XVIII International Fall Workshop on Geometry and Physics

WILLMORE SURFACES IN *Generalized Robertson-Walker spacetimes and static spacetimes*

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This talk is based on

M. Barros, _ and M. Ortega, *Rotational Surfaces in* $\mathbb{L}^3$ *and Solutions of the Nonlinear Sigma Model*. Commun. Math. Phys. 290, 437–477 (2009).

_, Willmore surfaces invariant under a 1-parameter group of isometries in Lorentzian 3-manifolds. In progress.

Introduction

The classical Willmore functional

The classical Willmore functional is defined as follows

$$\mathfrak{W}(\phi) = \int_M H^2 dA,$$

where

$S \longrightarrow$ compact, boundary free and connected surface

$\phi : S \to \mathbb{R}^3 \longrightarrow$ immersion

$H \longrightarrow$ mean curvature

$dA \longrightarrow$ element of area induced on S

Its critical points are called Willmore surfaces.

Its study was proposed by Willmore in 1965.

The Willmore functional in Riemannian setting

The Willmore functional in Riemannian setting

$$\mathfrak{W}(\phi) = \int_S (H^2 + \bar{R})\,dA + \int_{\partial S} k_g\, ds,$$

where

$$\begin{aligned}
&S \longrightarrow \text{surface (with boundary)}\\
&\phi : S \to \bar{M} \longrightarrow \text{immersion}\\
&\bar{M} \longrightarrow \text{Riemannian 3-manifold}\\
&H \longrightarrow \text{mean curvature of } \phi\\
&\bar{R} \longrightarrow \text{sectional curvature of } \phi(S) \text{ in } \bar{M}\\
&k_g \longrightarrow \text{geodesic curvature of } \partial S \text{ in } S\\
&\qquad\quad \text{(oriented as in the Stokes theorem).}
\end{aligned}$$

Proposed by Weiner in 1978.

The Willmore functional in Lorentzian setting

The Willmore functional in Lorentzian setting

$$\mathfrak{W}(\phi) = \int_S (H^2 + \varepsilon \bar{R}) dA - \int_{\partial S} k_g \, ds,$$

$S \longrightarrow$ surface (with boundary)

$\bar{M} \longrightarrow$ Lorentzian 3-manifold

$\phi : S \to \bar{M} \longrightarrow$ non-degenerate immersion with signature ε

$H \longrightarrow$ mean curvature of ϕ

$\bar{R} \longrightarrow$ sectional curvature of $\phi(S)$ in $\bar{M}$

$k_g \longrightarrow$ geodesic curvature of ∂S in S (oriented as in the Stokes theorem).

The Willmore functional in Lorentzian setting

Using the Gauss-Bonnet theorem for surfaces, in its Riemannian and its Lorentzian versions respectively, we get that

$\mathfrak{W}$ is invariant under conformal changes of the metric of $\bar{M}$.

$\phi : S \to \bar{M}$ is a Willmore surface if it is a critical point of $\mathfrak{W}$ under (compact support) variations fixing:

∂S and its Gauss map along ∂S

When $\bar{M} = \mathbb{L}^3$, the Gauss map of Willmore surfaces are the solutions of the 2-dimensional nonlinear sigma-model with symmetry $O(2,1)$.

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Elastic curves

Elastic energy:

$$\mathfrak{E}^{\lambda}(\alpha) = \int_{\alpha} (k^2 + \lambda) \qquad\qquad \lambda \in \mathbb{R}$$

$(\bar{M}, \bar{g}) \longrightarrow$ (semi-)Riemannian manifold

$\alpha : I \to \bar{M} \longrightarrow$ non-degenerate immersed curve with curvature k

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If α has no boundary & is a critical point of $\mathfrak{E}^\lambda \to$ closed elastic curve

Elastic curves

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$\alpha : I \to \bar{M} \quad \longrightarrow \quad$ non-degenerate immersed curve with curvature k

When α has boundary, it is called clamped elastic curve if it is a critical point of $\mathfrak{E}^\lambda$ under (compact support) variations fixing:

the boundary points of α and the tangents at them.

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the boundary points of α and the tangents at them.

$\lambda = 0 \longrightarrow$ free elastic curve

LINK

Willmore surfaces	generated by	free elastic curves	
of revolution in $\mathbb{R}^3$		hyperbolic plane	Barros
of revolution in $\mathbb{L}^3$ with spacelike axis		anti de Sitter plane	Barros, _ and Ortega
of revolution in $\mathbb{L}^3$ with null axis		anti de Sitter plane	Barros, _ and Ortega

LINK

(M, g) $\longrightarrow$ Riemannian or Lorentzian surface

γ $\longrightarrow$ nondegenerate curve immersed in (M, g)

$f : \mathbb{S}^1 \to R^+$ $\longrightarrow$ smooth function

$$\mathbb{S}^1 \times \gamma \text{ is Willmore in } (\mathbb{S}^1 \times M, \varepsilon dt^2 + f^2 g)$$

$$\Updownarrow$$

$$\gamma \text{ is a free elastic curve in } (M, g)$$

Barros

Natural question

Given:

$(\bar{M}, \bar{g})$ $\longrightarrow$ semi-Riemannian 3-manifold

G $\longrightarrow$ 1-parameter group of isometries

What must $(\bar{M}, \bar{g})$ and G satisfy to obtain that

G-invariant Willmore surfaces in $(\bar{M}, \bar{g})$

are generated by

elastic curves in certain surface

?

Technique

In all the previous results

G is COMPACT

Except for:

Rotational Willmore surfaces in $\mathbb{L}^3$ with null axis

Idea

Extend the technique to get results for

Lorentzian product manifolds of dimension 3,

G being a non necessarily compact 1-parameter group of isometries

WILLMORE SURFACES IN 3-DIM LORENTZIAN PRODUCT SPACES

THEOREM BARROS, _ AND ORTEGA

$\phi : S \to \bar{M}$ is a Willmore surface if and only if

$$\int_S \bar{g}(\mathfrak{R}(\mathbb{H}) + N(\bar{R}^{\mathbf{V}})N, \mathbf{V}^{\perp})dA = 0,$$

for any variational field $\mathbf{V}$ compatible with the boundary conditions.

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for any variational field $\mathbf{V}$ compatible with the boundary conditions.

$$\mathbb{H} \longrightarrow \text{mean curvature vector field}$$

$$\mathfrak{R} = \varepsilon(\triangle + \tilde{A}) + (\mathrm{Ric}(N, N) - 2(H^2 + \varepsilon\bar{R}))\,\mathbf{I}$$

is a kind of Schrödinger operator, being

$\triangle$	$\longrightarrow$	Laplacian respect to the normal conection
$\tilde{A}$	$\longrightarrow$	Simons' operator
Ric	$\longrightarrow$	Ricci curvature

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for any variational field $\mathbf{V}$ compatible with the boundary conditions.

$N \longrightarrow$ Gauss map along ϕ

$\bar{R}^{\mathbf{V}}(m, v) \to$ sectional curvature of $\bar{M}$ restricted to the level surface v, at the point m

Willmore surfaces in a 3-dim Lorentzian product space

(M^1, ds^2) $\longrightarrow$ 1-dimensional Riemannian manifold
(M, g) $\longrightarrow$ Riemannian or Lorentzian surface

$$(\bar{M}, \bar{g}) = (M \times M^1,\, g + \bar{\varepsilon}\, ds^2), \qquad \bar{\varepsilon} = \begin{cases} -1 & \text{if } g \text{ Riemannian} \\ 1 & \text{if } g \text{ Lorentzian} \end{cases}$$

$$S = \gamma \times M^1, \qquad \gamma \text{ non-degenerate curve in M}$$

Is $\gamma \times M^1$ Willmore?

$$N(\mathrm{R}^{\mathrm{V}}) = 0,$$

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Is $\gamma \times M^1$ Willmore?

$$N(\bar{R}^V) = 0,$$

so $\gamma \times M^1$ is Willmore if and only if

$$\int_{\gamma \times M^1} \bar{g}(\mathfrak{R}(\mathbb{H}), \mathbf{V}^{\perp})\, dA = 0,$$

if and only if

$$\mathfrak{R}(\mathbb{H}) = 0$$

if and only if

$$\tilde{\varepsilon} k'' + \varepsilon k^3 + 2R\, k = 0$$

where $\tilde{\varepsilon}$ is the signature of γ and R is the sectional curvature of M

Theorem

$\gamma \times M^1$ is a Willmore surface in $(M \times M^1, g + \bar{\varepsilon}\, ds^2)$

$$\Updownarrow$$

γ is a free elastic curve in (M, g)

Willmore surfaces in a 3-dim Lorentzian product space

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if and only if

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if and only if

$$\tilde{\varepsilon} k'' + \varepsilon k^3 + 2R\,k = 0 \quad \text{elastic curves equation}$$

where $\tilde{\varepsilon}$ is the signature of γ and R is the sectional curvature of M

THEOREM

$\gamma \times M^1$ is a Willmore surface in $(M \times M^1,\, g + \bar{\varepsilon}\, ds^2)$

$$\Updownarrow$$

γ is a free elastic curve in (M, g)

In a 3-dim warped product

Consider the warped product spacetimes

$$M^1 \times_f M = (M^1 \times M, \bar{\varepsilon} ds^2 + f^2 g) \quad \text{and} \quad M \times_h M^1 = (M^1 \times M, \bar{\varepsilon} h^2 ds^2 + g),$$

where $f : M^1 \longrightarrow \mathbb{R}^+$ and $h : M \longrightarrow \mathbb{R}^+$ are smooth

Since $\mathfrak{W}$ is invariant under conformal changes of the metric

Corollary

$M^1 \times \gamma$ is Willmore in $M^1 \times_f M$ $\Updownarrow$ γ is free elastic in (M, g)	$M^1 \times \gamma$ is Willmore in $M \times_h M^1$ $\Updownarrow$ γ is free elastic in $(M, \frac{1}{h^2} g)$

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WILLMORE SURFACES IN GENERALIZED ROBERTSON-WALKER AND STATIC SPACETIMES

In Generalized Robertson-Walker spacetimes

A Generalized Robertson-Walker spacetime is

$$I \times_f M = (I \times M, -dt^2 + f^2 g)$$

where (M, g) is Riemannian and $f : I \longrightarrow \mathbb{R}^+$

When $dimM = 2$

Corollary

$I \times \gamma$ is Willmore in the Generalized Robertson-Walker spacetime $I \times_f M$

$$\Updownarrow$$

γ is a free elastic curve in (M, g)

In Generalized Robertson-Walker spacetimes

A Generalized Robertson-Walker spacetime is

$$I \times_f M = (I \times M, -dt^2 + f^2 g)$$

where (M, g) is Riemannian and $f : I \longrightarrow \mathbb{R}^+$

When $dimM = 2$

Corollary

$$I \times \gamma \text{ is Willmore in the Generalized Robertson-Walker spacetime } I \times_f M$$

$$\Updownarrow$$

$$\gamma \text{ is a free elastic curve in } (M, g)$$

Standard Static and Static Spacetimes

A timelike Killing vector field ξ in a Lorentzian manifold $(\bar{M}, \bar{g})$ is

- static: if it is irrotational
- standard static: if static and there exists an isometry

$$\chi : (\bar{M}, \bar{g}) \longrightarrow (\mathbb{R} \times M, -h^2 dt^2 + g),$$

where $d\chi(\xi) = \partial_t$, $\xi(h \circ \chi) = 0$ and (M, g) is Riemannian

A standard static spacetime is

A spacetime admiting a standard static vector field.

A static spacetime is

A spacetime admiting a static vector field.
Locally, it is a standard static one.

In standard static spacetimes

Corollary

$$I \times \gamma \text{ is Willmore in the standard static spacetime } M \times_h I$$

$$\Updownarrow$$

$$\gamma \text{ is a free elastic curve in } (M, \frac{1}{h^2} g)$$

Given $G \longrightarrow$ 1-parameter group of isometries with timelike Killing vector field ξ

Corollary

If ξ is standard static,

$$G\text{-invariant Willmore surfaces in } (\bar{M}, \bar{g}) \text{ are generated by}$$
$$\text{elastic curves in } (M, \frac{-1}{\bar{g}(\xi,\xi)} \bar{g})$$

M being any maximal integral surface of the orthogonal distribution of ξ

Applying

Lemma M.Sánchez

Let ξ be a complete static vector field in $(\bar{M}, \bar{g})$. Its lift to the universal covering of $(\bar{M}, \bar{g})$ is standard static.

We get

Theorem _

If ξ is static,

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THEOREM _

If ξ is static,

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M being any maximal integral surface of the orthogonal distribution of ξ

With similar techniques, the following result is obtained

Given G $\longrightarrow$ 1-parameter subgroup of isometries
with spacelike Killing vector field ξ.

THEOREM

If ξ has no zero and it is irrotational, then

G-invariant Willmore surfaces are generated by
elastic curves in $\left(M, \dfrac{1}{g(\xi,\xi)}g\right)$

M being any maximal integral surface of the orthogonal distribution of ξ

The end