beta)^h\,,$$

by the use of (14), (13), (8) and (7) we get

$$\begin{aligned}
\mu_\alpha &= (\delta_\alpha)^h + \Gamma^\beta_\alpha\,(\delta_\beta)^v + (A^r_\alpha - P^r_\alpha)\,(\partial_r)^h + \left(\Gamma^r_\alpha + \Gamma^\beta_\alpha A^r_\beta - Q^r_\alpha - P^s_\alpha \Gamma^r_s\right)(\partial_r)^v\\
&= (\delta_\alpha)^h + \Gamma^\beta_\alpha\,(\delta_\beta)^v + K^r_\alpha\,(\partial_r)^v + L^r_\alpha\,(\partial_r)^h\,. &(16)
\end{aligned}$$

In similar way, from (15), by the use of (11), (8) and (7) we obtain

$$\begin{aligned}
\omega_\alpha &= (\delta_\alpha)^v + \left(A^r_\alpha - P^s_\alpha \Gamma^r_s - \overline{Q}^r_\alpha\right)(\partial_r)^v - \overline{P}^r_\alpha\,(\partial_r)^h\\
&= (\delta_\beta)^v + \overline{K}^r_\alpha\,(\partial_r)^v + \overline{L}^r_\alpha\,(\partial_r)^h\,. &(17)
\end{aligned}$$

b) Components of a g-natural metric with respect to a semi-holonomic frame field

Let G be a $g-$natural metric on TM. Let $\left\{\partial_j, \frac{\partial}{\partial u^r}, \mu_\alpha, \omega_\beta\right\}$ be a semi-holonomic frame field on $V \subset TM$ as above ((8),(7),(16),(17)). Making use of Lemma 4 and the fact that $\partial_j, \frac{\partial}{\partial u^r}$ are orthogonal to μ_α, ω_β we get in turn:

$$\begin{aligned}
&G\left(\frac{\partial}{\partial u^j}, \mu_\alpha\right) = G\left(\left(\frac{\partial}{\partial x^j}\right)^v, \mu_\alpha\right) =\\
&\alpha_1 g_{jr} K^r_\alpha + \alpha_2 g_{jr} L^r_\alpha + U_j\left[\beta_1\left(V_\beta \Gamma^\beta_\alpha + U_s K^s_\alpha\right) + \beta_2\left(V_\alpha + U_s L^s_\alpha\right)\right] = 0. \quad (18)
\end{aligned}$$

By the use of (18) we find

$$G\left(\frac{\partial}{\partial x^j},\mu_\alpha\right)=G\left((\partial_j)^h+\Gamma_j^r(\partial_r)^v,\mu_\alpha\right)=$$

$$\alpha_2 g_{jr}K_\alpha^r+(\alpha_1+\alpha_3)\,g_{jr}L_\alpha^r+U_j\left[\beta_2\left(V_\beta\Gamma_\alpha^\beta+U_sK_\alpha^s\right)+(\beta_1+\beta_3)\,(V_\alpha+U_sL_\alpha^s)\right]=0. \tag{19}$$

Moreover

$$G\left(\frac{\partial}{\partial u^j},\omega_\alpha\right)=G\left(\left(\frac{\partial}{\partial x^j}\right)^v,\omega_\alpha\right)=$$

$$\alpha_1 g_{jr}\overline{K}_\alpha^r+\alpha_2 g_{jr}\overline{L}_\alpha^r+U_j\left[\beta_1\left(V_\alpha+U_s\overline{K}_\alpha^s\right)+\beta_2U_s\overline{L}_\alpha^s\right]=0. \tag{20}$$

Making use of (20) we get

$$G\left(\frac{\partial}{\partial x^j},\omega_\alpha\right)=G\left((\partial_j)^h+\Gamma_j^r(\partial_r)^v,\omega_\alpha\right)=$$

$$\alpha_2 g_{jr}\overline{K}_\alpha^r+(\alpha_1+\alpha_3)g_{jr}\overline{L}_\alpha^r+U_j\left[\beta_2\left(V_\alpha+U_s\overline{K}_\alpha^s\right)+(\beta_1+\beta_3)\,U_s\overline{L}_\alpha^s\right]=0 \tag{21}$$

Let $R^2=g_{ij}U^iU^j$, $\Theta_j=\alpha_j+R^2\beta_j$, $j=1,2,3$, $\Theta=\Theta_1(\Theta_1+\Theta_3)-\Theta_2^2$.

Transvecting (18) and (19) with U^j we obtain

$$\Theta_1U_rK_\alpha^r+\Theta_2U_rL_\alpha^r=-R^2\left[\beta_1V_\beta\Gamma_\alpha^\beta+\beta_2V_\alpha\right],$$

$$\Theta_2U_rK_\alpha^r+(\Theta_1+\Theta_3)\,U_rL_\alpha^r=-R^2\left[\beta_2V_\beta\Gamma_\alpha^\beta+(\beta_1+\beta_3)\,V_\alpha\right],$$

whence, solving for $U_rK_\alpha^r$ and $U_rL_\alpha^r$, we easily get

$$\alpha_1 g_{jr}K_\alpha^r+\alpha_2 g_{jr}L_\alpha^r=$$

$$-U_j\left[\beta_1V_\beta\Gamma_\alpha^\beta+\beta_2V_\alpha-\frac{R^2}{\Theta}\left(aV_\beta\Gamma_\alpha^\beta+bV_\alpha\right)\right], \tag{22}$$

where

$$\begin{aligned} a &= \beta_2\left(\beta_2\Theta_1-\beta_1\Theta_2\right)+\beta_1\left[\beta_1\left(\Theta_1+\Theta_3\right)-\beta_2\Theta_2\right],\\ b &= \beta_2\left[(\beta_1+\beta_3)\,\Theta_1-\beta_2\Theta_2\right]+\beta_1\left[\beta_2\left(\Theta_1+\Theta_3\right)-(\beta_1+\beta_3)\,\Theta_2\right]\end{aligned}$$

and

$$\alpha_2 g_{jr}K_\alpha^r+(\alpha_1+\alpha_3)\,g_{jr}L_\alpha^r=$$

$$-U_j\left[\beta_2V_\beta\Gamma_\alpha^\beta+(\beta_1+\beta_3)\,V_\alpha-\frac{R^2}{\Theta}\left(cV_\beta\Gamma_\alpha^\beta+dV_\alpha\right)\right], \tag{23}$$

where

$$\begin{aligned} c &= \beta_2\left[\beta_1\left(\Theta_1+\Theta_3\right)-\beta_2\Theta_2\right]+\left(\beta_1+\beta_3\right)\left(\beta_2\Theta_1-\beta_1\Theta_2\right), \\ d &= \beta_2\left[\beta_2\left(\Theta_1+\Theta_3\right)-\left(\beta_1+\beta_3\right)\Theta_2\right]+\left(\beta_1+\beta_3\right)\left[\left(\beta_1+\beta_3\right)\Theta_1-\beta_2\Theta_2\right]. \end{aligned}$$

Notice that

$$b = c$$

In a similar way, from (20) and (21), we deduce

$$\alpha_1 g_{jr}\overline{K}^r_\alpha+\alpha_2 g_{jr}\overline{L}^r_\alpha=-\left(\beta_1-\frac{R^2}{\Theta}a\right)U_j V_\alpha, \tag{24}$$

$$\alpha_2 g_{jr}\overline{K}^r_\alpha+(\alpha_1+\alpha_3) g_{jr}\overline{L}^r_\alpha=-\left(\beta_2-\frac{R^2}{\Theta}c\right)U_j V_\alpha. \tag{25}$$

From (18) - (21) we also get

$$G\left(\left(\partial_j\right)^v,\mu_\alpha\right)=G\left(\left(\partial_j\right)^h,\mu_\alpha\right)=G\left(\left(\partial_j\right)^v,\omega_\alpha\right)=G\left(\left(\partial_j\right)^h,\omega_\alpha\right)=0. \tag{26}$$

From the definition of G and the decompositions of μ_α and ω_β, by the use of (26), we find

$$\begin{aligned} &G\left(\mu_\alpha,\mu_\beta\right)= \\ &\qquad (\alpha_1+\alpha_3)g_{\alpha\beta}+\alpha_2\left(g_{\alpha\omega}\Gamma^\omega_\beta+g_{\beta\omega}\Gamma^\omega_\alpha\right)+\alpha_1 g_{\mu\omega}\Gamma^\mu_\alpha\Gamma^\omega_\beta+ \\ &\qquad V_\alpha\left[\beta_2\left(V_\omega\Gamma^\omega_\beta+U_sK^s_\beta\right)+\left(\beta_1+\beta_3\right)\left(V_\beta+U_sL^s_\beta\right)\right]+ \\ &\qquad\qquad V_\mu\Gamma^\mu_\alpha\left[\beta_1\left(V_\omega\Gamma^\omega_\beta+U_sK^s_\beta\right)+\beta_2\left(V_\beta+U_sL^s_\beta\right)\right], \end{aligned} \tag{27}$$

$$G\left(\omega_\alpha,\omega_\beta\right)=\alpha_1 g_{\alpha\beta}+V_\alpha\left[\beta_1\left(V_\beta+U_s\overline{K}^s_\beta\right)+\beta_2 U_s\overline{L}^s_\beta\right], \tag{28}$$

and either

$$\begin{aligned} &G\left(\mu_\alpha,\omega_\beta\right)= \\ &\qquad \alpha_2 g_{\alpha\beta}+\alpha_1 g_{\beta\omega}\Gamma^\omega_\alpha+V_\alpha\left[\beta_2\left(V_\beta+U_s\overline{K}^s_\beta\right)+\left(\beta_1+\beta_3\right)U_s\overline{L}^s_\beta\right]+ \\ &\qquad\qquad V_\mu\Gamma^\mu_\alpha\left[\beta_1\left(V_\beta+U_s\overline{K}^s_\beta\right)+\beta_2 U_s\overline{L}^s_\beta\right] \end{aligned} \tag{29}$$

or

$$\begin{aligned} &G\left(\mu_\alpha,\omega_\beta\right)= \\ &\qquad \alpha_2 g_{\alpha\beta}+\alpha_1 g_{\beta\omega}\Gamma^\omega_\alpha+V_\beta\left[\beta_1\left(V_\mu\Gamma^\mu_\alpha+U_sK^s_\alpha\right)+\beta_2\left(V_\alpha+U_sL^s_\alpha\right)\right]. \end{aligned} \tag{30}$$

Finally, comparing (18) to (22) and (19) to (23), from (27) we have

$$\begin{aligned} &G\left(\mu_\alpha,\mu_\beta\right)= \\ &\qquad (\alpha_1+\alpha_3)g_{\alpha\beta}+\alpha_2\left(g_{\alpha\omega}\Gamma^\omega_\beta+g_{\beta\omega}\Gamma^\omega_\alpha\right)+\alpha_1 g_{\mu\omega}\Gamma^\mu_\alpha\Gamma^\omega_\beta+ \\ &(\beta_1+\beta_3-\frac{R^2}{\Theta}d)V_\alpha V_\beta+\left(\beta_2-\frac{R^2}{\Theta}b\right)\left(V_\alpha T_\beta+V_\beta T_\alpha\right)+\left(\beta_1-\frac{R^2}{\Theta}a\right)T_\alpha T_\beta \end{aligned} \tag{31}$$

where

$$T_\alpha = V_\mu \Gamma^\mu_\alpha$$

Comparing (20) to (24), from (28) we get

$$G(\omega_\alpha, \omega_\beta) = \alpha_1 g_{\alpha\beta} + \left(\beta_1 - \frac{R^2}{\Theta} a\right) V_\alpha V_\beta. \tag{32}$$

At last, comparing (21) to (22), from (30) or (29), by the use of (22), we find

$$G(\mu_\alpha, \omega_\beta) = \alpha_2 g_{\alpha\beta} + \alpha_1 g_{\beta\omega}\Gamma^\omega_\alpha + \left(\beta_2 - \frac{R^2}{\Theta} b\right) V_\alpha V_\beta + \left(\beta_1 - \frac{R^2}{\Theta} a\right) T_\alpha V_\beta. \tag{33}$$

Notice that

$$R^2 = r^2 - V^\alpha V^\beta g_{\alpha\beta}. \tag{34}$$

Put

$$A_j = \partial_j\left(r^2\right) = \partial_j\left(R^2\right). \tag{35}$$

We also have

$$\frac{\partial}{\partial u^j}\left(r^2\right) = \frac{\partial}{\partial u^j}\left(R^2\right) = 2g_{ji}u^i = 2u_j. \tag{36}$$

To simplify (31) and (33), observe that

$$G(\mu_\alpha, \omega_\beta) = \alpha_2 g_{\alpha\beta} + \left(\beta_2 - \frac{R^2}{\Theta} b\right) V_\alpha V_\beta + G(\omega_\beta, \omega_\mu)\Gamma^\mu_\alpha. \tag{37}$$

$$G\left(\mu_\alpha, \mu_\beta\right) = (\alpha_1 + \alpha_3) g_{\alpha\beta} + (\beta_1 + \beta_3 - \frac{R^2}{\Theta} d) V_\alpha V_\beta + G(\omega_\alpha, \omega_\mu)\Gamma^\mu_\beta + G(\omega_\beta, \omega_\mu)\Gamma^\mu_\alpha - G(\omega_\nu, \omega_\mu)\Gamma^\nu_\alpha \Gamma^\mu_\beta. \tag{38}$$

By the Theorem 3 it is clear that G is bundle-like if and only if α_j, $j = 1, 2, 3$ are constants, $\alpha_1 \neq 0$ and $\beta_j - \frac{R^2}{\Theta} b_j$ are constants, where we have put $b_1 = a$, $b_2 = b$ and $b_3 = d - a$.

Taking into consideration Lemma 5 we obtain

Theorem 6 *Let $(M, g, \mathcal{F})$ be a Riemannian manifold with metric g that is bundle-like for a non-degenerated foliation $\mathcal{F}$. Then the semi-Riemannian $g-$natural metric G on TM is bundle like for a non-degenerated distribution if and only if*

$$for\ j = 1, 2, 3,\ \ \alpha_j(r^2) = C_j,$$

$$\beta_1 - \frac{R^2}{\Theta}\left\{\beta_2\left(\beta_2\Theta_1 - \beta_1\Theta_2\right) + \beta_1\left[\beta_1\left(\Theta_1 + \Theta_3\right) - \beta_2\Theta_2\right]\right\} = C_4,$$

$$\beta_2 - \frac{R^2}{\Theta}\left\{\beta_2\left[\beta_1\left(\Theta_1 + \Theta_3\right) - \beta_2\Theta_2\right] + \left(\beta_1 + \beta_3\right)\left(\beta_2\Theta_1 - \beta_1\Theta_2\right)\right\} = C_5,$$

$$\beta_3 - \frac{R^2}{\Theta}\left[(\beta_1 + \beta_3)^2\Theta_1 + \beta_1^2\left(\Theta_1 + \Theta_3\right) + \beta_2^2\Theta_3 - 2\beta_2(\beta_1 + \beta_3)\Theta_2\right] = C_6$$

and

$$\alpha(r^2) \neq 0, \quad \Phi(r^2) \neq 0,$$

C_k *being constants.*

It is easily seen that the Sasaki metric satisfies these conditions with $C_4 = C_5 = C_6 = 0$ while the Cheeger-Gromoll one does not.

3 Bundle-like g-natural metrics determined by the vertical distribution

Let $(U,(x,u)) = \big(U,(x^j,u^j)\big)$ be a coordinate neighbourhood on a manifold TM. $T_{(x,u)}TM$ is spanned by

$$\frac{\partial}{\partial u^j} = \left(\frac{\partial}{\partial x^j}\right)^v \tag{39}$$

and

$$\left(\frac{\partial}{\partial x^j}\right)^h = \frac{\partial}{\partial x^j} - \Gamma^r_{jt} u^t \frac{\partial}{\partial u^r}. \tag{40}$$

Then $\left\{\frac{\partial}{\partial u^j}, \left(\frac{\partial}{\partial x^j}\right)^h\right\}$ is a non-holonomic frame field on U with respect to the decomposition $T_{(x,u)}TM = V_{(x,u)}TM \oplus H_{(x,u)}TM$. Recall that the distribution $VTM = \cup_{(x,u)\in TM} V_{(x,u)}TM$ is involutive. Express each $\frac{\partial}{\partial x^j}$ with respect to this frame field:

$$\frac{\partial}{\partial x^j} = A^i_j \frac{\partial}{\partial u^i} + C^i_j \left(\frac{\partial}{\partial x^i}\right)^h. \tag{41}$$

Fields $\frac{\delta}{\delta x^j} = C^i_j \left(\frac{\partial}{\partial x^i}\right)^h$ also represents HTM on U and

$$\frac{\delta}{\delta x^j} = \frac{\partial}{\partial x^j} - A^i_j \frac{\partial}{\partial u^i}. \tag{42}$$

Following [BF] we call $\left\{\frac{\partial}{\partial u^j}, \frac{\delta}{\delta x^j}\right\}$ a semi-holonomic frame field on U. Making use of (39) and (40) we get at a point (x,u)

$$\frac{\delta}{\delta x^j} = \frac{\partial}{\partial x^j} - A^i_j \left(\frac{\partial}{\partial x^i}\right)^v = \left(\frac{\partial}{\partial x^j}\right)^h + \Gamma^r_{jt} u^t \left(\frac{\partial}{\partial x^r}\right)^v - A^i_j \left(\frac{\partial}{\partial x^i}\right)^v = \left(\frac{\partial}{\partial x^j}\right)^h + B^r_j \left(\frac{\partial}{\partial x^r}\right)^v. \tag{43}$$

Let G be a $g-natural$ metric on U. Put $u_j = u^r g_{rj}$. Then, by the use of (43) and (39), we find

$$G_{(x,u)}\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial u^j}\right) = \alpha_2 g_{ij} + \beta_2 u_i u_j + B^r_j(\alpha_1 g_{ri} + \beta_1 u_r u_i) = 0. \tag{44}$$

Hence, by transvecting with B^i_k, we get

$$\alpha_2 B^i_k g_{ij} + \beta_2 B^i_k u_i u_j + B^r_j B^i_k (\alpha_1 g_{ri} + \beta_1 u_r u_i) = 0 \tag{45}$$

which yields to

$$G_{(x,u)}(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = (\alpha_1 + \alpha_3) g_{ij} + (\beta_1 + \beta_3)\, u_i u_j - B^r_i B^s_j\, (\alpha_2 g_{rs} + \beta_2 u_r u_s)\,. \tag{46}$$

Transvecting (44) with u^i we obtain $(\alpha_2 + r^2\beta_2) u_j + (\alpha_1 + r^2\beta_1) B^r_j u_r = 0$ or

$$\Phi_2 u_j + \Phi_1 B^r_j u_r = 0. \tag{47}$$

If $\Phi_1(p) = 0$, then $\Phi_2(p) = 0$ and $\Phi(p) = \Phi_1(p)(\Phi_1(p) + \Phi_3(p)) - \Phi_2^2(p) = 0$. Consequently, G would not be regular at p. Transvecting (44) with g^{il} and applying (47) we obtain

$$\alpha_1 B^l_j = (\beta_1 \frac{\Phi_2}{\Phi 1} - \beta_2) u_j u^l - \alpha_2 \delta^l_j. \tag{48}$$

Suppose $\alpha_1 = 0$ at some point p. From (48) it follows

$$\alpha_2 B^j_m \delta^l_j = (\beta_1 \frac{\Phi_2}{\Phi 1} - \beta_2) B^j_m u_j u^l, \tag{49}$$

whence, by the use of (47) and (48)

$$\alpha_2 B^l_m = -(\beta_1 \frac{\Phi_2}{\Phi 1} - \beta_2) \frac{\Phi_2}{\Phi 1} u_m u^l = -\alpha_2 \frac{\Phi_2}{\Phi 1} \delta^l_m. \tag{50}$$

But $\alpha_1 = \alpha_2 = 0$ yields irregularity of the metric ($\alpha(p) = \alpha_1(p)(\alpha_1(p) + \alpha_3(p)) - \alpha_2^2(p)$). Therefore, if $\alpha_1 = 0$, then $\alpha_2 \neq 0$ and $B^l_m = -\frac{\Phi_2}{\Phi 1}\delta^l_m$. Substituting the last result into (44) we obtain

$$g_{ij} = \frac{1}{\alpha_2}(\beta_1 \frac{\Phi_2}{\Phi 1} - \beta_2) u_i u_j. \tag{51}$$

But in the case $\dim M = 1$, (ie. $rank\,[g_{ij}] = 1$), the functions β_j vanish, a contradiction. Thus (48) holds good.

Applying (48) to (46) we find

$$G_{(x,u)}(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = A g_{ij} + B u_i u_j \tag{52}$$

where

$$A = \alpha_1 + \alpha_3 - \frac{\alpha_2^3}{\alpha_1^2},$$

$$B = \beta_1 + \beta_3 + \frac{\alpha_2 \beta_1}{\alpha_1} \left(\frac{\Phi_2}{\Phi 1} - \frac{\beta_2}{\beta_1}\right) \left(\frac{\Phi_2}{\Phi 1} + \frac{\alpha_2}{\alpha_1}\right) - \beta_2 \left(\frac{\Phi_2}{\Phi 1}\right)^2$$

and all functions on the right hand sides depend on $r^2 = g_{rs}u^r u^s$. Differentiating (52) with respect to u^k we easily obtain

$$\begin{aligned} nA' + B'r^2 + B &= 0, \\ 2A' + 2B'r^2 + (n+1)B &= 0 \end{aligned}$$

whence, for arbitrary constants C_1, C_2,

$$B = \frac{C_1}{r^{n+2}}, \qquad A = -\frac{C_1}{n}\frac{1}{r^n} + C_2.$$

Taking into consideration Lemma 5 we obtain

Theorem 7 *Let $(TM, G, \mathcal{F})$ be a tangent bundle of a Riemannian manifold (M, g), with a g-natural metric G and foliation $\mathcal{F}$ determined by the vertical distribution VTM. A semi-Riemannian metric G on TM is bundle-like for the foliation $\mathcal{F}$ if and only if*

$$\begin{aligned} \beta_1 + \beta_3 + \frac{\alpha_2\beta_1}{\alpha_1}\left(\frac{\Phi_2}{\Phi 1} - \frac{\beta_2}{\beta_1}\right)\left(\frac{\Phi_2}{\Phi 1} + \frac{\alpha_2}{\alpha_1}\right) - \beta_2\left(\frac{\Phi_2}{\Phi 1}\right)^2 &= \frac{C_1}{r^{n+2}}, \\ \alpha_1 + \alpha_3 - \frac{\alpha_2^3}{\alpha_1^2} &= -\frac{C_1}{n}\frac{1}{r^n} + C_2 \neq 0, \\ \alpha(r^2) &\neq 0 \text{ and } \Phi(r^2) \neq 0. \end{aligned}$$

It is easily seen that the Sasaki metric as well as the Cheeger-Gromoll one satisfy these conditions with $C_1 = 0$.

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