Moving Frames

Peter J. Olver

University of Minnesota

http://www.math.umn.edu/ ∼ olver

Benasque, September 2009

History of Moving Frames

Classical contributions:

M. Bartels ($\sim$1800), J. Serret, J. Frénet, G. Darboux, É. Cotton,

Élie Cartan

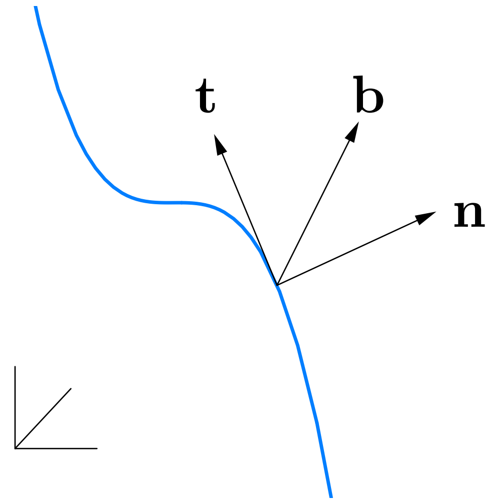
Modern developments: (1970's)

S.S. Chern, M. Green, P. Griffiths, G. Jensen, T. Ivey, J. Landsberg, ...

The equivariant approach: (1997 –)

PJO, M. Fels, G. Marí–Beffa, I. Kogan, J. Cheh, J. Pohjanpelto, P. Kim, M. Boutin, D. Lewis, E. Mansfield, E. Hubert, E. Shemyakova, O. Morozov, R. McLenaghan, R. Smirnov, J. Yue, A. Nikitin, J. Patera, ...

Moving Frame — Space Curves



tangent	normal	binormal
$\mathbf{t} = \dfrac{dz}{ds}$	$\mathbf{n} = \dfrac{d^2 z}{ds^2}$	$\mathbf{b} = \mathbf{t} \times \mathbf{n}$

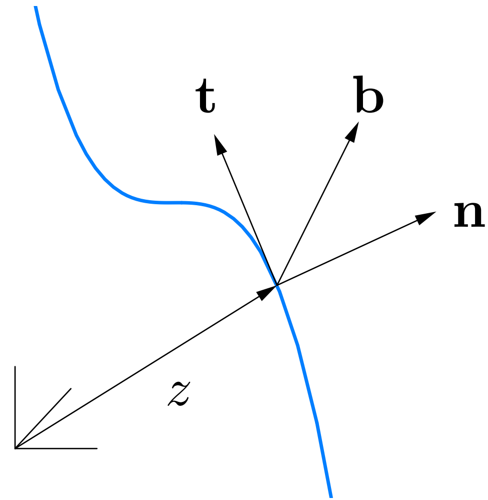
s — arc length

Frénet–Serret equations

$$\frac{d\mathbf{t}}{ds} = \kappa\,\mathbf{n} \qquad \frac{d\mathbf{n}}{ds} = -\,\kappa\,\mathbf{t} + \tau\,\mathbf{b} \qquad \frac{d\mathbf{b}}{ds} = -\,\tau\,\mathbf{n}$$

κ — curvature $\qquad$ τ — torsion

Moving Frame — Space Curves



tangent   normal   binormal

$$\mathbf{t} = \frac{dz}{ds} \qquad \mathbf{n} = \frac{d^2 z}{ds^2} \qquad \mathbf{b} = \mathbf{t} \times \mathbf{n}$$

s — arc length

Frénet–Serret equations

$$\frac{d\mathbf{t}}{ds} = \kappa\,\mathbf{n} \qquad \frac{d\mathbf{n}}{ds} = -\kappa\,\mathbf{t} + \tau\,\mathbf{b} \qquad \frac{d\mathbf{b}}{ds} = -\tau\,\mathbf{n}$$

κ — curvature τ — torsion

"I did not quite understand how he [Cartan] does this in general, though in the examples he gives the procedure is clear."

"Nevertheless, I must admit I found the book, like most of Cartan's papers, hard reading."

— Hermann Weyl

"Cartan on groups and differential geometry"
Bull. Amer. Math. Soc. **44** (1938) 598–601

Applications of Moving Frames

- Differential geometry
- Equivalence
- Symmetry
- Differential invariants
- Rigidity
- Identities and syzygies
- Joint invariants and semi-differential invariants
- Invariant differential forms and tensors
- Integral invariants
- Classical invariant theory

- Computer vision
    - object recognition
    - symmetry detection
- Invariant variational problems
- Invariant numerical methods
- Mechanics, including DNA
- Poisson geometry & solitons
- Killing tensors in relativity
- Invariants of Lie algebras in quantum mechanics
- Control theory
- Lie pseudo-groups

The Basic Equivalence Problem

M — smooth m-dimensional manifold.

G — transformation group acting on M

- finite-dimensional Lie group
- infinite-dimensional Lie pseudo-group

Equivalence:

Determine when two p-dimensional submanifolds

$$N \quad \text{and} \quad \overline{N} \subset M$$

are *congruent*:

$$\overline{N} = g \cdot N \qquad \text{for} \qquad g \in G$$

Symmetry:

Find all symmetries,

i.e., self-equivalences or *self-congruences*:

$$N = g \cdot N$$

Classical Geometry — *F. Klein*

- **Euclidean group:** $$G = \begin{cases} \mathrm{SE}(m) = \mathrm{SO}(m) \ltimes \mathbb{R}^m \\ \mathrm{E}(m) = \mathrm{O}(m) \ltimes \mathbb{R}^m \end{cases}$$

  $$\boxed{z \longmapsto A \cdot z + b} \qquad A \in \mathrm{SO}(m) \text{ or } \mathrm{O}(m), \quad b \in \mathbb{R}^m, \quad z \in \mathbb{R}^m$$

  $\Rightarrow$ isometries: rotations, translations , (reflections)

- **Equi-affine group:** $$G = \mathrm{SA}(m) = \mathrm{SL}(m) \ltimes \mathbb{R}^m$$

  $A \in \mathrm{SL}(m)$ — volume-preserving

- **Affine group:** $$G = \mathrm{A}(m) = \mathrm{GL}(m) \ltimes \mathbb{R}^m$$

  $A \in \mathrm{GL}(m)$

- **Projective group:** $$G = \mathrm{PSL}(m+1)$$

  acting on $\mathbb{R}^m \subset \mathbb{R}\boldsymbol{\xi}^m$

$\Longrightarrow$ Applications in computer vision

Tennis, Anyone?

Classical Invariant Theory

Binary form:

$$Q(x) = \sum_{k=0}^{n} \binom{n}{k} a_k \, x^k$$

Equivalence of polynomials (binary forms):

$$Q(x) = (\gamma x + \delta)^n \, \overline{Q}\left(\frac{\alpha x + \beta}{\gamma x + \delta}\right) \qquad g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}(2)$$

- multiplier representation of GL(2)
- modular forms

$$Q(x) = (\gamma x + \delta)^n \, \overline{Q}\left(\frac{\alpha x + \beta}{\gamma x + \delta}\right)$$

Transformation group:

$$g : (x, u) \longmapsto \left(\frac{\alpha x + \beta}{\gamma x + \delta}, \; \frac{u}{(\gamma x + \delta)^n}\right)$$

Equivalence of functions $\Longleftrightarrow$ equivalence of graphs

$$\Gamma_Q = \{\, (x, u) = (x, Q(x)) \,\} \subset \mathbb{C}^2$$

Moving Frames

Definition.

A moving frame is a G-equivariant map

$$\rho : M \longrightarrow G$$

Equivariance:

$$\rho(g \cdot z) = \begin{cases} g \cdot \rho(z) & \text{left moving frame} \\ \rho(z) \cdot g^{-1} & \text{right moving frame} \end{cases}$$

$$\rho_{left}(z) = \rho_{right}(z)^{-1}$$

The Main Result

Theorem. A moving frame exists in a neighborhood of a point $z \in M$ if and only if G acts freely and regularly near z.

Isotropy & Freeness

Isotropy subgroup: $G_z = \{ g \mid g \cdot z = z \}$ for $z \in M$

- free — the only group element $g \in G$ which fixes *one* point $z \in M$ is the identity: $\Longrightarrow$ $G_z = \{e\}$ for all $z \in M$.
- locally free — the orbits all have the same dimension as G: $\Longrightarrow$ G_z is a discrete subgroup of G.
- regular — all orbits have the same dimension and intersect sufficiently small coordinate charts only once

  $\not\approx$ irrational flow on the torus
- effective — the only group element which fixes *every* point in M is the identity: $g \cdot z = z$ for all $z \in M$ iff $g = e$:

$$G_M^* = \bigcap_{z \in M} G_z = \{e\}$$

Proof of the Main Theorem

Necessity: Let $\rho : M \to G$ be a left moving frame.

Proof of the Main Theorem

Necessity: Let $\rho : M \to G$ be a left moving frame.

Freeness: If $g \in G_z$, so $g \cdot z = z$, then by left equivariance:

$$\rho(z) = \rho(g \cdot z) = g \cdot \rho(z).$$

Therefore $g = e$, and hence $G_z = \{e\}$ for all $z \in M$.

Proof of the Main Theorem

Necessity: Let $\rho : M \to G$ be a left moving frame.

Freeness: If $g \in G_z$, so $g \cdot z = z$, then by left equivariance:

$$\rho(z) = \rho(g \cdot z) = g \cdot \rho(z).$$

Therefore $g = e$, and hence $G_z = \{e\}$ for all $z \in M$.

Regularity: Suppose $z_n = g_n \cdot z \longrightarrow z$ as $n \to \infty$.

By continuity, $\rho(z_n) = \rho(g_n \cdot z) = g_n \cdot \rho(z) \longrightarrow \rho(z)$.
Hence $g_n \longrightarrow e$ in G.

Proof of the Main Theorem

Necessity: Let $\rho : M \to G$ be a left moving frame.

Freeness: If $g \in G_z$, so $g \cdot z = z$, then by left equivariance:

$$\rho(z) = \rho(g \cdot z) = g \cdot \rho(z).$$

Therefore $g = e$, and hence $G_z = \{e\}$ for all $z \in M$.

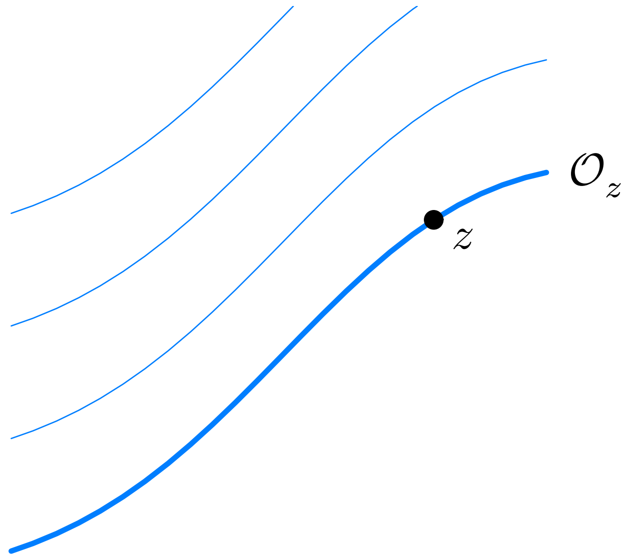
Regularity: Suppose $z_n = g_n \cdot z \longrightarrow z$ as $n \to \infty$.

By continuity, $\rho(z_n) = \rho(g_n \cdot z) = g_n \cdot \rho(z) \longrightarrow \rho(z)$.
Hence $g_n \longrightarrow e$ in G.

Sufficiency: By direct construction — “normalization”.

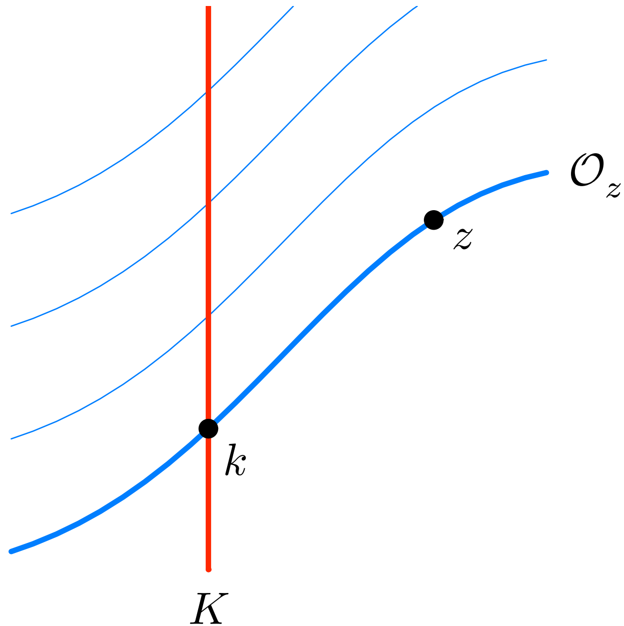
Q.E.D.

Geometric Construction



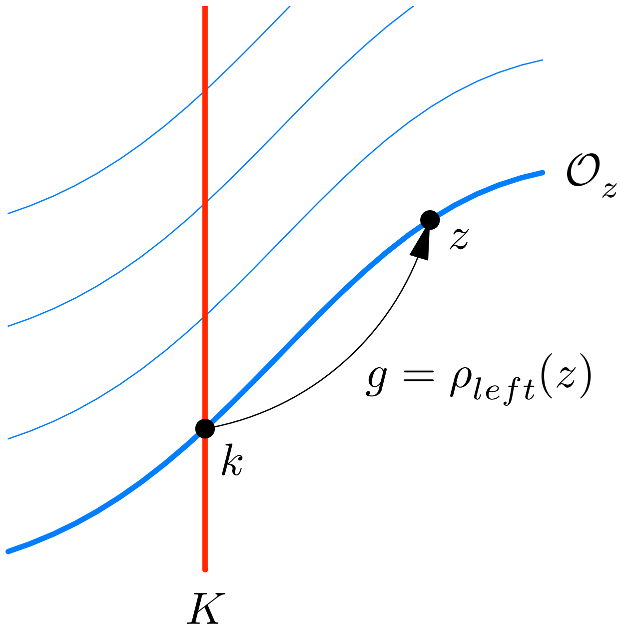
Normalization = choice of cross-section to the group orbits

Geometric Construction



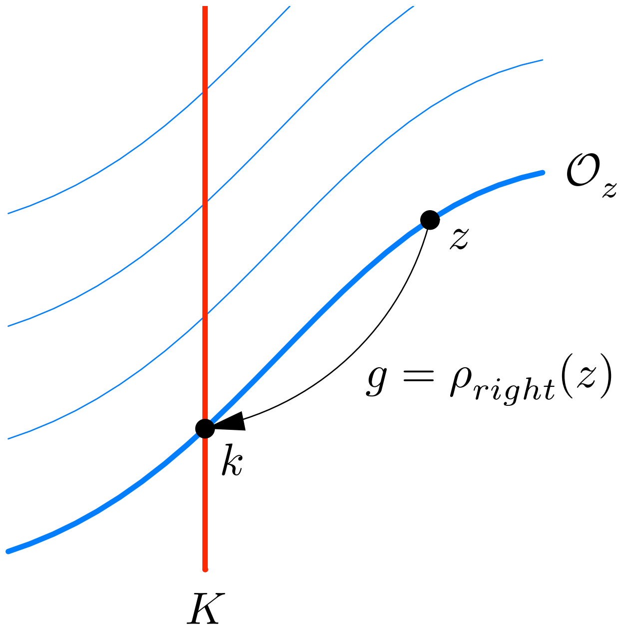
Normalization = choice of cross-section to the group orbits

Geometric Construction



Normalization = choice of cross-section to the group orbits

Geometric Construction



Normalization = choice of cross-section to the group orbits

K — cross-section to the group orbits

$\mathcal{O}_z$ — orbit through $z \in M$

$k \in K \cap \mathcal{O}_z$ — unique point in the intersection

- k is the *canonical or normal form* of z
- the (nonconstant) coordinates of k are the fundamental invariants

$g \in G$ — *unique* group element mapping k to z

$\Longrightarrow$ freeness

$\rho(z) = g$ left moving frame $\rho(h \cdot z) = h \cdot \rho(z)$

$$k = \rho^{-1}(z) \cdot z = \rho_{right}(z) \cdot z$$

Algebraic Construction

$$r = \dim G \ \leq \ m = \dim M$$

Coordinate cross-section

$$K = \{ z_1 = c_1, \ \ldots \ , z_r = c_r \}$$

left	right
$w(g, z) = g^{-1} \cdot z$	$w(g, z) = g \cdot z$

$g = (g_1, \ldots, g_r)$ — group parameters

$z = (z_1, \ldots, z_m)$ — coordinates on M

Choose $r = \dim G$ components to *normalize*:

$$w_1(g, z) = c_1 \qquad \ldots \qquad w_r(g, z) = c_r$$

Solve for the group parameters $g = (g_1, \ldots, g_r)$

$\Longrightarrow$ Implicit Function Theorem

The solution

$$g = \rho(z)$$

is a (local) moving frame.

The Fundamental Invariants

Substituting the moving frame formulae

$$g = \rho(z)$$

into the unnormalized components of $w(g, z)$ produces the fundamental invariants

$$I_1(z) = w_{r+1}(\rho(z), z) \quad \ldots \quad I_{m-r}(z) = w_m(\rho(z), z)$$

Theorem. Every invariant $I(z)$ can be (locally) uniquely written as a function of the fundamental invariants:

$$I(z) = H(I_1(z), \ \ldots \ , I_{m-r}(z))$$

Prolongation

Most interesting group actions (Euclidean, affine, projective, etc.) are *not* free!

Freeness typically fails because the dimension of the underlying manifold is not large enough, i.e., $m < r = \dim G$.

Thus, to make the action free, we must increase the dimension of the space via some natural prolongation procedure.

- An effective action can usually be made free by:

- Prolonging to derivatives (jet space)

$$G^{(n)} : \mathrm{J}^n(M,p) \longrightarrow \mathrm{J}^n(M,p)$$

$\Longrightarrow$ differential invariants

- Prolonging to Cartesian product actions

$$G^{\times n} : M \times \cdots \times M \longrightarrow M \times \cdots \times M$$

$\Longrightarrow$ joint invariants

- Prolonging to "multi-space"

$$G^{(n)} : M^{(n)} \longrightarrow M^{(n)}$$

$\Longrightarrow$ joint or semi-differential invariants
$\Longrightarrow$ invariant numerical approximations

- Prolonging to derivatives (jet space)

$$G^{(n)} : \mathrm{J}^n(M,p) \longrightarrow \mathrm{J}^n(M,p)$$

$\Longrightarrow$ differential invariants

- Prolonging to Cartesian product actions

$$G^{\times n} : M \times \cdots \times M \longrightarrow M \times \cdots \times M$$

$\Longrightarrow$ joint invariants

- Prolonging to "multi-space"

$$G^{(n)} : M^{(n)} \longrightarrow M^{(n)}$$

$\Longrightarrow$ joint or semi-differential invariants

$\Longrightarrow$ invariant numerical approximations

Euclidean Plane Curves

Special Euclidean group: $G = \mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$

acts on $M = \mathbb{R}^2$ via rigid motions: $w = R\, z + c$

To obtain the classical (left) moving frame we invert the group transformations:

$$\left.\begin{aligned} y &= \quad \cos\phi\,(x-a) + \sin\phi\,(u-b) \\ v &= -\sin\phi\,(x-a) + \cos\phi\,(u-b) \end{aligned}\right\} \qquad w = R^{-1}(z-c)$$

Assume for simplicity the curve is (locally) a graph:

$$\mathcal{C} = \{u = f(x)\}$$

$\Longrightarrow$ extensions to parametrized curves are straightforward

Prolong the action to J^n via implicit differentiation:

$$y = \cos\phi\,(x-a) + \sin\phi\,(u-b)$$

$$v = -\sin\phi\,(x-a) + \cos\phi\,(u-b)$$

$$v_y = \frac{-\sin\phi + u_x\cos\phi}{\cos\phi + u_x\sin\phi}$$

$$v_{yy} = \frac{u_{xx}}{(\cos\phi + u_x\sin\phi)^3}$$

$$v_{yyy} = \frac{(\cos\phi + u_x\sin\phi)u_{xxx} - 3u_{xx}^2\sin\phi}{(\cos\phi + u_x\sin\phi)^5}$$

$$\vdots$$

Choose a cross-section, or, equivalently a set of
$r = \dim G = 3$ normalization equations:

$$y = \cos\phi\,(x - a) + \sin\phi\,(u - b) = 0$$

$$v = -\sin\phi\,(x - a) + \cos\phi\,(u - b) = 0$$

$$v_y = \frac{-\sin\phi + u_x \cos\phi}{\cos\phi + u_x \sin\phi} = 0$$

$$v_{yy} = \frac{u_{xx}}{(\cos\phi + u_x \sin\phi)^3}$$

$$v_{yyy} = \frac{(\cos\phi + u_x \sin\phi)u_{xxx} - 3u_{xx}^2 \sin\phi}{(\cos\phi + u_x \sin\phi)^5}$$

$$\vdots$$

Solve the normalization equations for the group parameters:

$$y = \quad \cos\phi\,(x - a) + \sin\phi\,(u - b) = 0$$

$$v = -\sin\phi\,(x - a) + \cos\phi\,(u - b) = 0$$

$$v_y = \frac{-\sin\phi + u_x \cos\phi}{\cos\phi + u_x \sin\phi} = 0$$

The result is the left moving frame $\rho : \mathrm{J}^1 \longrightarrow \mathrm{SE}(2)$

$$a = x \qquad b = u \qquad \phi = \tan^{-1} u_x$$

$$a = x \qquad b = u \qquad \phi = \tan^{-1} u_x$$

Substitute into the moving frame formulas for the group parameters into the remaining prolonged transformation formulae to produce the basic differential invariants:

$$v_{yy} = \frac{u_{xx}}{(\cos\phi + u_x \sin\phi)^3} \longmapsto \kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}}$$

$$v_{yyy} = \cdots \longmapsto \frac{d\kappa}{ds} = \frac{(1+u_x^2)u_{xxx} - 3u_x u_{xx}^2}{(1+u_x^2)^3}$$

$$v_{yyyy} = \cdots \longmapsto \frac{d^2\kappa}{ds^2} - 3\kappa^3 = \cdots$$

Theorem. All differential invariants are functions of the derivatives of curvature with respect to arc length:

$$\kappa \qquad \frac{d\kappa}{ds} \qquad \frac{d^2\kappa}{ds^2} \qquad \cdots$$

The invariant differential operators and invariant differential forms are also substituting the moving frame formulas for the group parameters:

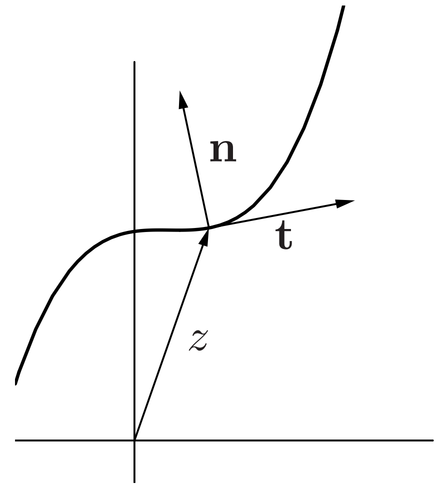
(Contact-)invariant one-form — arc length element

$$dy = (\cos\phi + u_x \sin\phi)\, dx \quad \longmapsto \quad ds = \sqrt{1 + u_x^2}\; dx$$

Invariant differential operator — arc length derivative

$$\frac{d}{dy} = \frac{1}{\cos\phi + u_x \sin\phi}\,\frac{d}{dx} \quad \longmapsto \quad \frac{d}{ds} = \frac{1}{\sqrt{1 + u_x^2}}\,\frac{d}{dx}$$

The Classical Picture:



Moving frame $\qquad \rho : \ (x, u, u_x) \longmapsto (R, \mathbf{a}) \in \mathrm{SE}(2)$

$$R = \frac{1}{\sqrt{1+u_x^2}} \begin{pmatrix} 1 & -u_x \\ u_x & 1 \end{pmatrix} = (\,\mathbf{t},\ \mathbf{n}\,) \qquad \mathbf{a} = \begin{pmatrix} x \\ u \end{pmatrix}$$

Equi-affine Curves $\qquad G = \mathrm{SA}(2)$

$$z \longmapsto A\,z + \mathbf{b} \qquad\qquad A \in \mathrm{SL}(2), \qquad \mathbf{b} \in \mathbb{R}^2$$

Invert for left moving frame:

$$\left.\begin{aligned} y &= \delta\,(x-a) - \beta\,(u-b) \\ v &= -\,\gamma\,(x-a) + \alpha\,(u-b) \end{aligned}\right\} \qquad w = A^{-1}(z-b)$$

$$\alpha\,\delta - \beta\,\gamma = 1$$

Prolong to J^3 via implicit differentiation

$$dy = (\delta - \beta\,u_x)\;dx \qquad\qquad D_y = \frac{1}{\delta - \beta\,u_x}\,D_x$$

Prolongation:

$$y = \delta\,(x - a) - \beta\,(u - b)$$

$$v = -\,\gamma\,(x - a) + \alpha\,(u - b)$$

$$v_y = -\,\frac{\gamma - \alpha\,u_x}{\delta - \beta\,u_x}$$

$$v_{yy} = -\,\frac{u_{xx}}{(\delta - \beta\,u_x)^3}$$

$$v_{yyy} = -\,\frac{(\delta - \beta\,u_x)\,u_{xxx} + 3\,\beta\,u_{xx}^2}{(\delta - \beta\,u_x)^5}$$

$$v_{yyyy} = -\,\frac{u_{xxxx}(\delta - \beta\,u_x)^2 + 10\,\beta\,(\delta - \beta\,u_x)\,u_{xx}\,u_{xxx} + 15\,\beta^2\,u_{xx}^3}{(\delta - \beta\,u_x)^7}$$

$$v_{yyyyy} = \;\ldots$$

Normalization: $\qquad r = \dim G = 5$

$$y = \delta\,(x - a) - \beta\,(u - b) = 0$$

$$v = -\gamma\,(x - a) + \alpha\,(u - b) = 0$$

$$v_y = -\frac{\gamma - \alpha\,u_x}{\delta - \beta\,u_x} = 0$$

$$v_{yy} = -\frac{u_{xx}}{(\delta - \beta\,u_x)^3} = 1$$

$$v_{yyy} = -\frac{(\delta - \beta\,u_x)\,u_{xxx} + 3\,\beta\,u_{xx}^2}{(\delta - \beta\,u_x)^5} = 0$$

$$v_{yyyy} = -\frac{u_{xxxx}(\delta - \beta\,u_x)^2 + 10\,\beta\,(\delta - \beta\,u_x)\,u_{xx}\,u_{xxx} + 15\,\beta^2\,u_{xx}^3}{(\delta - \beta\,u_x)^7}$$

$$v_{yyyyy} = \;\ldots$$

Equi-affine Moving Frame

$$\rho : \ (x, u, u_x, u_{xx}, u_{xxx}) \longmapsto (A, \mathbf{b}) \in \mathrm{SA}(2)$$

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \sqrt[3]{u_{xx}} & -\frac{1}{3}\, u_{xx}^{-5/3}\, u_{xxx} \\ u_x \sqrt[3]{u_{xx}} & u_{xx}^{-1/3} - \frac{1}{3}\, u_{xx}^{-5/3}\, u_{xxx} \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x \\ u \end{pmatrix}$$

Nondegeneracy condition: $u_{xx} \neq 0.$

Equi-affine arc length

$$dy = (\delta - \beta\, u_x)\ dx \quad \longmapsto \quad ds = \sqrt[3]{u_{xx}}\ dx$$

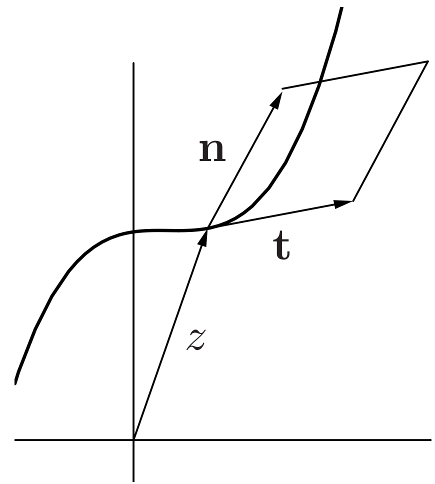
Equi-affine curvature

$$v_{yyyy} \quad \longmapsto \quad \kappa = \frac{5\, u_{xx} u_{xxxx} - 3\, u_{xxx}^2}{9\, u_{xx}^{8/3}}$$

$$v_{yyyyy} \quad \longmapsto \quad \frac{d\kappa}{ds}$$

$$v_{yyyyyy} \quad \longmapsto \quad \frac{d^2\kappa}{ds^2} - 5\kappa^2$$

The Classical Picture:



$$A = \begin{pmatrix} \sqrt[3]{u_{xx}} & -\frac{1}{3}\, u_{xx}^{-5/3}\, u_{xxx} \\ u_x \sqrt[3]{u_{xx}} & u_{xx}^{-1/3} - \frac{1}{3}\, u_{xx}^{-5/3}\, u_{xxx} \end{pmatrix} = (\, \mathbf{t},\ \mathbf{n}\,) \qquad \mathbf{b} = \begin{pmatrix} x \\ u \end{pmatrix}$$

Equivalence & Invariants

- Equivalent submanifolds $N \approx \overline{N}$
  must have the same invariants: $I = \bar{I}$.

Constant invariants provide immediate information:

$$\text{e.g.} \qquad \kappa = 2 \qquad \Longleftrightarrow \qquad \overline{\kappa} = 2$$

Non-constant invariants are not useful in isolation, because an equivalence map can drastically alter the dependence on the submanifold parameters:

$$\text{e.g.} \qquad \kappa = x^3 \qquad \text{versus} \qquad \overline{\kappa} = \sinh x$$

Syzygies

However, a functional dependency or syzygy among the invariants *is* intrinsic:

$$\text{e.g.} \qquad \kappa_s = \kappa^3 - 1 \quad \Longleftrightarrow \quad \overline{\kappa}_{\overline{s}} = \overline{\kappa}^3 - 1$$

- Universal syzygies — Gauss–Codazzi
- Distinguishing syzygies.

Equivalence & Syzygies

Theorem. (Cartan) Two smooth submanifolds are (locally) equivalent if and only if they have identical syzygies among *all* their differential invariants.

Proof:

Cartan's technique of the graph:

Construct the graph of the equivalence map as the solution to a (Frobenius) integrable differential system, which can be integrated by solving ordinary differential equations.

Finiteness of Generators and Syzygies

♠ There are, in general, an infinite number of differential invariants and hence an infinite number of syzygies must be compared to establish equivalence.

♡ But the higher order syzygies are all consequences of a finite number of low order syzygies!

Example — Plane Curves

If non-constant, both κ and κ_s depend on a single parameter, and so, locally, are subject to a syzygy:

$$\kappa_s = H(\kappa) \qquad (*)$$

But then

$$\kappa_{ss} = \frac{d}{ds} H(\kappa) = H'(\kappa)\,\kappa_s = H'(\kappa)\,H(\kappa)$$

and similarly for κ_{sss}, etc.

Consequently, all the higher order syzygies are generated by the fundamental first order syzygy $(*)$.

Thus, for Euclidean (or equi-affine or projective or ...) plane curves we need only know a single syzygy between κ and κ_s in order to establish equivalence!

The Signature Map

The generating syzygies are encoded by the signature map

$$\Sigma : N \longrightarrow \mathcal{S}$$

of the submanifold N, which is parametrized by the fundamental differential invariants:

$$\Sigma(x) = (I_1(x), \ldots, I_m(x))$$

The image

$$\mathcal{S} = \operatorname{Im} \Sigma$$

is the signature subset (or submanifold) of N.

Equivalence & Signature

Theorem. Two smooth submanifolds are equivalent

$$\overline{N} = g \cdot N$$

if and only if their signatures are identical

$$\overline{\mathcal{S}} = \mathcal{S}$$

Signature Curves

Definition. The *signature curve* $\mathcal{S} \subset \mathbb{R}^2$ of a curve $\mathcal{C} \subset \mathbb{R}^2$ is parametrized by the two lowest order differential invariants

$$\mathcal{S} = \left\{ \left(\kappa \, , \, \frac{d\kappa}{ds} \right) \right\} \quad \subset \quad \mathbb{R}^2$$

Equivalence & Signature Curves

Theorem. Two smooth curves $\mathcal{C}$ and $\overline{\mathcal{C}}$ are equivalent:

$$\overline{\mathcal{C}} = g \cdot \mathcal{C}$$

if and only if their signature curves are identical:

$$\overline{\mathcal{S}} = \mathcal{S}$$

$\Longrightarrow$ object recognition

Symmetry and Signature

Theorem. The dimension of the symmetry group

$$G_N = \{ g \mid g \cdot N \subset N \}$$

of a nonsingular submanifold $N \subset M$ equals the codimension of its signature:

$$\dim G_N = \dim N - \dim \mathcal{S}$$

Corollary. For a nonsingular submanifold $N \subset M$,

$$0 \leq \dim G_N \leq \dim N$$

$\Longrightarrow$ Only totally singular submanifolds can have larger symmetry groups!

Maximally Symmetric Submanifolds

Theorem. The following are equivalent:

- The submanifold N has a p-dimensional symmetry group
- The signature $\mathcal{S}$ degenerates to a point: $\dim \mathcal{S} = 0$
- The submanifold has all constant differential invariants
- $N = H \cdot \{ z_0 \}$ is the orbit of a p-dimensional subgroup $H \subset G$

$\Longrightarrow$ Euclidean geometry: circles, lines, helices, spheres, cylinders, planes, . .

$\Longrightarrow$ Equi-affine plane geometry: conic sections.

$\Longrightarrow$ Projective plane geometry: W curves (*Lie & Klein*)

Discrete Symmetries

Definition. The index of a submanifold N equals the number of points in N which map to a generic point of its signature:

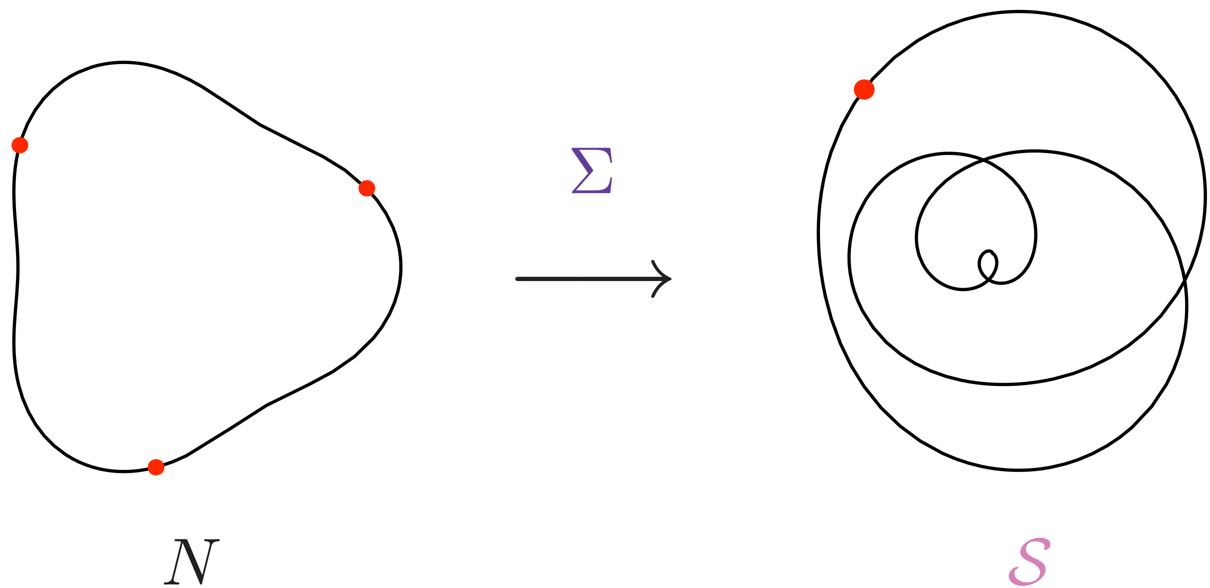
$$\iota_N = \min\left\{ \# \Sigma^{-1}\{w\} \;\middle|\; w \in \mathcal{S} \right\}$$

$\Longrightarrow$ Self–intersections

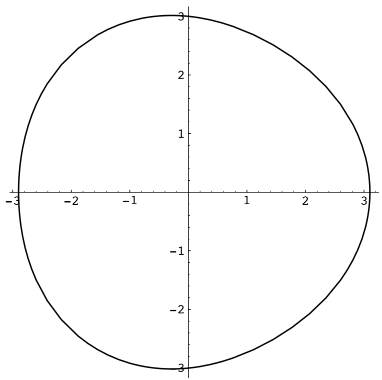
Theorem. The cardinality of the symmetry group of a submanifold N equals its index ι_N.

$\Longrightarrow$ Approximate symmetries

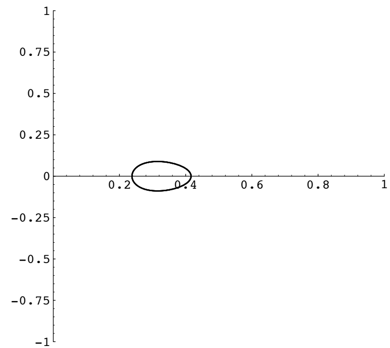
The Index



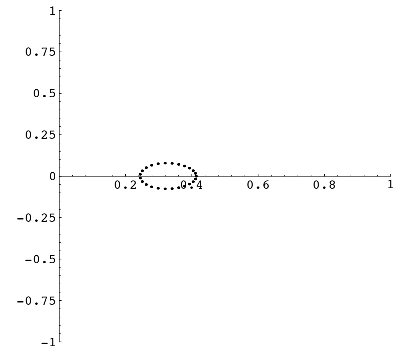
The polar curve $r = 3 + \frac{1}{10}\cos 3\theta$



The Original Curve

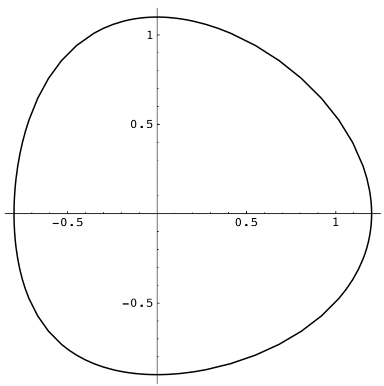


Euclidean Signature

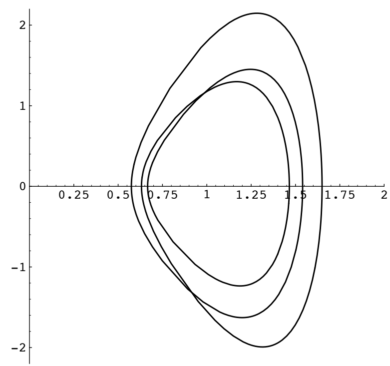


Numerical Signature

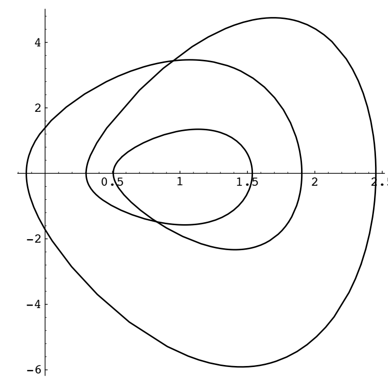
The Curve $x = \cos t + \frac{1}{5}\cos^2 t, \quad y = \sin t + \frac{1}{10}\sin^2 t$



The Original Curve

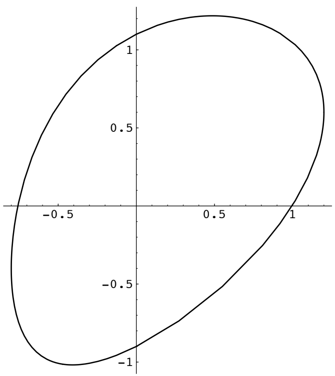


Euclidean Signature

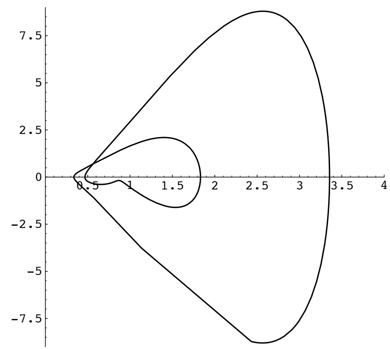


Affine Signature

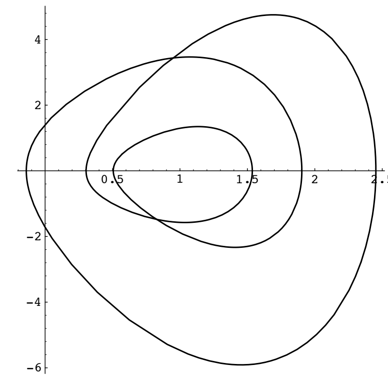
The Curve $x = \cos t + \frac{1}{5}\cos^2 t, \quad y = \frac{1}{2}x + \sin t + \frac{1}{10}\sin^2 t$



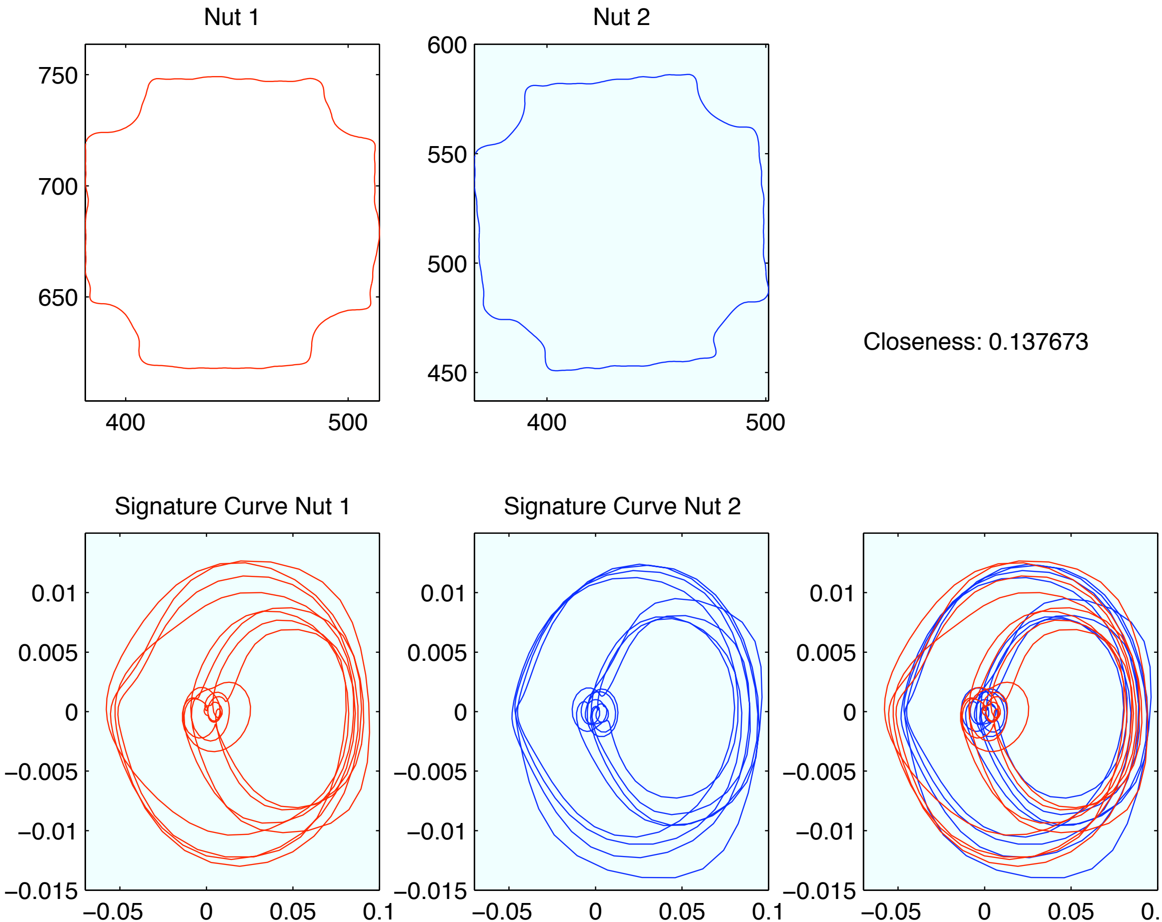
The Original Curve

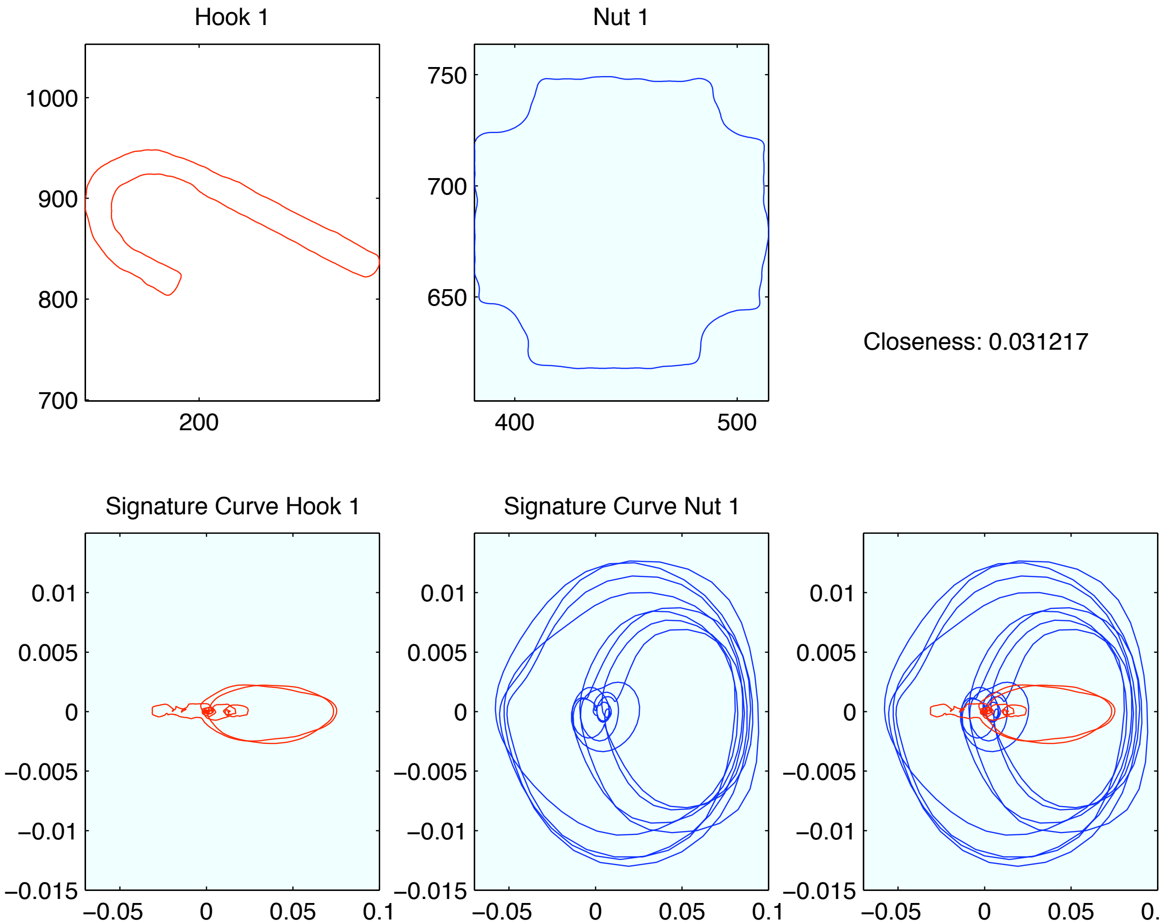


Euclidean Signature



Affine Signature


Nut 1
750
700
650
400
500
Nut 2
600
550
500
450
400
500
Closeness: 0.137673
Signature Curve Nut 1
Signature Curve Nut 2
0.01
0.005
0
−0.005
−0.01
−0.015
−0.05
0
0.05
0.1


Hook 1
Nut 1
Closeness: 0.031217
Signature Curve Hook 1
Signature Curve Nut 1
1000
900
800
700
200
750
700
650
400
500
0.01
0.005
0
−0.005
−0.01
−0.015
−0.05
0
0.05
0.1

Basic Jet Space Notation

M — $m = p + q$ dimensional manifold

$z = (x, u)$ — local coordinates on M

$x = (x^1, \ldots, x^p)$ — independent variables

$u = (u^1, \ldots, u^q)$ — dependent variables

$\mathrm{J}^n = \mathrm{J}^n(M, p)$ — jet space of p-dimensional submanifolds

$u^\alpha_J = \partial_J u^\alpha$ — partial derivatives (jet coordinates)

$F(x, u^{(n)}) = F(\ \ldots \ x^k \ \ldots \ u^\alpha_J \ \ldots \ )$ — differential function
$F : \mathrm{J}^n \to \mathbb{R}$

Invariantization

The process of replacing group parameters in transformation rules by their moving frame formulae is known as invariantization:

$$\iota : \begin{cases} \text{Functions} & \longrightarrow & \text{Invariants} \\ \text{Forms} & \longrightarrow & \text{Invariant Forms} \\ \begin{array}{c}\text{Differential} \\ \text{Operators}\end{array} & \longrightarrow & \begin{array}{c}\text{Invariant Differential} \\ \text{Operators}\end{array} \\ \vdots & & \vdots \end{cases}$$

- The invariantization $I = \iota(F)$ is the unique invariant function that agrees with F on the cross-section: $I \mid K = F \mid K$.
- Invariantization defines an (exterior) algebra morphism.
- Invariantization does not affect invariants: $\iota(I) = I$

The Fundamental Differential Invariants

Invariantized jet coordinate functions:

$$H^i(x, u^{(n)}) = \iota(x^i) \qquad I^\alpha_J(x, u^{(l)}) = \iota(u^\alpha_J)$$

- The constant differential invariants, as dictated by the moving frame normalizations, are known as the phantom invariants.
- The remaining non-constant differential invariants are the basic invariants and form a complete system of functionally independent differential invariants for the prolonged group action.

Invariantization of general differential functions:

$$\iota \, [\, F(\; \ldots \; x^i \; \ldots \; u^\alpha_J \; \ldots \;) \,] = F(\; \ldots \; H^i \; \ldots \; I^\alpha_J \; \ldots \;)$$

Invariantization of general differential functions:

$$\iota\,[\,F(\ \ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\ )\,] = F(\ \ldots\ H^i\ \ldots\ I^\alpha_J\ \ldots\ )$$

The Replacement Theorem:

If $I(x, u^{(n)})$ is any differential invariant, then $\iota(I) = I$.

$$I(\ \ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\ ) = I(\ \ldots\ H^i\ \ldots\ I^\alpha_J\ \ldots\ )$$

Invariantization of general differential functions:

$$\iota\,[\,F(\;\ldots\;x^i\;\ldots\;u^\alpha_J\;\ldots\;)\,] = F(\;\ldots\;H^i\;\ldots\;I^\alpha_J\;\ldots\;)$$

The Replacement Theorem:

If $I(x, u^{(n)})$ is any differential invariant, then $\iota(I) = I$.

$$I(\;\ldots\;x^i\;\ldots\;u^\alpha_J\;\ldots\;) = I(\;\ldots\;H^i\;\ldots\;I^\alpha_J\;\ldots\;)$$

Key fact: Invariantization and differentiation do not commute:

$$\iota(D_i F) \;\neq\; \mathcal{D}_i \iota(F)$$

$\star\star$ Recurrence Formulae $\star\star$

The Differential Invariant Algebra

A differential invariant is an invariant function $I : \mathrm{J}^n \to \mathbb{R}$ for the prolonged pseudo-group action

$$I(g^{(n)} \cdot (x, u^{(n)})) = I(x, u^{(n)})$$

$$\Longrightarrow \quad \text{curvature, torsion, } \ldots$$

Invariant differential operators:

$$\mathcal{D}_1, \ldots, \mathcal{D}_p \qquad \Longrightarrow \quad \text{arc length derivative}$$

- If I is a differential invariant, so is $\mathcal{D}_j I$.

$\mathcal{I}(G)$ — the algebra of differential invariants

Applications

- Equivalence and signatures of submanifolds
- Characterization of moduli spaces
- Invariant differential equations:

$$H(\ \ldots\ \mathcal{D}_J I_\kappa\ \ldots\ ) = 0$$

- Group splitting of PDEs and explicit solutions
- Invariant variational problems:

$$\int L(\ \ldots\ \mathcal{D}_J I_\kappa\ \ldots\ )\,\boldsymbol{\omega}$$

- Invariant geometric flows

The Basis Theorem

Theorem. The differential invariant algebra $\mathcal{I}(G)$ is locally generated by a finite number of differential invariants

$$I_1, \ \ldots \ , I_\ell$$

and $p = \dim S$ invariant differential operators

$$\mathcal{D}_1, \ \ldots \ , \mathcal{D}_p$$

meaning that *every* differential invariant can be locally expressed as a function of the generating invariants and their invariant derivatives:

$$\mathcal{D}_J I_\kappa = \mathcal{D}_{j_1} \mathcal{D}_{j_2} \cdots \mathcal{D}_{j_n} I_\kappa .$$

$\Longrightarrow$ Lie groups: *Lie, Ovsiannikov*

$\Longrightarrow$ Lie pseudo-groups: *Tresse, Kumpera, Kruglikov–Lychagin, Muñoz–Muriel–Rodríguez, Pohjanpelto–O*

Key Issues

- Minimal basis of generating invariants: $I_1, \ldots, I_\ell$
- Commutation formulae for

  the invariant differential operators:

$$[\mathcal{D}_j, \mathcal{D}_k] = \sum_{i=1}^{p} Y_{jk}^i \, \mathcal{D}_i$$

$\Longrightarrow$ Non-commutative differential algebra

- Syzygies (functional relations) among

  the differentiated invariants:

$$\Phi(\; \ldots \; \mathcal{D}_J I_\kappa \; \ldots \;) \equiv 0$$

$\Longrightarrow$ Codazzi relations

Computing Differential Invariants

♠ The infinitesimal method:

$$\mathbf{v}(I) = 0 \qquad \text{for every infinitesimal generator} \qquad \mathbf{v} \in \mathfrak{g}$$

$\Longrightarrow$ Requires solving differential equations.

♡ Moving frames.

- Completely algebraic.
- Can be adapted to arbitrary group and pseudo-group actions.
- Describes the complete structure of the differential invariant algebra $\mathcal{I}(G)$ — using only linear algebra & differentiation!
- Prescribes differential invariant signatures for equivalence and symmetry detection.

Infinitesimal Generators

Infinitesimal generators of action of G on M:

$$\mathbf{v}_\kappa = \sum_{i=1}^{p} \xi_\kappa^i(x,u)\,\frac{\partial}{\partial x^i} \;+\; \sum_{\alpha=1}^{q} \varphi_\kappa^\alpha(x,u)\,\frac{\partial}{\partial u^\alpha} \qquad \kappa = 1,\ldots,r$$

Prolonged infinitesimal generators on J^n:

$$\mathbf{v}_\kappa^{(n)} = \mathbf{v}_\kappa \;+\; \sum_{\alpha=1}^{q} \; \sum_{j=\#J=1}^{n} \varphi_{J,\kappa}^\alpha(x,u^{(j)})\;\frac{\partial}{\partial u_J^\alpha}$$

Prolongation formula:

$$\varphi_{J,\kappa}^\alpha = D_K\left(\varphi_\kappa^\alpha - \sum_{i=1}^{p} u_i^\alpha\,\xi_\kappa^i\right) + \sum_{i=1}^{p} u_{J,i}^\alpha\,\xi_\kappa^i$$

$D_1,\ldots,D_p$ — total derivatives

Recurrence Formulae

$$\mathcal{D}_j \, \iota(F) = \iota(D_j F) \; + \; \sum_{\kappa=1}^{r} R_j^\kappa \, \iota(\mathbf{v}_\kappa^{(n)}(F))$$

$\omega^i = \iota(dx^i)$ — invariant coframe

$\mathcal{D}_i = \iota(D_{x^i})$ — dual invariant differential operators

R_j^κ — Maurer–Cartan invariants

$\mathbf{v}_1, \ \ldots \ \mathbf{v}_r \in \mathfrak{g}$ — infinitesimal generators

$\mu^1, \ \ldots \ \mu^r \in \mathfrak{g}^*$ — dual Maurer–Cartan forms

The Maurer–Cartan Invariants

Invariantized Maurer–Cartan forms:

$$\gamma^\kappa = \rho^*(\mu^\kappa) \equiv \sum_{j=1}^{p} R_j^\kappa \, \omega^j$$

The Maurer–Cartan Invariants

Invariantized Maurer–Cartan forms:

$$\gamma^\kappa = \rho^*(\mu^\kappa) \equiv \sum_{j=1}^{p} R_j^\kappa \, \omega^j$$

Remark: When $G \subset \mathrm{GL}(N)$, the Maurer–Cartan invariants R_j^κ are the entries of the Frenet matrices

$$\mathcal{D}_i \, \rho(x, u^{(n)}) \cdot \rho(x, u^{(n)})^{-1}$$

The Maurer–Cartan Invariants

Invariantized Maurer–Cartan forms:

$$\gamma^\kappa = \rho^*(\mu^\kappa) \equiv \sum_{j=1}^{p} R_j^\kappa \, \omega^j$$

Remark: When $G \subset \mathrm{GL}(N)$, the Maurer–Cartan invariants R_j^κ are the entries of the Frenet matrices

$$\mathcal{D}_i \, \rho(x, u^{(n)}) \cdot \rho(x, u^{(n)})^{-1}$$

Theorem. (*E. Hubert*) The Maurer–Cartan invariants and, in the intransitive case, the order zero invariants serve to generate the differential invariant algebra $\mathcal{I}(G)$.

Recurrence Formulae

$$\mathcal{D}_j \, \iota(F) = \iota(D_j F) \; + \; \sum_{\kappa=1}^{r} R_j^{\kappa} \, \iota(\mathbf{v}_{\kappa}^{(n)}(F))$$

Recurrence Formulae

$$\mathcal{D}_j\, \iota(F) = \iota(D_j F) \;+\; \sum_{\kappa=1}^{r} R_j^\kappa\, \iota(\mathbf{v}_\kappa^{(n)}(F))$$

♠ If $\iota(F) = c$ is a phantom differential invariant, then the left hand side of the recurrence formula is zero. The collection of all such phantom recurrence formulae form a linear algebraic system of equations that can be uniquely solved for the Maurer–Cartan invariants R_j^κ!

Recurrence Formulae

$$\mathcal{D}_j\, \iota(F) = \iota(D_j F) \; + \; \sum_{\kappa=1}^{r} R_j^{\kappa}\, \iota(\mathbf{v}_{\kappa}^{(n)}(F))$$

♠ If $\iota(F) = c$ is a phantom differential invariant, then the left hand side of the recurrence formula is zero. The collection of all such phantom recurrence formulae form a linear algebraic system of equations that can be uniquely solved for the Maurer–Cartan invariants R_j^{κ}!

♡ Once the Maurer–Cartan invariants are replaced by their explicit formulae, the induced recurrence relations completely determine the structure of the differential invariant algebra $\mathcal{I}(G)$!

The Universal Recurrence Formula

Let Ω be any differential form on J^n.

$$d\,\iota(\Omega) = \iota(d\Omega) + \sum_{\kappa=1}^{r} \gamma^\kappa \wedge \iota[\mathbf{v}_\kappa(\Omega)]$$

$\Longrightarrow$ *The invariant variational bicomplex*

The Universal Recurrence Formula

Let Ω be any differential form on J^n.

$$d\,\iota(\Omega) = \iota(d\Omega) + \sum_{\kappa=1}^{r} \gamma^\kappa \wedge \iota[\mathbf{v}_\kappa(\Omega)]$$

$\Longrightarrow$ *The invariant variational bicomplex*

Commutator invariants:

$$d\omega^i = d[\iota(dx^i)] = \iota(d^2x^i) + \sum_{\kappa=1}^{r} \gamma^\kappa \wedge \iota[\mathbf{v}_\kappa(dx^i)]$$

$$= -\sum_{j<k} Y^i_{jk}\,\omega^j \wedge \omega^k + \cdots$$

$$[\,\mathcal{D}_j, \mathcal{D}_k\,] = \sum_{i=1}^{p} Y^i_{jk}\,\mathcal{D}_i$$

The Differential Invariant Algebra

Thus, remarkably, the structure of $\mathcal{I}(G)$ can be completely determined without knowing the explicit formulae for either the moving frame, or the differential invariants, or the invariant differential operators!

The only required ingredients are the specification of the cross-section, and the standard formulae for the prolonged infinitesimal generators.

Theorem. If G acts transitively on M, or if the infinitesimal generator coefficients depend rationally in the coordinates, then all recurrence formulae are rational in the basic differential invariants and so $\mathcal{I}(G)$ is a rational, non-commutative differential algebra.

Euclidean Surfaces

$$M = \mathbb{R}^3 \qquad G = \mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3 \qquad \dim G = 6.$$

$$g \cdot z = R\,z + b, \qquad R^T R = I, \qquad z = \begin{pmatrix} x \\ y \\ u \end{pmatrix} \in \mathbb{R}^3.$$

Assume (for simplicity) that $S \subset \mathbb{R}^3$ is the graph of a function:

$$u = f(x, y)$$

Cross-section to prolonged action on J^2:

$$x = y = u = u_x = u_y = u_{xy} = 0, \qquad u_{xx} \neq u_{yy}.$$

Invariantization — differential invariants: $\quad I_{jk} = \iota(u_{jk})$

Phantom differential invariants:

$$\iota(x) = \iota(y) = \iota(u) = \iota(u_x) = \iota(u_y) = \iota(u_{xy}) = 0.$$

Principal curvatures:

$$\kappa_1 = I_{20} = \iota(u_{xx}), \qquad \kappa_2 = I_{02} = \iota(u_{yy}),$$

$\star\star$ non-umbilic point: $\kappa_1 \neq \kappa_2$ $\star\star$

Mean and Gauss curvatures:

$$H = \tfrac{1}{2}(\kappa_1 + \kappa_2), \qquad K = \kappa_1 \kappa_2.$$

Invariant differential operators:

$$\mathcal{D}_1 = \iota(D_x), \qquad \mathcal{D}_2 = \iota(D_y).$$

$\Longrightarrow$ diagonalizing Frenet frame

To obtain the recurrence formulae for the higher order differential invariants, we need the infinitesimal generators of $\mathfrak{g} = \mathfrak{se}(3)$:

$$\mathbf{v}_1 = -\, y\,\partial_x + x\,\partial_y$$

$$\mathbf{v}_2 = -\, u\,\partial_x + x\,\partial_u,$$

$$\mathbf{v}_3 = -\, u\,\partial_y + y\,\partial_u$$

$$\mathbf{w}_1 = \partial_x \qquad \mathbf{w}_2 = \partial_y \qquad \mathbf{w}_3 = \partial_u$$

- The translations will be ignored, as they play no role in the higher order recurrence formulae.

Recurrence formulae

$$\boxed{\mathcal{D}_i \iota(u_{jk}) = \iota(D_i u_{jk}) + \sum_{\nu=1}^{3} \iota[\,\varphi_\nu^{jk}(x,y,u^{(j+k)})\,]\,R_i^\nu, \qquad j+k \geq 1}$$

$$\mathcal{D}_1 I_{jk} = I_{j+1,k} + \sum_{\nu=1}^{3} \varphi_\nu^{jk}(0,0,I^{(j+k)})\,R_1^\nu$$

$$\mathcal{D}_2 I_{jk} = I_{j,k+1} + \sum_{\nu=1}^{3} \varphi_\nu^{jk}(0,0,I^{(j+k)})\,R_2^\nu$$

$\varphi_\nu^{jk}(0,0,I^{(j+k)}) = \iota[\,\varphi_\nu^{jk}(x,y,u^{(j+k)})\,]$ — invariantized prolonged infinitesimal generator coefficients

R_i^ν — Maurer–Cartan invariants

Phantom recurrence formulae:

$$0 = \mathcal{D}_1 I_{10} = I_{20} + R_1^2 \qquad 0 = \mathcal{D}_2 I_{10} = R_2^2$$

$$0 = \mathcal{D}_1 I_{01} = R_1^3 \qquad 0 = \mathcal{D}_2 I_{01} = I_{02} + R_2^3$$

$$0 = \mathcal{D}_1 I_{11} = I_{21} + (I_{20} - I_{02}) R_1^1 \qquad 0 = \mathcal{D}_2 I_{11} = I_{12} + (I_{20} - I_{02}) R_2^1$$

Maurer–Cartan invariants:

$$R_1 = (Y_2, -\kappa_1, 0) \qquad R_2 = (-Y_1, 0, -\kappa_2)$$

where

$$Y_1 = \frac{I_{12}}{I_{20} - I_{02}} = \frac{\mathcal{D}_1 \kappa_2}{\kappa_1 - \kappa_2} \qquad Y_2 = \frac{I_{21}}{I_{02} - I_{20}} = \frac{\mathcal{D}_2 \kappa_1}{\kappa_2 - \kappa_1}$$

are also the commutator invariants:

$$[\,\mathcal{D}_1, \mathcal{D}_2\,] = \mathcal{D}_1 \mathcal{D}_2 - \mathcal{D}_2 \mathcal{D}_1 = Y_2 \mathcal{D}_1 - Y_1 \mathcal{D}_2.$$

Second order recurrence formulae:

$$I_{30} = \mathcal{D}_1 I_{20} = \kappa_{1,1} \qquad I_{21} = \mathcal{D}_2 I_{20} = \kappa_{1,2}$$

$$I_{12} = \mathcal{D}_1 I_{02} = \kappa_{2,1} \qquad I_{03} = \mathcal{D}_2 I_{02} = \kappa_{2,2}$$

The fourth order recurrence formulae

$$\mathcal{D}_2 I_{21} + \frac{I_{30} I_{12} - 2\, I_{12}^2}{\kappa_1 - \kappa_2} + \kappa_1 \kappa_2^2 = I_{22} = \mathcal{D}_1 I_{12} - \frac{I_{21} I_{03} - 2\, I_{21}^2}{\kappa_1 - \kappa_2} + \kappa_1^2 \kappa_2$$

lead to the Codazzi syzygy

$$\kappa_{1,22} - \kappa_{2,11} + \frac{\kappa_{1,1}\kappa_{2,1} + \kappa_{1,2}\kappa_{2,2} - 2\,\kappa_{2,1}^2 - 2\,\kappa_{1,2}^2}{\kappa_1 - \kappa_2} - \kappa_1 \kappa_2 (\kappa_1 - \kappa_2) = 0$$

- The principal curvatures κ_1, κ_2, or, equivalently, the Gauss and mean curvatures H, K, form a generating system for the differential invariant algebra.

Second order recurrence formulae:

$$I_{30} = \mathcal{D}_1 I_{20} = \kappa_{1,1} \qquad I_{21} = \mathcal{D}_2 I_{20} = \kappa_{1,2}$$

$$I_{12} = \mathcal{D}_1 I_{02} = \kappa_{2,1} \qquad I_{03} = \mathcal{D}_2 I_{02} = \kappa_{2,2}$$

The fourth order recurrence formulae

$$\mathcal{D}_2 I_{21} + \frac{I_{30} I_{12} - 2\, I_{12}^2}{\kappa_1 - \kappa_2} + \kappa_1 \kappa_2^2 = I_{22} = \mathcal{D}_1 I_{12} - \frac{I_{21} I_{03} - 2\, I_{21}^2}{\kappa_1 - \kappa_2} + \kappa_1^2 \kappa_2$$

lead to the Codazzi syzygy

$$\kappa_{1,22} - \kappa_{2,11} + \frac{\kappa_{1,1}\kappa_{2,1} + \kappa_{1,2}\kappa_{2,2} - 2\,\kappa_{2,1}^2 - 2\,\kappa_{1,2}^2}{\kappa_1 - \kappa_2} - \kappa_1 \kappa_2 (\kappa_1 - \kappa_2) = 0$$

- The principal curvatures κ_1, κ_2, or, equivalently, the Gauss and mean curvatures H, K, form a generating system for the differential invariant algebra.

★★ Neither is a minimal generating set! ★★

Codazzi syzygy:

$$K = \kappa_1 \kappa_2 = -(\mathcal{D}_1 + Y_1)Y_1 - (\mathcal{D}_2 + Y_2)Y_2$$

Codazzi syzygy:

$$K = \kappa_1 \kappa_2 = -(\mathcal{D}_1 + Y_1)Y_1 - (\mathcal{D}_2 + Y_2)Y_2$$

Gauss' Theorema Egregium

The Gauss curvature is intrinsic.

Codazzi syzygy:

$$K = \kappa_1 \kappa_2 = -(\mathcal{D}_1 + Y_1)Y_1 - (\mathcal{D}_2 + Y_2)Y_2$$

Gauss' Theorema Egregium

The Gauss curvature is intrinsic.

Proof: The Frenet frome is intrinsic, hence so are the invariant differentiations and also commutator invariants. *Q.E.D.*

Codazzi syzygy:

$$K = \kappa_1 \kappa_2 = -(\mathcal{D}_1 + Y_1)Y_1 - (\mathcal{D}_2 + Y_2)Y_2$$

Gauss' Theorema Egregium

The Gauss curvature is intrinsic.

Proof: The Frenet frome is intrinsic, hence so are the invariant differentiations and also commutator invariants. *Q.E.D.*

Theorem. For suitably nondegenerate surfaces, the mean curvature H is a generating differential invariant, i.e., all other Euclidean surface differential invariants can be expressed as functions of H and its invariant derivatives.

Proof: Since H, K generate the differential invariant algebra, it suffices to express the Gauss curvature K as a function of H and its derivatives. For this, the Codazzi syzygy implies that we need only express the commutator invariants in terms of H.

The commutator identity can be applied to any differential invariant. In particular,

$$\begin{aligned} \mathcal{D}_1\mathcal{D}_2 H - \mathcal{D}_2\mathcal{D}_1 H &= Y_2\,\mathcal{D}_1 H - Y_1\,\mathcal{D}_2 H \\ \mathcal{D}_1\mathcal{D}_2\mathcal{D}_j H - \mathcal{D}_2\mathcal{D}_1\mathcal{D}_j H &= Y_2\,\mathcal{D}_1\mathcal{D}_j H - Y_1\,\mathcal{D}_2\mathcal{D}_j H \end{aligned} \qquad (*)$$

Provided the nondegeneracy condition

$$(\mathcal{D}_1 H)(\mathcal{D}_2\mathcal{D}_j H) \neq (\mathcal{D}_2 H)(\mathcal{D}_1\mathcal{D}_j H), \qquad \text{for } j = 1 \text{ or } 2$$

holds, we can solve $(*)$ for the commutator invariants as rational functions of invariant derivatives of H. *Q.E.D.*

Note: Constant Mean Curvature surfaces are degenerate.
Are there others?

Theorem. $G = \mathrm{SA}(3) = \mathrm{SL}(3) \ltimes \mathbb{R}^3$ acts on $S \subset M = \mathbb{R}^3$:

The algebra of differential invariants of generic equiaffine surfaces is generated by a single third order invariant, the Pick invariant.

Theorem. $G = \mathrm{SO}(4,1)$ acts on $S \subset M = \mathbb{R}^3$:

The algebra of differential invariants of generic conformal surfaces is generated by a single third order invariant.

Theorem. $G = \mathrm{PSL}(4)$ acts on $S \subset M = \mathbb{R}^3$:

The algebra of differential invariants of generic projective surfaces is generated by a single fourth order invariant.

Variational Problems

$\mathcal{I}[u] = \int L(x, u^{(n)})\, d\mathbf{x}$ — variational problem

$L(x, u^{(n)})$ — Lagrangian

Variational derivative — Euler-Lagrange equations: $\mathbf{E}(L) = 0$

components: $\mathbf{E}_\alpha(L) = \sum_J (-D)^J \dfrac{\partial L}{\partial u^\alpha_J}$

$$D_k F = \frac{\partial F}{\partial x^k} + \sum_{\alpha, J} u^\alpha_{J,k} \frac{\partial F}{\partial u^\alpha_J}$$

— total derivative of F with respect to x^k

Invariant Variational Problems

According to Lie, any G–invariant variational problem can be written in terms of the differential invariants:

$$\boxed{\mathcal{I}[u] = \int L(x, u^{(n)})\, d\mathbf{x} = \int P(\ \ldots\ \mathcal{D}_K I^\alpha\ \ldots\ )\, \boldsymbol{\omega}}$$

$I^1, \ldots, I^\ell$ — fundamental differential invariants

$\mathcal{D}_1, \ldots, \mathcal{D}_p$ — invariant differential operators

$\mathcal{D}_K I^\alpha$ — differentiated invariants

$\boldsymbol{\omega} = \omega^1 \wedge \cdots \wedge \omega^p$ — invariant volume form

If the variational problem is G-invariant, so

$$\mathcal{I}[u] = \int L(x, u^{(n)})\, d\mathbf{x} = \int P(\,\ldots\, \mathcal{D}_K I^\alpha \,\ldots\,)\, \boldsymbol{\omega}$$

then its Euler–Lagrange equations admit G as a symmetry group, and hence can also be expressed in terms of the differential invariants:

$$\mathbf{E}(L) \simeq F(\,\ldots\, \mathcal{D}_K I^\alpha \,\ldots\,) = 0$$

Main Problem:

Construct F directly from P.

(*P. Griffiths, I. Anderson*)

Planar Euclidean group $\quad G = \mathrm{SE}(2)$

$$\kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}} \qquad \text{— curvature (differential invariant)}$$

$$ds = \sqrt{1+u_x^2}\,dx \qquad \text{— arc length}$$

$$\mathcal{D} = \frac{d}{ds} = \frac{1}{\sqrt{1+u_x^2}}\,\frac{d}{dx} \qquad \text{— arc length derivative}$$

Euclidean–invariant variational problem

$$\mathcal{I}[u] = \int L(x, u^{(n)})\,dx = \int P(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ )\,ds$$

Euler-Lagrange equations

$$\mathbf{E}(L) \simeq F(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ ) = 0$$

Euclidean Curve Examples

Minimal curves (geodesics):

$$\mathcal{I}[\,u\,] = \int ds = \int \sqrt{1+u_x^2}\;dx$$

$$\mathbf{E}(L) = -\,\kappa = 0$$

$\Longrightarrow$ straight lines

The Elastica (Euler):

$$\mathcal{I}[\,u\,] = \int \tfrac{1}{2}\,\kappa^2\,ds = \int \frac{u_{xx}^2\,dx}{(1+u_x^2)^{5/2}}$$

$$\mathbf{E}(L) = \kappa_{ss} + \tfrac{1}{2}\,\kappa^3 = 0$$

$\Longrightarrow$ elliptic functions

General Euclidean–invariant variational problem

$$\mathcal{I}[u] = \int L(x, u^{(n)})\, dx = \int P(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ )\, ds$$

General Euclidean–invariant variational problem

$$\mathcal{I}[u] = \int L(x, u^{(n)})\, dx = \int P(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ )\, ds$$

Invariantized Euler–Lagrange expression

$$\mathcal{E}(P) = \sum_{n=0}^{\infty} (-\mathcal{D})^n \frac{\partial P}{\partial \kappa_n} \qquad\qquad \mathcal{D} = \frac{d}{ds}$$

General Euclidean–invariant variational problem

$$\mathcal{I}[u] = \int L(x, u^{(n)})\, dx = \int P(\kappa, \kappa_s, \kappa_{ss}, \;\ldots\;)\, ds$$

Invariantized Euler–Lagrange expression

$$\mathcal{E}(P) = \sum_{n=0}^{\infty} (-\mathcal{D})^n \frac{\partial P}{\partial \kappa_n} \qquad\qquad \mathcal{D} = \frac{d}{ds}$$

Invariantized Hamiltonian

$$H^i(P) = \sum_{i>j} \kappa_{i-j}\; (-\mathcal{D})^j \frac{\partial P}{\partial \kappa_i} - P$$

$$\mathcal{I}[u] = \int L(x, u^{(n)})\, dx = \int P(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ )\, ds$$

Euclidean–invariant Euler-Lagrange formula

$$\boxed{\mathbf{E}(L) = (\mathcal{D}^2 + \kappa^2)\ \mathcal{E}(P) + \kappa\, H^i(P) = 0}$$

$$\mathcal{I}[u] = \int L(x, u^{(n)})\, dx = \int P(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ )\, ds$$

Euclidean–invariant Euler-Lagrange formula

$$\boxed{\mathbf{E}(L) = (\mathcal{D}^2 + \kappa^2)\, \mathcal{E}(P) + \kappa\, H^i(P) = 0}$$

The Elastica: $\mathcal{I}[u] = \int \frac{1}{2}\kappa^2\, ds$ $\qquad P = \frac{1}{2}\kappa^2$

$$\mathcal{E}(P) = \kappa \qquad H^i(P) = -P = -\frac{1}{2}\kappa^2$$

$$\boxed{\begin{aligned} \mathbf{E}(L) &= (\mathcal{D}^2 + \kappa^2)\,\kappa + \kappa\,(-\tfrac{1}{2}\kappa^2) \\ &= \kappa_{ss} + \tfrac{1}{2}\kappa^3 = 0 \end{aligned}}$$

The shape of a Möbius strip

E. L. STAROSTIN AND G. H. M. VAN DER HEIJDEN*
Centre for Nonlinear Dynamics, Department of Civil and Environmental Engineering, University College London, London WC1E 6BT, UK
*e-mail: g.heijden@ucl.ac.uk

Published online: 15 July 2007; doi:10.1038/nmat1929

The Möbius strip, obtained by taking a rectangular strip of plastic or paper, twisting one end through 180°, and then joining the ends, is the canonical example of a one-sided surface. Finding its characteristic developable shape has been an open problem ever since its first formulation in refs 1,2. Here we use the invariant variational bicomplex formalism to derive the first equilibrium equations for a wide developable strip undergoing large deformations, thereby giving the first non-trivial demonstration of the potential of this approach. We then formulate the boundary-value problem for the Möbius strip and solve it numerically. Solutions for increasing width show the formation of creases bounding nearly flat triangular regions, a feature also familiar from fabric draping[3] and paper crumpling[4,5]. This could give new insight into energy localization phenomena in unstretchable sheets[6], which might help to predict points of onset of tearing. It could also aid our understanding of the relationship between geometry and physical properties of nano- and microscopic Möbius strip structures[7–9].

It is fair to say that the Möbius strip is one of the few icons of mathematics that have been absorbed into wider culture. It has mathematical beauty and inspired artists such as Escher[10]. In engineering, pulley belts are often used in the form of Möbius strips to wear 'both' sides equally. At a much smaller scale, Möbius strips have recently been formed in ribbon-shaped $NbSe_3$ crystals under certain growth conditions involving a large temperature gradient[7,8].

Figure 1 Photo of a paper Möbius strip of aspect ratio 2π. The strip adopts a characteristic shape. Inextensibility of the material causes the surface to be developable. Its straight generators are drawn and the colouring varies according to the bending energy density.

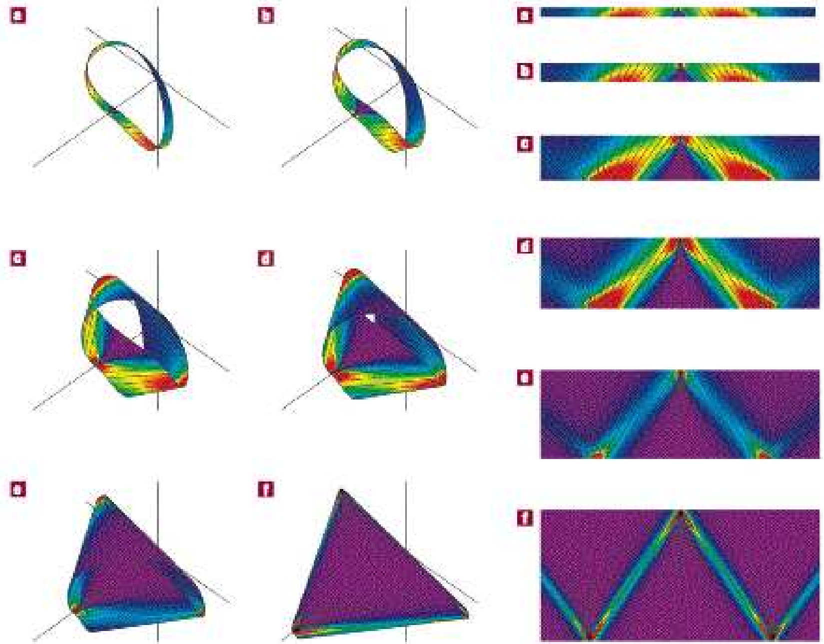


Figure 2 Computed Möbius strips. The left panel shows their three-dimensional shapes for $w = 0.1$ (**a**), 0.2 (**b**), 0.5 (**c**), 0.8 (**d**), 1.0 (**e**) and 1.5 (**f**), and the right panel the corresponding developments on the plane. The colouring changes according to the local bending energy density, from violet for regions of low bending to red for regions of high bending (scales are individually adjusted). Solution **c** may be compared with the paper model in Fig. 1 on which the generator field and density colouring have been printed.

The Infinite Jet Bundle

Jet bundles

$$M = \mathrm{J}^0 \longleftarrow \mathrm{J}^1 \longleftarrow \mathrm{J}^2 \longleftarrow \cdots$$

Inverse limit

$$\mathrm{J}^\infty = \lim_{n \to \infty} \mathrm{J}^n$$

Local coordinates

$$z^{(\infty)} = (x, u^{(\infty)}) = (\ \ldots \ x^i \ \ldots \ u^\alpha_J \ \ldots \ )$$

$\Longrightarrow$ Taylor series

Differential Forms

Coframe — basis for the cotangent space $T^*\mathrm{J}^\infty$:

- Horizontal one-forms

$$dx^1, \ldots, dx^p$$

- Contact (vertical) one-forms

$$\theta^\alpha_J = du^\alpha_J - \sum_{i=1}^{p} u^\alpha_{J,i}\, dx^i$$

Intrinsic definition of contact form

$$\theta \mid \mathrm{j}_\infty N = 0 \qquad \Longleftrightarrow \qquad \theta = \sum A^\alpha_J\, \theta^\alpha_J$$

The Variational Bicomplex

$\Longrightarrow$ *Dedecker, Vinogradov, Tsujishita, I. Anderson, ...*

Bigrading of the differential forms on J^∞:

$$\Omega^* = \bigoplus_{r,s} \Omega^{r,s}$$

$r = \#$ horizontal forms

$s = \#$ contact forms

Vertical and Horizontal Differentials

$$d = d_H + d_V$$

$$d_H : \Omega^{r,s} \longrightarrow \Omega^{r+1,s}$$

$$d_V : \Omega^{r,s} \longrightarrow \Omega^{r,s+1}$$

Vertical and Horizontal Differentials

$$F(x, u^{(n)}) \qquad \text{— differential function}$$

$$d_H F = \sum_{i=1}^{p} (D_i F)\, dx^i \qquad \text{— total differential}$$

$$d_V F = \sum_{\alpha, J} \frac{\partial F}{\partial u_J^\alpha}\, \theta_J^\alpha \qquad \text{— first variation}$$

$$d_H (dx^i) = d_V (dx^i) = 0,$$

$$d_H (\theta_J^\alpha) = \sum_{i=1}^{p} dx^i \wedge \theta_{J,i}^\alpha \qquad d_V (\theta_J^\alpha) = 0$$

The Simplest Example

$$(x, u) \in M = \mathbb{R}^2$$

x — independent variable

u — dependent variable

Horizontal form

$$dx$$

Contact (vertical) forms

$$\theta = du - u_x \, dx$$

$$\theta_x = du_x - u_{xx} \, dx$$

$$\theta_{xx} = du_{xx} - u_{xxx} \, dx$$

$$\vdots$$

$$\theta = du - u_x\,dx, \qquad \theta_x = du_x - u_{xx}\,dx, \qquad \theta_{xx} = du_{xx} - u_{xxx}\,dx$$

Differential:

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial u}\,du + \frac{\partial F}{\partial u_x}\,du_x + \frac{\partial F}{\partial u_{xx}}\,du_{xx} + \cdots$$

$$= (D_x F)\,dx + \frac{\partial F}{\partial u}\theta + \frac{\partial F}{\partial u_x}\theta_x + \frac{\partial F}{\partial u_{xx}}\theta_{xx} + \cdots$$

$$= d_H F + d_V F$$

Total derivative:

$$D_x F = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial u}\,u_x + \frac{\partial F}{\partial u_x}\,u_{xx} + \frac{\partial F}{\partial u_{xx}}\,u_{xxx} + \cdots$$

The Variational Bicomplex

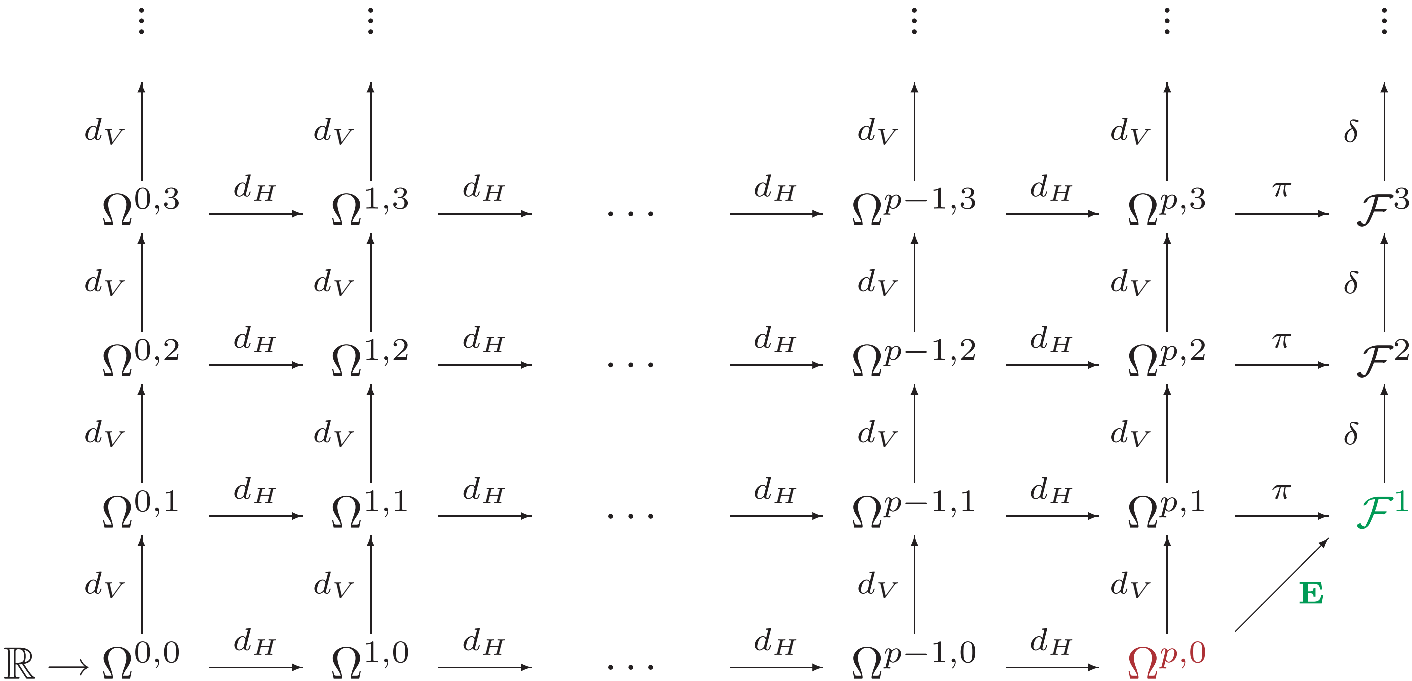
$$
\begin{array}{ccccccccccc}
 & \vdots & & \vdots & & & & \vdots & & \vdots & & \vdots \\
 & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle \delta} \\
 & \Omega^{0,3} & \xrightarrow{d_H} & \Omega^{1,3} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,3} & \xrightarrow{d_H} & \Omega^{p,3} & \xrightarrow{\pi} & \mathcal{F}^3 \\
 & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle \delta} \\
 & \Omega^{0,2} & \xrightarrow{d_H} & \Omega^{1,2} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,2} & \xrightarrow{d_H} & \Omega^{p,2} & \xrightarrow{\pi} & \mathcal{F}^2 \\
 & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle \delta} \\
 & \Omega^{0,1} & \xrightarrow{d_H} & \Omega^{1,1} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,1} & \xrightarrow{d_H} & \Omega^{p,1} & \xrightarrow{\pi} & \mathcal{F}^1 \\
 & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & \nearrow{\scriptstyle \mathbf{E}} & \\
\mathbb{R} \longrightarrow & \Omega^{0,0} & \xrightarrow{d_H} & \Omega^{1,0} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,0} & \xrightarrow{d_H} & \Omega^{p,0} & &
\end{array}
$$

The Variational Bicomplex

$$
\begin{array}{ccccccccccccc}
 & & \vdots & & \vdots & & & & \vdots & & \vdots & & \vdots \\
 & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle \delta} \\
 & & \Omega^{0,3} & \xrightarrow{d_H} & \Omega^{1,3} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,3} & \xrightarrow{d_H} & \Omega^{p,3} & \xrightarrow{\pi} & \mathcal{F}^3 \\
 & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle \delta} \\
 & & \Omega^{0,2} & \xrightarrow{d_H} & \Omega^{1,2} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,2} & \xrightarrow{d_H} & \Omega^{p,2} & \xrightarrow{\pi} & \mathcal{F}^2 \\
 & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle \delta} \\
 & & \Omega^{0,1} & \xrightarrow{d_H} & \Omega^{1,1} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,1} & \xrightarrow{d_H} & \Omega^{p,1} & \xrightarrow{\pi} & \mathcal{F}^1 \\
 & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & \nearrow{\scriptstyle \mathbf{E}} & \\
\mathbb{R} & \longrightarrow & \Omega^{0,0} & \xrightarrow{d_H} & \Omega^{1,0} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,0} & \xrightarrow{d_H} & \Omega^{p,0} & &
\end{array}
$$

Lagrangians

The Variational Bicomplex



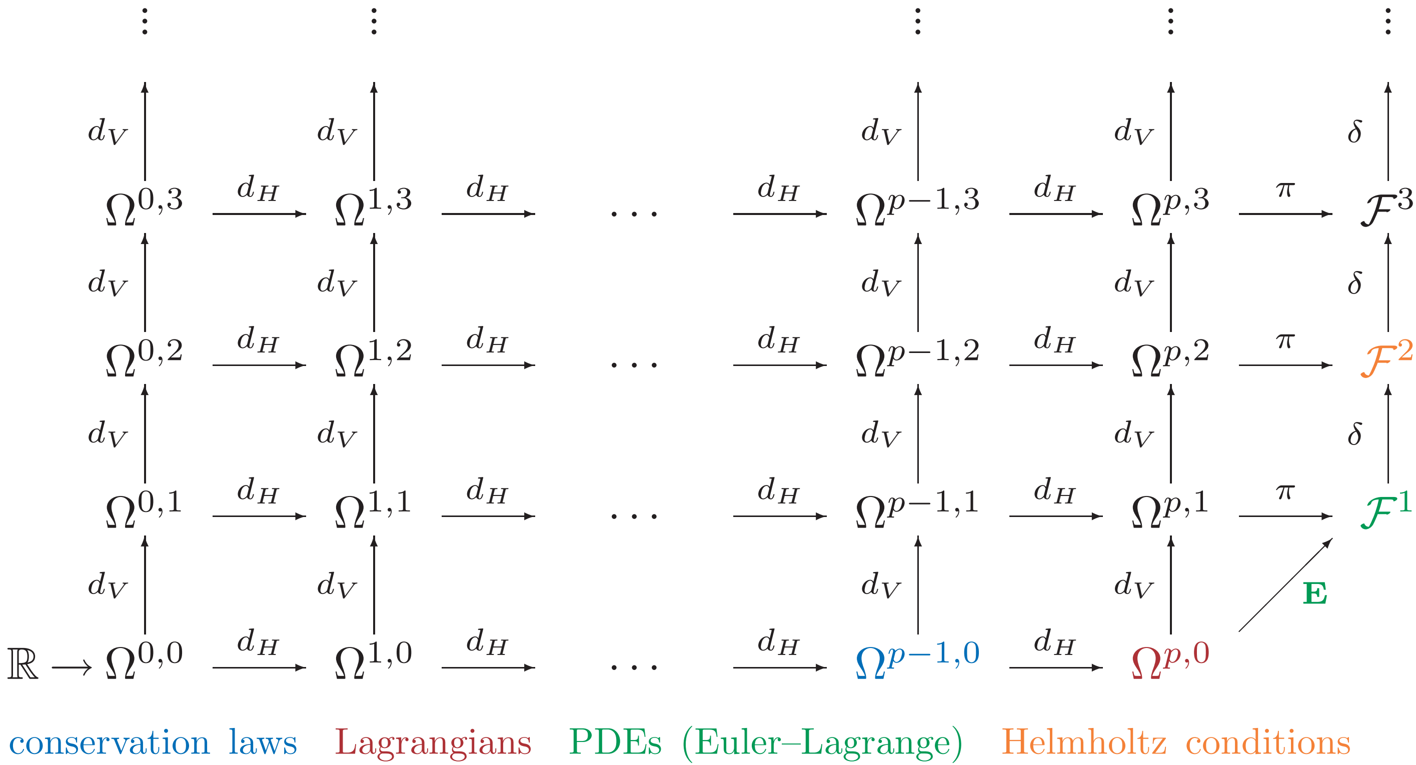
Lagrangians   PDEs (Euler–Lagrange)

The Variational Bicomplex

$$
\begin{array}{ccccccccccc}
 & \vdots & & \vdots & & & & \vdots & & \vdots & & \vdots \\
 & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & & & & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & & {\scriptstyle \delta}\uparrow \\
 & \Omega^{0,3} & \xrightarrow{d_H} & \Omega^{1,3} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,3} & \xrightarrow{d_H} & \Omega^{p,3} & \xrightarrow{\pi} & \mathcal{F}^3 \\
 & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & & & & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & & {\scriptstyle \delta}\uparrow \\
 & \Omega^{0,2} & \xrightarrow{d_H} & \Omega^{1,2} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,2} & \xrightarrow{d_H} & \Omega^{p,2} & \xrightarrow{\pi} & \mathcal{F}^2 \\
 & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & & & & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & & {\scriptstyle \delta}\uparrow \\
 & \Omega^{0,1} & \xrightarrow{d_H} & \Omega^{1,1} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,1} & \xrightarrow{d_H} & \Omega^{p,1} & \xrightarrow{\pi} & \mathcal{F}^1 \\
 & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & & & & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & \nearrow \mathbf{E} & \\
\mathbb{R} \longrightarrow & \Omega^{0,0} & \xrightarrow{d_H} & \Omega^{1,0} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,0} & \xrightarrow{d_H} & \Omega^{p,0} & &
\end{array}
$$

Lagrangians   PDEs (Euler–Lagrange)   Helmholtz conditions

The Variational Bicomplex



The Variational Derivative

$$\boxed{\mathbf{E} = \pi \circ d_V}$$

d_V — first variation

π — integration by parts = mod out by image of d_H

$$\Omega^{p,0} \xrightarrow{\ d_V\ } \Omega^{p,1} \xrightarrow{\ \pi\ } \mathcal{F}^1 = \Omega^{p,1} / \, d_H \, \Omega^{p-1,1}$$

$$\lambda = L \, d\mathbf{x} \longrightarrow \sum_{\alpha, J} \frac{\partial L}{\partial u^\alpha_J} \, \theta^\alpha_J \wedge \, d\mathbf{x} \longrightarrow \sum_{\alpha=1}^{q} \mathbf{E}_\alpha(L) \;\, \theta^\alpha \wedge \, d\mathbf{x}$$

Variational problem ⟶ First variation ⟶ Euler–Lagrange source form

The Simplest Example: $(x, u) \in M = \mathbb{R}^2$

Lagrangian form: $\lambda = L(x, u^{(n)})\, dx \in \Omega^{1,0}$

The Simplest Example: $(x, u) \in M = \mathbb{R}^2$

Lagrangian form: $\qquad \lambda = L(x, u^{(n)})\, dx \;\in\; \Omega^{1,0}$

First variation — vertical derivative:

$$d\lambda = d_V\,\lambda = d_V\,L \wedge dx$$

$$= \left(\frac{\partial L}{\partial u}\,\theta + \frac{\partial L}{\partial u_x}\,\theta_x + \frac{\partial L}{\partial u_{xx}}\,\theta_{xx} + \cdots \right) \wedge\, dx \;\in\; \Omega^{1,1}$$

The Simplest Example: $(x,u) \in M = \mathbb{R}^2$

Lagrangian form: $\lambda = L(x,u^{(n)})\,dx \;\in\; \Omega^{1,0}$

First variation — vertical derivative:

$$d\lambda = d_V\,\lambda = d_V\,L \wedge dx$$

$$= \left(\frac{\partial L}{\partial u}\,\theta + \frac{\partial L}{\partial u_x}\,\theta_x + \frac{\partial L}{\partial u_{xx}}\,\theta_{xx} + \cdots \right) \wedge dx \;\in\; \Omega^{1,1}$$

Integration by parts — compute modulo im d_H :

$$d\lambda \;\sim\; \delta\lambda = \left(\frac{\partial L}{\partial u} - D_x \frac{\partial L}{\partial u_x} + D_x^2 \frac{\partial L}{\partial u_{xx}} - \cdots \right) \theta \wedge dx \;\in\; \mathcal{F}^1$$

$$= \mathbf{E}(L)\,\theta \wedge dx$$

$\Longrightarrow$ Euler-Lagrange source form.

To analyze invariant variational problems, invariant conservation laws, invariant flows, etc., we apply the moving frame invariantization process to the variational bicomplex:

Differential Invariants and Invariant Differential Forms

ι — invariantization associated with moving frame ρ.

- Fundamental differential invariants

$$H^i(x,u^{(n)}) = \iota(x^i) \qquad I^\alpha_K(x,u^{(n)}) = \iota(u^\alpha_K)$$

- Invariant horizontal forms

$$\varpi^i = \iota(dx^i)$$

- Invariant contact forms

$$\vartheta^\alpha_J = \iota(\theta^\alpha_J)$$

The Invariant "Quasi–Tricomplex"

Differential forms

$$\Omega^* = \bigoplus_{r,s} \widehat{\Omega}^{r,s}$$

Differential

$$d = d_{\mathcal{H}} + d_{\mathcal{V}} + d_{\mathcal{W}}$$

$$d_{\mathcal{H}} : \quad \widehat{\Omega}^{r,s} \longrightarrow \widehat{\Omega}^{r+1,s}$$

$$d_{\mathcal{V}} : \quad \widehat{\Omega}^{r,s} \longrightarrow \widehat{\Omega}^{r,s+1}$$

$$d_{\mathcal{W}} : \quad \widehat{\Omega}^{r,s} \longrightarrow \widehat{\Omega}^{r-1,s+2}$$

Key fact: invariantization and differentiation *do not commute*:

$$d\,\iota(\Omega) \neq \iota(d\Omega)$$

The Universal Recurrence Formula

$$d\,\iota(\Omega) = \iota(d\Omega) + \sum_{\kappa=1}^{r} \nu^{\kappa} \wedge \iota\,[\mathbf{v}_{\kappa}(\Omega)]$$

$\mathbf{v}_1, \ldots, \mathbf{v}_r$ — basis for $\mathfrak{g}$ — infinitesimal generators

$\nu^1, \ldots, \nu^r$ — invariantized dual Maurer–Cartan forms

$\Longrightarrow$ uniquely determined by the recurrence formulae for the phantom differential invariants

$$d\,\iota(\Omega) = \iota(d\,\Omega) + \sum_{\kappa=1}^{r} \nu^\kappa \wedge \iota\,[\mathbf{v}_\kappa(\Omega)]$$

★ ★ ★ All identities, commutation formulae, syzygies, etc., among differential invariants and, more generally, the invariant variational bicomplex follow from this universal formula by letting Ω range over the basic functions and differential forms!

★ ★ ★ Moreover, determining the structure of the differential invariant algebra and invariant variational bicomplex requires only linear differential algebra, and not any explicit formulas for the moving frame, the differential invariants, the invariant differential forms, or the group transformations!

Euclidean plane curves

Fundamental normalized differential invariants

$$\left.\begin{aligned} \iota(x) &= H = 0 \\ \iota(u) &= I_0 = 0 \\ \iota(u_x) &= I_1 = 0 \end{aligned}\right\} \qquad \text{phantom diff. invs.}$$

$$\iota(u_{xx}) = I_2 = \kappa \qquad \iota(u_{xxx}) = I_3 = \kappa_s \qquad \iota(u_{xxxx}) = I_4 = \kappa_{ss} + 3\kappa^3$$

In general:

$$\iota(\, F(x, u, u_x, u_{xx}, u_{xxx}, u_{xxxx}, \ldots\,)) = F(0, 0, 0, \kappa, \kappa_s, \kappa_{ss} + 3\kappa^3, \ldots\ )$$

Invariant arc length form

$$dy = (\cos\phi - u_x \sin\phi)\, dx - (\sin\phi)\, \theta$$

$$\begin{aligned} \varpi = \iota(dx) \quad &= \quad \omega \quad + \quad \eta \\ &= \quad \sqrt{1+u_x^2}\, dx \quad + \quad \frac{u_x}{\sqrt{1+u_x^2}}\, \theta \end{aligned}$$

$$\Longrightarrow \quad \theta = du - u_x\, dx$$

Invariant contact forms

$$\vartheta = \iota(\theta) = \frac{\theta}{\sqrt{1+u_x^2}} \qquad \vartheta_1 = \iota(\theta_x) = \frac{(1+u_x^2)\,\theta_x - u_x u_{xx}\theta}{(1+u_x^2)^2}$$

Prolonged infinitesimal generators

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = \partial_u, \qquad \mathbf{v}_3 = -u\,\partial_x + x\,\partial_u + (1+u_x^2)\,\partial_{u_x} + 3\,u_x u_{xx}\,\partial_{u_{xx}} + \cdots$$

Basic recurrence formula

$$d\iota(F) = \iota(dF) + \iota(\mathbf{v}_1(F))\,\nu^1 + \iota(\mathbf{v}_2(F))\,\nu^2 + \iota(\mathbf{v}_3(F))\,\nu^3$$

Use phantom invariants

$$0 = dH = \iota(dx) + \iota(\mathbf{v}_1(x))\,\nu^1 + \iota(\mathbf{v}_2(x))\,\nu^2 + \iota(\mathbf{v}_3(x))\,\nu^3 = \varpi + \nu^1,$$

$$0 = dI_0 = \iota(du) + \iota(\mathbf{v}_1(u))\,\nu^1 + \iota(\mathbf{v}_2(u))\,\nu^2 + \iota(\mathbf{v}_3(u))\,\nu^3 = \vartheta + \nu^2,$$

$$0 = dI_1 = \iota(du_x) + \iota(\mathbf{v}_1(u_x))\,\nu^1 + \iota(\mathbf{v}_2(u_x))\,\nu^2 + \iota(\mathbf{v}_3(u_x))\,\nu^3 = \kappa\,\varpi + \vartheta_1 + \nu^3,$$

to solve for the Maurer–Cartan forms:

$$\boxed{\nu^1 = -\varpi, \qquad \nu^2 = -\vartheta, \qquad \nu^3 = -\kappa\,\varpi - \vartheta_1.}$$

$$\boxed{\nu^1 = -\varpi, \qquad \nu^2 = -\vartheta, \qquad \nu^3 = -\kappa\,\varpi - \vartheta_1.}$$

Recurrence formulae:

$$d\kappa = d\iota(u_{xx}) = \iota(du_{xx}) + \iota(\mathbf{v}_1(u_{xx}))\,\nu^1 + \iota(\mathbf{v}_2(u_{xx}))\,\nu^2 + \iota(\mathbf{v}_3(u_{xx}))\,\nu^3$$

$$= \iota(u_{xxx}\,dx + \theta_{xx}) - \iota(3\,u_x u_{xx})\,(\kappa\,\varpi + \vartheta_1) = I_3\varpi + \vartheta_2.$$

Therefore,

$$\mathcal{D}\kappa = \kappa_s = I_3, \qquad d_{\mathcal{V}}\,\kappa = \vartheta_2 = (\mathcal{D}^2 + \kappa^2)\,\vartheta$$

where the final formula follows from the contact form recurrence formulae

$$d\vartheta = d\iota(\theta_x) = \varpi \wedge \vartheta_1, \qquad d\vartheta_1 = d\iota(\theta) = \varpi \wedge (\vartheta_2 - \kappa^2\,\vartheta) - \kappa\,\vartheta_1 \wedge \vartheta$$

which imply

$$\vartheta_1 = \mathcal{D}\vartheta, \qquad \vartheta_2 = \mathcal{D}\vartheta_1 + \kappa^2\,\vartheta = (\mathcal{D}^2 + \kappa^2)\,\vartheta$$

Similarly,

$$\begin{aligned} d\varpi &= \iota(d^2x) + \nu^1 \wedge \iota(\mathbf{v}_1(dx)) + \nu^2 \wedge \iota(\mathbf{v}_2(dx)) + \nu^3 \wedge \iota(\mathbf{v}_3(dx)) \\ &= (\kappa\,\varpi + \vartheta_1) \wedge \iota(u_x\,dx + \theta) = \kappa\,\varpi \wedge \vartheta + \vartheta_1 \wedge \vartheta. \end{aligned}$$

In particular,

$$d_{\mathcal{V}}\,\varpi = -\,\kappa\,\vartheta \wedge \varpi$$

Key recurrence formulae:

$$\boxed{d_{\mathcal{V}}\,\kappa = (\mathcal{D}^2 + \kappa^2)\,\vartheta} \qquad \boxed{d_{\mathcal{V}}\,\varpi = -\,\kappa\,\vartheta \wedge \varpi}$$

Plane Curves

Invariant Lagrangian:

$$\widetilde{\lambda} = L(x, u^{(n)})\, dx = P(\kappa, \kappa_s, \ldots)\, \varpi$$

Euler–Lagrange form:

$$d_{\mathcal{V}}\, \widetilde{\lambda} \;\sim\; \mathbf{E}(L)\, \vartheta \wedge \varpi$$

Invariant Integration by Parts Formula

$$F\, d_{\mathcal{V}}\, (\mathcal{D}H) \wedge \varpi \;\sim\; -\, (\mathcal{D}F)\, d_{\mathcal{V}}\, H \wedge \varpi - (F \cdot \mathcal{D}H)\, d_{\mathcal{V}}\, \varpi$$

$$\begin{aligned} d_{\mathcal{V}}\, \widetilde{\lambda} &= d_{\mathcal{V}}\, P \wedge \varpi + P\, d_{\mathcal{V}}\, \varpi \\ &= \sum_n \frac{\partial P}{\partial \kappa_n}\; d_{\mathcal{V}}\, \kappa_n \wedge \varpi + P\, d_{\mathcal{V}}\, \varpi \\ &\sim\; \mathcal{E}(P)\; d_{\mathcal{V}}\, \kappa \wedge \varpi + H^i(P)\; d_{\mathcal{V}}\, \varpi \end{aligned}$$

Vertical differentiation formulae

$$d_{\mathcal{V}}\,\kappa = \mathcal{A}(\vartheta) \qquad \mathcal{A} \; \text{— “Eulerian operator”}$$

$$d_{\mathcal{V}}\,\varpi = \mathcal{B}(\vartheta) \wedge \varpi \qquad \mathcal{B} \; \text{— “Hamiltonian operator”}$$

$$\begin{aligned} d_{\mathcal{V}}\,\tilde{\lambda} \;&\sim\; \mathcal{E}(P)\,\mathcal{A}(\vartheta) \wedge \varpi + H^i(P)\,\mathcal{B}(\vartheta) \wedge \varpi \\ &\sim\; \left[\,\mathcal{A}^*\mathcal{E}(P) - \mathcal{B}^* H^i(P)\,\right]\,\vartheta \wedge \varpi \end{aligned}$$

Invariant Euler-Lagrange equation

$$\boxed{\mathcal{A}^*\mathcal{E}(P) - \mathcal{B}^* H^i(P) = 0}$$

Euclidean Plane Curves

$$d_{\mathcal{V}}\,\kappa = (\mathcal{D}^2 + \kappa^2)\,\vartheta$$

Eulerian operator

$$\mathcal{A} = \mathcal{D}^2 + \kappa^2 \qquad\qquad \mathcal{A}^* = \mathcal{D}^2 + \kappa^2$$

$$d_{\mathcal{V}}\,\varpi = -\,\kappa\,\vartheta \wedge \varpi$$

Hamiltonian operator

$$\mathcal{B} = -\,\kappa \qquad\qquad \mathcal{B}^* = -\,\kappa$$

Euclidean–invariant Euler-Lagrange formula

$$\mathbf{E}(L) = \mathcal{A}^*\mathcal{E}(P) - \mathcal{B}^* H^i(P) = (\mathcal{D}^2 + \kappa^2)\;\mathcal{E}(P) + \kappa\,H^i(P).$$

Invariant Plane Curve Flows

G — Lie group acting on $\mathbb{R}^2$

$C(t)$ — parametrized family of plane curves

G–invariant curve flow:

$$\frac{dC}{dt} = \mathbf{V} = I\,\mathbf{t} + J\,\mathbf{n}$$

- I, J — differential invariants
- $\mathbf{t}$ — "unit tangent"
- $\mathbf{n}$ — "unit normal"

$\mathbf{t}, \ \mathbf{n}$ — basis of the invariant vector fields dual to the invariant one-forms:

$$\langle\, \mathbf{t} \,;\varpi \,\rangle = 1, \qquad \langle\, \mathbf{n} \,;\varpi \,\rangle = 0,$$
$$\langle\, \mathbf{t} \,;\vartheta \,\rangle = 0, \qquad \langle\, \mathbf{n} \,;\vartheta \,\rangle = 1.$$

$$C_t = \mathbf{V} = I\,\mathbf{t} + J\,\mathbf{n}$$

- The tangential component $I\,\mathbf{t}$ only affects the underlying parametrization of the curve. Thus, we can set I to be anything we like without affecting the curve evolution.
- There are two principal choices of tangential component:

Normal Curve Flows

$$C_t = J\,\mathbf{n}$$

Examples — Euclidean–invariant curve flows

- $C_t = \mathbf{n}$ — geometric optics or grassfire flow;
- $C_t = \kappa\,\mathbf{n}$ — curve shortening flow;
- $C_t = \kappa^{1/3}\,\mathbf{n}$ — equi-affine invariant curve shortening flow:
  $$C_t = \mathbf{n}_{\text{equi}-\text{affine}}\,;$$
- $C_t = \kappa_s\,\mathbf{n}$ — modified Korteweg–deVries flow;
- $C_t = \kappa_{ss}\,\mathbf{n}$ — thermal grooving of metals.

Intrinsic Curve Flows

Theorem. The curve flow generated by

$$\mathbf{v} = I\,\mathbf{t} + J\,\mathbf{n}$$

preserves arc length if and only if

$$\mathcal{B}(J) + \mathcal{D}\,I = 0.$$

$\mathcal{D}$ — invariant arc length derivative

$$d_{\mathcal{V}}\,\varpi = \mathcal{B}(\vartheta) \wedge \varpi$$

$\mathcal{B}$ — invariant Hamiltonian operator

Normal Evolution of Differential Invariants

Theorem. Under a normal flow $C_t = J\,\mathbf{n}$,

$$\frac{\partial \kappa}{\partial t} = \mathcal{A}_\kappa(J), \qquad \frac{\partial \kappa_s}{\partial t} = \mathcal{A}_{\kappa_s}(J).$$

Invariant variations:

$$d_{\mathcal{V}}\,\kappa = \mathcal{A}_\kappa(\vartheta), \qquad d_{\mathcal{V}}\,\kappa_s = \mathcal{A}_{\kappa_s}(\vartheta).$$

$\mathcal{A}_\kappa = \mathcal{A}$ — invariant linearization operator of curvature;

$\mathcal{A}_{\kappa_s} = \mathcal{D}\,\mathcal{A}_\kappa + \kappa\,\kappa_s$ — invariant linearization operator of κ_s.

Euclidean–invariant Curve Evolution

Normal flow: $C_t = J\,\mathbf{n}$

$$\frac{\partial \kappa}{\partial t} = \mathcal{A}_\kappa(J) = (\mathcal{D}^2 + \kappa^2)\,J,$$

$$\frac{\partial \kappa_s}{\partial t} = \mathcal{A}_{\kappa_s}(J) = (\mathcal{D}^3 + \kappa^2\mathcal{D} + 3\,\kappa\,\kappa_s)\,J.$$

Warning: For non-intrinsic flows, ∂_t and ∂_s *do not commute*!

Grassfire flow: $J = 1$

$$\frac{\partial \kappa}{\partial t} = \kappa^2, \qquad \frac{\partial \kappa_s}{\partial t} = 3\,\kappa\,\kappa_s, \qquad \ldots$$

$\Longrightarrow$ caustics

Euclidean Signature Evolution

Evolution of the Euclidean signature curve

$$\kappa_s = \Phi(t, \kappa).$$

Grassfire flow:

$$\frac{\partial \Phi}{\partial t} = 3\,\kappa\,\Phi - \kappa^2\,\frac{\partial \Phi}{\partial \kappa}\ .$$

Curve shortening flow:

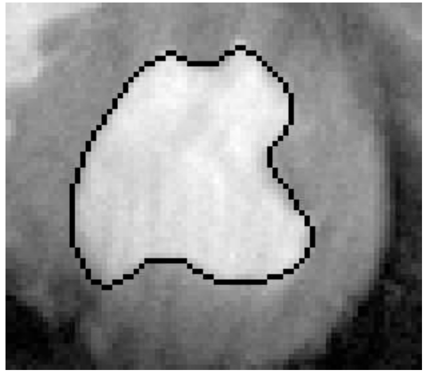
$$\frac{\partial \Phi}{\partial t} = \Phi^2\,\Phi_{\kappa\kappa} - \kappa^3 \Phi_\kappa + 4\,\kappa^2 \Phi.$$

Modified Korteweg-deVries flow:

$$\frac{\partial \Phi}{\partial t} = \Phi^3\,\Phi_{\kappa\kappa\kappa} + 3\,\Phi^2\,\Phi_\kappa\,\Phi_{\kappa\kappa} + 3\,\kappa\,\Phi^2.$$

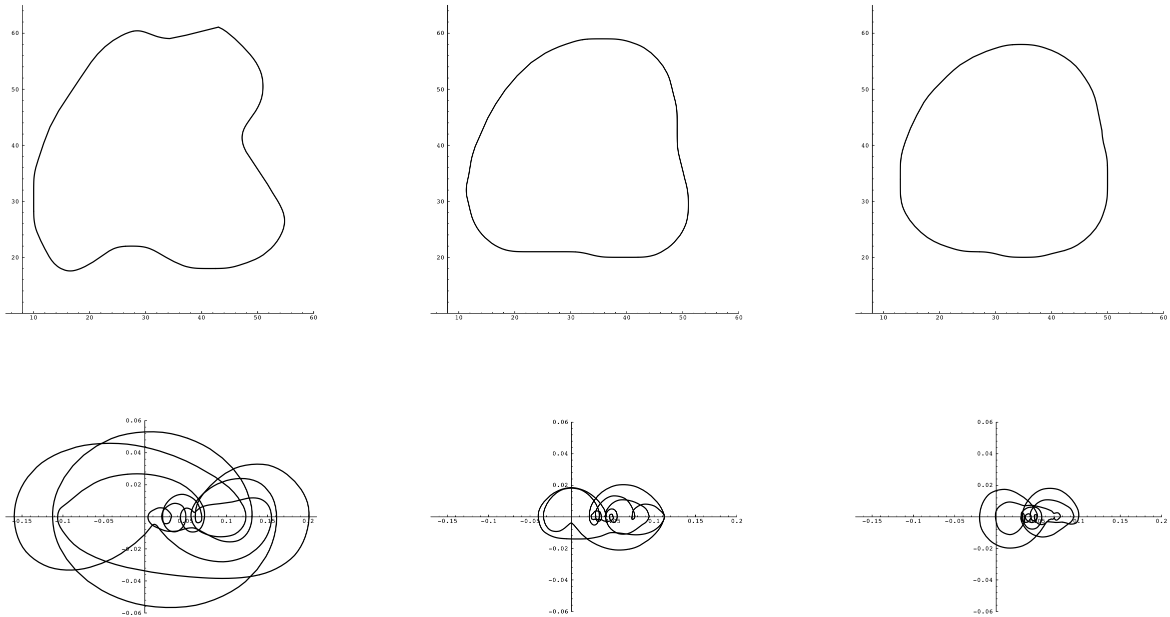
Canine Left Ventricle Signature

Original Canine Heart
MRI Image



Boundary of Left Ventricle

Smoothed Ventricle Signature



Intrinsic Evolution of Differential Invariants

Theorem.

Under an arc-length preserving flow,

$$\kappa_t = \mathcal{R}(J) \qquad \text{where} \qquad \mathcal{R} = \mathcal{A} - \kappa_s \mathcal{D}^{-1} \mathcal{B} \qquad (*)$$

In surprisingly many situations, (*) is a well-known integrable evolution equation, and $\mathcal{R}$ is its recursion operator!

$\Longrightarrow$ Hasimoto

$\Longrightarrow$ Langer, Singer, Perline

$\Longrightarrow$ Marí–Beffa, Sanders, Wang

$\Longrightarrow$ Qu, Chou, and many more ...

Euclidean plane curves

$$G = \mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$$

$$d_{\mathcal{V}}\,\kappa = (\mathcal{D}^2 + \kappa^2)\,\vartheta, \qquad d_{\mathcal{V}}\,\varpi = -\,\kappa\,\vartheta \wedge \varpi$$

$$\Longrightarrow \qquad \mathcal{A} = \mathcal{D}^2 + \kappa^2, \qquad \mathcal{B} = -\,\kappa$$

$$\mathcal{R} = \mathcal{A} - \kappa_s \mathcal{D}^{-1}\mathcal{B} = \mathcal{D}^2 + \kappa^2 + \kappa_s \mathcal{D}^{-1} \cdot \kappa$$

$$\kappa_t = \mathcal{R}(\kappa_s) = \kappa_{sss} + \tfrac{3}{2}\,\kappa^2\kappa_s$$

$\Longrightarrow$ modified Korteweg-deVries equation

Equi-affine plane curves

$$G = \mathrm{SA}(2) = \mathrm{SL}(2) \ltimes \mathbb{R}^2$$

$$d_{\mathcal{V}}\,\kappa = \mathcal{A}(\vartheta), \qquad d_{\mathcal{V}}\,\varpi = \mathcal{B}(\vartheta) \wedge \varpi$$

$$\mathcal{A} = \mathcal{D}^4 + \tfrac{5}{3}\,\kappa\,\mathcal{D}^2 + \tfrac{5}{3}\,\kappa_s\mathcal{D} + \tfrac{1}{3}\,\kappa_{ss} + \tfrac{4}{9}\,\kappa^2,$$

$$\mathcal{B} = \tfrac{1}{3}\,\mathcal{D}^2 - \tfrac{2}{9}\,\kappa,$$

$$\mathcal{R} = \mathcal{A} - \kappa_s\mathcal{D}^{-1}\mathcal{B}$$

$$= \mathcal{D}^4 + \tfrac{5}{3}\,\kappa\,\mathcal{D}^2 + \tfrac{4}{3}\,\kappa_s\mathcal{D} + \tfrac{1}{3}\,\kappa_{ss} + \tfrac{4}{9}\,\kappa^2 + \tfrac{2}{9}\,\kappa_s\mathcal{D}^{-1} \cdot \kappa$$

$$\kappa_t = \mathcal{R}(\kappa_s) = \kappa_{5s} + 2\,\kappa\,\kappa_{ss} + \tfrac{4}{3}\,\kappa_s^2 + + \tfrac{5}{9}\,\kappa^2\,\kappa_s$$

$\Longrightarrow$ Sawada–Kotera equation

Euclidean space curves

$$G = \mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$$

$$\begin{pmatrix} d_{\mathcal{V}}\,\kappa \\ d_{\mathcal{V}}\,\tau \end{pmatrix} = \mathcal{A} \begin{pmatrix} \vartheta_1 \\ \vartheta_2 \end{pmatrix} \qquad d_{\mathcal{V}}\,\varpi = \mathcal{B} \begin{pmatrix} \vartheta_1 \\ \vartheta_2 \end{pmatrix} \wedge \varpi$$

$$\mathcal{A} = \begin{pmatrix} D_s^2 + (\kappa^2 - \tau^2) & -2\tau D_s - \tau_s \\ \dfrac{2\tau}{\kappa} D_s^2 + \dfrac{3\kappa\tau_s - 2\kappa_s\tau}{\kappa^2} D_s + \dfrac{\kappa\tau_{ss} - \kappa_s\tau_s + 2\kappa^3\tau}{\kappa^2} & \dfrac{1}{\kappa} D_s^3 - \dfrac{\kappa_s}{\kappa^2} D_s^2 + \dfrac{\kappa^2 - \tau^2}{\kappa} D_s + \dfrac{\kappa_s\tau^2 - 2\kappa\tau\tau_s}{\kappa^2} \end{pmatrix}$$

$$\mathcal{B} = \begin{pmatrix} \kappa & 0 \end{pmatrix}$$

Recursion operator:

$$\mathcal{R} = \mathcal{A} - \begin{pmatrix} \kappa_s \\ \tau_s \end{pmatrix} \mathcal{D}^{-1} \mathcal{B}$$

$$\begin{pmatrix} \kappa_t \\ \tau_t \end{pmatrix} = \mathcal{R} \begin{pmatrix} \kappa_s \\ \tau_s \end{pmatrix}$$

$\Longrightarrow$ vortex filament flow

$\Longrightarrow$ nonlinear Schrödinger equation (Hasimoto)

Moving Frames for Lie Pseudo–Groups

Peter J. Olver

University of Minnesota

http://www.math.umn.edu/ ∼ olver

Benasque, September, 2009

Sur la théorie, si importante sans doute, mais pour nous si obscure, des ≪groupes de Lie infinis≫, nous ne savons rien que ce qui trouve dans les mémoires de Cartan, première exploration à travers une jungle presque impénétrable; mais celle-ci menace de se refermer sur les sentiers déjà tracés, si l'on ne procède bientôt à un indispensable travail de défrichement.

— André Weil, 1947

What's the Deal with Infinite–Dimensional Groups?

- Lie invented Lie groups to study symmetry and solution of differential equations.

♢ In Lie's time, there were no abstract Lie groups. All groups were realized by their action on a space.

♠ Therefore, Lie saw no essential distinction between finite-dimensional and infinite-dimensional group actions.

However, with the advent of abstract Lie groups, the two subjects have gone in radically different directions.

♡ The general theory of finite-dimensional Lie groups has been rigorously formalized and applied.

♣ But there is still no generally accepted abstract object that represents an infinite-dimensional Lie pseudo-group!

Ehresmann's Trinity

1953:

Ehresmann's Trinity

1953:

- Lie Pseudo-groups

Ehresmann's Trinity

1953:

- Lie Pseudo-groups
- Jets

Ehresmann's Trinity

1953:

- Lie Pseudo-groups
- Jets
- Groupoids

Lie Pseudo-groups in Action

- Lie — Medolaghi — Vessiot
- Cartan
- Ehresmann
- Kuranishi, Spencer, Goldschmidt, Guillemin, Sternberg, Kumpera, . . .

Lie Pseudo-groups in Action

- Lie — Medolaghi — Vessiot
- Cartan
- Ehresmann
- Kuranishi, Spencer, Goldschmidt, Guillemin, Sternberg, Kumpera, . . .

- Relativity
- Noether's (Second) Theorem

- Gauge theory and field theories:
  Maxwell, Yang–Mills, conformal, string, ...
- Fluid mechanics, metereology: Navier–Stokes,
  Euler, boundary layer, quasi-geostropic, ...
- Solitons (in $2 + 1$ dimensions):
  K–P, Davey-Stewartson, ...
- Kac–Moody
- Morphology and shape recognition
- Control theory
- Linear and linearizable PDEs
- Geometric numerical integration
- *Lie groups*!

Moving Frames

In collaboration with Juha Pohjanpelto and Jeongoo Cheh, I have recently established a moving frame theory for infinite-dimensional Lie pseudo-groups mimicking the earlier equivariant approach for finite-dimensional Lie groups developed with Mark Fels and others.

The finite-dimensional theory and algorithms have had a very wide range of significant applications, including differential geometry, differential equations, calculus of variations, computer vision, Poisson geometry and solitons, numerical methods, relativity, classical invariant theory, ...

What's New?

In the infinite-dimensional case, the moving frame approach provides new constructive algorithms for:

- Invariant Maurer–Cartan forms
- Structure equations
- Moving frames
- Differential invariants
- Invariant differential operators
- Basis Theorem
- Syzygies and recurrence formulae

- Further applications:

  $\Longrightarrow$ Symmetry groups of differential equations

  $\Longrightarrow$ Vessiot group splitting; explicit solutions

  $\Longrightarrow$ Gauge theories

  $\Longrightarrow$ Calculus of variations

  $\Longrightarrow$ Invariant geometric flows

Symmetry Groups — Review

System of differential equations:

$$\Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, 2, \ldots, k$$

By a symmetry, we mean a transformation that maps solutions to solutions.

Lie: To find the symmetry group of the differential equations, work infinitesimally.

The vector field

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi_\alpha(x,u) \frac{\partial}{\partial u^\alpha}$$

is an infinitesimal symmetry if its flow $\exp(t\,\mathbf{v})$ is a one-parameter symmetry group of the differential equation.

To find the infinitesimal symmetry conditions, we prolong $\mathbf{v}$ to the jet space whose coordinates are the derivatives appearing in the differential equation:

$$\mathbf{v}^{(n)} = \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{\#J=0}^{n} \varphi_\alpha^J \frac{\partial}{\partial u_J^\alpha}$$

where

$$\varphi_\alpha^J = D_J \left(\varphi^\alpha - \sum_{i=1}^{p} u_i^\alpha \, \xi^i \right) + \sum_{i=1}^{p} u_{J,i}^\alpha \, \xi^i$$

Infinitesimal invariance criterion:

$$\mathbf{v}^{(n)}(\Delta_\nu) = 0 \qquad \text{whenever} \qquad \Delta = 0.$$

Infinitesimal determining equations:

$$\mathcal{L}(x, u; \xi^{(n)}, \varphi^{(n)}) = 0$$

The Heat Equation

$$\boxed{u_t = u_{xx}}$$

Symmetry generator:

$$\mathbf{v} = \tau(t,x,u)\,\frac{\partial}{\partial t} + \xi(t,x,u)\,\frac{\partial}{\partial x} + \varphi(t,x,u)\,\frac{\partial}{\partial u}$$

Prolongation:

$$\mathbf{v}^{(2)} = \mathbf{v} + \varphi^t\,\frac{\partial}{\partial u_t} + \varphi^x\,\frac{\partial}{\partial u_x} + \varphi^{xx}\,\frac{\partial}{\partial u_{xx}} + \;\cdots$$

$$\varphi^t = \varphi_t + u_t\varphi_u - u_t\tau_t - u_t^2\tau_u - u_x\xi_t - u_t u_x\xi_u$$

$$\varphi^x = \varphi_x + u_x\varphi_u - u_t\tau_x - u_t u_x\tau_u - u_x\xi_x - u_x^2\xi_u$$

$$\varphi^{xx} = \varphi_{xx} + u_x(2\,\varphi_{xu} - \xi_{xx}) - u_t\tau_{xx} + u_x^2(\varphi_{uu} - 2\,\xi_{xu})$$
$$- 2\,u_x u_t\tau_{xu} - u_x^3\xi_{uu} - u_x^2 u_t\tau_{uu} + u_{xx}\varphi_u - u_x u_{xx}\xi_u - u_t u_{xx}\tau_u$$

Infinitesimal invariance:

$$\mathbf{v}^{(3)}(u_t - u_{xx}) = \varphi^t - \varphi^{xx} = 0 \qquad \text{whenever} \qquad u_t = u_{xx}$$

Determining equations:

Coefficient	*Monomial*
$0 = -2\tau_u$	$u_x u_{xt}$
$0 = -2\tau_x$	u_{xt}
$0 = -\tau_{uu}$	$u_x^2 u_{xx}$
$-\xi_u = -2\tau_{xu} - 3\xi_u$	$u_x u_{xx}$
$\varphi_u - \tau_t = -\tau_{xx} + \varphi_u - 2\xi_x$	u_{xx}
$0 = -\xi_{uu}$	u_x^3
$0 = \varphi_{uu} - 2\xi_{xu}$	u_x^2
$-\xi_t = 2\varphi_{xu} - \xi_{xx}$	u_x
$\varphi_t = \varphi_{xx}$	1

General solution:

$$\xi = c_1 + c_4 x + 2c_5 t + 4c_6 x t,$$

$$\tau = c_2 + 2c_4 t + 4c_6 t^2,$$

$$\varphi = (c_3 - c_5 x - 2c_6 t - c_6 x^2) u + \alpha(x,t),$$

where $\alpha_t = \alpha_{xx}$ is an arbitrary solution to the heat equation.

Basis for the (infinite-dimensional) symmetry algebra:

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = \partial_t, \qquad \mathbf{v}_3 = u\partial_u, \qquad \mathbf{v}_4 = x\partial_x + 2t\partial_t,$$
$$\mathbf{v}_5 = 2t\partial_x - xu\partial_u, \qquad \mathbf{v}_6 = 4xt\partial_x + 4t^2\partial_t - (x^2 + 2t)u\partial_u,$$
$$\mathbf{v}_\alpha = \alpha(x,t)\partial_u, \qquad \text{where} \qquad \alpha_t = \alpha_{xx}.$$

- x and t translations, scalings: λu, and $(\lambda x, \lambda^2 t)$, Galilean boosts, inversions, and the addition of solutions stemming from the linearity of the equation.

The Korteweg–deVries equation

$$\boxed{u_t + u_{xxx} + u u_x = 0}$$

Symmetry generator:

$$\mathbf{v} = \tau(t,x,u)\,\frac{\partial}{\partial t} + \xi(t,x,u)\,\frac{\partial}{\partial x} + \varphi(t,x,u)\,\frac{\partial}{\partial u}$$

Prolongation:

$$\mathbf{v}^{(3)} = \mathbf{v} + \varphi^t\,\frac{\partial}{\partial u_t} + \varphi^x\,\frac{\partial}{\partial u_x} + \;\cdots\; + \varphi^{xxx}\,\frac{\partial}{\partial u_{xxx}}$$

where

$$\varphi^t = \varphi_t + u_t\varphi_u - u_t\tau_t - u_t^2\tau_u - u_x\xi_t - u_t u_x\xi_u$$

$$\varphi^x = \varphi_x + u_x\varphi_u - u_t\tau_x - u_t u_x\tau_u - u_x\xi_x - u_x^2\xi_u$$

$$\varphi^{xxx} = \varphi_{xxx} + 3\,u_x\varphi_u + \;\cdots$$

Infinitesimal invariance:

$$\mathbf{v}^{(3)}(u_t + u_{xxx} + u u_x) = \varphi^t + \varphi^{xxx} + u\,\varphi^x + u_x\,\varphi = 0$$

on solutions

Infinitesimal determining equations:

$$\tau_x = \tau_u = \xi_u = \varphi_t = \varphi_x = 0$$

$$\varphi = \xi_t - \tfrac{2}{3}\,u\tau_t \qquad \varphi_u = -\tfrac{2}{3}\,\tau_t = -\,2\,\xi_x$$

$$\tau_{tt} = \tau_{tx} = \tau_{xx} = \;\cdots\; = \varphi_{uu} = 0$$

General solution:

$$\tau = c_1 + 3\,c_4 t, \qquad \xi = c_2 + c_3 t + c_4 x, \qquad \varphi = c_3 - 2c_4 u.$$

Basis for symmetry algebra $\mathfrak{g}_{KdV}$:

$$\mathbf{v}_1 = \partial_t,$$

$$\mathbf{v}_2 = \partial_x,$$

$$\mathbf{v}_3 = t\,\partial_x + \partial_u,$$

$$\mathbf{v}_4 = 3\,t\,\partial_t + x\,\partial_x - 2\,u\,\partial_u.$$

The symmetry group $\mathcal{G}_{KdV}$ is four-dimensional

$$(x, t, u) \longmapsto (\lambda^3\, t + a, \lambda\, x + c\, t + b, \lambda^{-2}\, u + c)$$

$$\mathbf{v}_1 = \partial_t, \qquad \mathbf{v}_2 = \partial_x,$$

$$\mathbf{v}_3 = t\,\partial_x + \partial_u, \qquad \mathbf{v}_4 = 3\,t\,\partial_t + x\,\partial_x - 2\,u\,\partial_u.$$

Commutator table:

	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$
$\mathbf{v}_1$	0	0	0	$\mathbf{v}_1$
$\mathbf{v}_2$	0	0	$\mathbf{v}_1$	$3\,\mathbf{v}_2$
$\mathbf{v}_3$	0	$-\,\mathbf{v}_1$	0	$-2\,\mathbf{v}_3$
$\mathbf{v}_4$	$-\,\mathbf{v}_1$	$-3\,\mathbf{v}_2$	$2\,\mathbf{v}_3$	0

Entries: $[\mathbf{v}_i, \mathbf{v}_j] = \sum_k C_{ij}^k \mathbf{v}_k.$ C_{ij}^k — structure constants of $\mathfrak{g}$

Navier–Stokes Equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu\, \Delta \mathbf{u}, \qquad \nabla \cdot \mathbf{u} = 0.$$

Symmetry generators:

$$\mathbf{v}_{\boldsymbol{\alpha}} = \boldsymbol{\alpha}(t) \cdot \partial_{\mathbf{x}} + \boldsymbol{\alpha}'(t) \cdot \partial_{\mathbf{u}} - \boldsymbol{\alpha}''(t) \cdot \mathbf{x}\, \partial_p$$

$$\mathbf{v}_0 = \partial_t$$

$$\mathbf{s} = \mathbf{x} \cdot \partial_{\mathbf{x}} + 2\, t\, \partial_t - \mathbf{u} \cdot \partial_{\mathbf{u}} - 2\, p\, \partial_p$$

$$\mathbf{r} = \mathbf{x} \wedge \partial_{\mathbf{x}} + \mathbf{u} \wedge \partial_{\mathbf{u}}$$

$$\mathbf{w}_h = h(t)\, \partial_p$$

Kadomtsev–Petviashvili (KP) Equation

$$(u_t + \tfrac{3}{2}\, u\, u_x + \tfrac{1}{4}\, u_{xxx})_x \pm \tfrac{3}{4}\, u_{yy} = 0$$

Symmetry generators:

$$\begin{aligned}
\mathbf{v}_f &= f(t)\,\partial_t + \tfrac{2}{3}\, y\, f'(t)\,\partial_y + \left(\tfrac{1}{3}\, x\, f'(t) \mp \tfrac{2}{9}\, y^2 f''(t) \right) \partial_x \\
&\qquad + \left(-\tfrac{2}{3}\, u\, f'(t) + \tfrac{2}{9}\, x\, f''(t) \mp \tfrac{4}{27}\, y^2 f'''(t) \right) \partial_u, \\
\mathbf{w}_g &= g(t)\,\partial_y \mp \tfrac{2}{3}\, y\, g'(t)\,\partial_x \mp \tfrac{4}{9}\, y\, g''(t)\,\partial_u, \\
\mathbf{z}_h &= h(t)\,\partial_x + \tfrac{2}{3}\, h'(t)\,\partial_u.
\end{aligned}$$

$\Longrightarrow$ Kac–Moody loop algebra $A_4^{(1)}$

Main Goals

Given a system of partial differential equations:

Main Goals

Given a system of partial differential equations:

- Find the structure of its symmetry (pseudo-) group $\mathcal{G}$ directly from the determining equations.

Main Goals

Given a system of partial differential equations:

- Find the structure of its symmetry (pseudo-) group $\mathcal{G}$ directly from the determining equations.
- Find and classify its differential invariants.

Main Goals

Given a system of partial differential equations:

- Find the structure of its symmetry (pseudo-) group $\mathcal{G}$ directly from the determining equations.
- Find and classify its differential invariants.
- Use symmetry reduction or group splitting to construct explicit solutions.

Pseudo-groups

M — smooth (analytic) manifold

Definition. A pseudo-group is a collection of local diffeomorphisms $\varphi\colon M \to M$ such that

- *Identity*: $\mathbf{1}_M \in \mathcal{G}$,
- *Inverses*: $\varphi^{-1} \in \mathcal{G}$,
- *Restriction*: $U \subset \operatorname{dom}\varphi \implies \varphi \mid U \in \mathcal{G}$,
- *Continuation*: $\operatorname{dom}\varphi = \bigcup U_\kappa$ and $\varphi \mid U_\kappa \in \mathcal{G} \implies \varphi \in \mathcal{G}$,
- *Composition*: $\operatorname{im}\varphi \subset \operatorname{dom}\psi \implies \psi \circ \varphi \in \mathcal{G}$.

Lie Pseudo-groups

Definition. A Lie pseudo-group $\mathcal{G}$ is a pseudo-group whose transformations are the solutions to an involutive system of partial differential equations:

$$F(z, \varphi^{(n)}) = 0.$$

called the nonlinear determining equations.

$\Longrightarrow$ *analytic* (*Cartan–Kähler*)

$\star\star$ Key complication: $\nexists$ abstract object $\mathcal{G}$ $\star\star$

A Non-Lie Pseudo-group

Acting on $M = \mathbb{R}^2$:

$$\boxed{X = \varphi(x) \qquad Y = \varphi(y)}$$

where $\varphi \in \mathcal{D}(\mathbb{R})$ is any local diffeomorphism.

♠ Cannot be characterized by a system of partial differential equations

$$\Delta(x, y, X^{(n)}, Y^{(n)}) = 0$$

Theorem. Any regular non-Lie pseudo-group can be completed to a Lie pseudo-group with the same differential invariants.

Completion of previous example:

$$\boxed{X = \varphi(x), \qquad Y = \psi(y)}$$

where $\varphi, \psi \in \mathcal{D}(\mathbb{R})$.

Infinitesimal Generators

$\mathfrak{g}$ — Lie algebra of infinitesimal generators of the pseudo-group $\mathcal{G}$

$z = (x, u)$ — local coordinates on M

Vector field:

$$\mathbf{v} = \sum_{a=1}^{m} \zeta^a(z) \frac{\partial}{\partial z^a} = \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha \frac{\partial}{\partial u^\alpha}$$

Vector field jet:

$$\mathrm{j}_n \mathbf{v} \longmapsto \zeta^{(n)} = (\ \ldots\ \zeta^b_A\ \ldots\ )$$

$$\zeta^b_A = \frac{\partial^{\#A} \zeta^b}{\partial z^A} = \frac{\partial^k \zeta^b}{\partial z^{a_1} \cdots \partial z^{a_k}}$$

The infinitesimal generators of $\mathcal{G}$ are the solutions to the *Infinitesimal (Linearized) Determining Equations*

$$\mathcal{L}(z, \zeta^{(n)}) = 0 \tag{$*$}$$

Remark: If $\mathcal{G}$ is the symmetry group of a system of differential equations $\Delta(x, u^{(n)}) = 0$, then $(*)$ is the (involutive completion of) the usual Lie determining equations for the symmetry group.

The Diffeomorphism Pseudo-group

M — smooth m-dimensional manifold

$\mathcal{D} = \mathcal{D}(M)$ — pseudo-group of all local diffeomorphisms

$$Z = \varphi(z)$$

$$\begin{cases} z = (z^1, \ldots, z^m) \text{ — source coordinates} \\ Z = (Z^1, \ldots, Z^m) \text{ — target coordinates} \end{cases}$$

Jets

For $0 \leq n \leq \infty$:

Given a smooth map $\varphi \colon M \to M$, written in local coordinates as

$Z = \varphi(z)$, let $\mathrm{j}_n\varphi|_z$ denote its n–jet at $z \in M$, i.e., its n^{th} order Taylor polynomial or series based at z.

$\mathrm{J}^n(M, M)$ is the n^{th} order jet bundle, whose points are the jets.

Local coordinates on $\mathrm{J}^n(M, M)$:

$$(z, Z^{(n)}) = (\ \ldots\ z^a\ \ldots\ Z^b_A\ \ldots\ ), \qquad Z^b_A = \frac{\partial^k Z^b}{\partial z^{a_1} \cdots \partial z^{a_k}}$$

Diffeomorphism Jets

The n^{th} order diffeomorphism jet bundle is the subbundle

$$\mathcal{D}^{(n)} = \mathcal{D}^{(n)}(M) \subset \mathrm{J}^n(M, M)$$

consisting of n^{th} order jets of local diffeomorphisms $\varphi \colon M \longrightarrow M$.

The Inverse Function Theorem tells us that $\mathcal{D}^{(n)}$ is defined by the non-vanishing of the Jacobian determinant:

$$\det(\, Z^a_b \,) = \det(\, \partial Z^a / \partial z^b \,) \neq 0$$

Pseudo-group Jets

A Lie pseudo-group $\mathcal{G} \subset \mathcal{D}$ defines the subbundle

$$\mathcal{G}^{(n)} = \{\, F(z, Z^{(n)}) = 0 \,\} \subset \mathcal{D}^{(n)}$$

consisting of the jets of pseudo-group diffeomorphisms, and therefore characterized by the pseudo-group's nonlinear determining equations.

Pseudo-group Jets

A Lie pseudo-group $\mathcal{G} \subset \mathcal{D}$ defines the subbundle

$$\mathcal{G}^{(n)} = \{\, F(z, Z^{(n)}) = 0 \,\} \subset \mathcal{D}^{(n)}$$

consisting of the jets of pseudo-group diffeomorphisms, and therefore characterized by the pseudo-group's nonlinear determining equations.

♡ Local coordinates on $\mathcal{G}^{(n)}$, e.g., the restricted diffeomorphism jet coordinates z^c, Z^a_B, are viewed as the pseudo-group parameters, playing the same role as the local coordinates on a Lie group G.

Pseudo-group Jets

A Lie pseudo-group $\mathcal{G} \subset \mathcal{D}$ defines the subbundle

$$\mathcal{G}^{(n)} = \{\, F(z, Z^{(n)}) = 0 \,\} \subset \mathcal{D}^{(n)}$$

consisting of the jets of pseudo-group diffeomorphisms, and therefore characterized by the pseudo-group's nonlinear determining equations.

♡ Local coordinates on $\mathcal{G}^{(n)}$, e.g., the restricted diffeomorphism jet coordinates z^c, Z^a_B, are viewed as the pseudo-group parameters, playing the same role as the local coordinates on a Lie group G.

♠ The pseudo-group jet bundle $\mathcal{G}^{(n)}$ does not form a group, but rather a groupoid under composition of Taylor polynomials/series.

Groupoid Structure

Double fibration:

$$
\begin{array}{ccc}
 & \mathcal{G}^{(n)} & \\
\boldsymbol{\sigma}^{(n)} \swarrow & & \searrow \boldsymbol{\tau}^{(n)} \\
M & & M
\end{array}
$$

$$\boldsymbol{\sigma}^{(n)}(z, Z^{(n)}) = z \qquad \text{— source map}$$

$$\boldsymbol{\tau}^{(n)}(z, Z^{(n)}) = Z \qquad \text{— target map}$$

You are only allowed to multiply $h^{(n)} \cdot g^{(n)}$ if

$$\boldsymbol{\sigma}^{(n)}(h^{(n)}) = \boldsymbol{\tau}^{(n)}(g^{(n)})$$

⋆ ⋆ Composition of Taylor polynomials/series is well-defined only when the source of the second matches the target of the first.

One-dimensional case: $M = \mathbb{R}$

Source coordinate: x Target coordinate: X

Local coordinates on $\mathcal{D}^{(n)}(\mathbb{R})$

$$g^{(n)} = (x, X, X_x, X_{xx}, X_{xxx}, \ \ldots \ , X_n)$$

Diffeomorphism jet:

$$X[\![\, h \,]\!] = X + X_x\, h + \tfrac{1}{2}\, X_{xx}\, h^2 + \tfrac{1}{6}\, X_{xxx}\, h^3 + \ \cdots$$

$\Longrightarrow$ Taylor polynomial/series at a source point x

Groupoid multiplication of diffeomorphism jets:

$$(X, \mathbf{X}, \mathbf{X}_X, \mathbf{X}_{XX}, \ \ldots\ ) \cdot (x, X, X_x, X_{xx}, \ \ldots\ )$$

$$= (x, \mathbf{X}, \mathbf{X}_X\, X_x, \mathbf{X}_X\, X_{xx} + \mathbf{X}_{XX}\, X_x^2, \ \ldots\ )$$

$\Longrightarrow$ Composition of Taylor polynomials/series

The higher order terms are expressed in terms of Bell polynomials according to the general Fàa–di–Bruno formula.

- The groupoid multiplication (or Taylor composition) is only defined when the source coordinate X of the first multiplicand matches the target coordinate X of the second.

Structure of Lie Pseudo-groups

The structure of a finite-dimensional Lie group G is specified by its Maurer–Cartan forms — a basis $\mu^1, \ldots, \mu^r$ for the right-invariant one-forms:

$$d\mu^k = \sum_{i<j} C^k_{ij}\, \mu^i \wedge \mu^j$$

What should be the Maurer–Cartan forms of a Lie pseudo-group?

What should be the Maurer–Cartan forms of a Lie pseudo-group?

Cartan: Use exterior differential systems and prolongation to determine the structure equations.

What should be the Maurer–Cartan forms of a Lie pseudo-group?

Cartan: Use exterior differential systems and prolongation to determine the structure equations.

I propose a direct approach based on the following observation:

.

What should be the Maurer–Cartan forms of a Lie pseudo-group?

Cartan: Use exterior differential systems and prolongation to determine the structure equations.

I propose a direct approach based on the following observation:

The Maurer–Cartan forms for a pseudo-group can be identified with the right-invariant one-forms on the jet groupoid $\mathcal{G}^{(\infty)}$.

What should be the Maurer–Cartan forms of a Lie pseudo-group?

Cartan: Use exterior differential systems and prolongation to determine the structure equations.

I propose a direct approach based on the following observation:

The Maurer–Cartan forms for a pseudo-group can be identified with the right-invariant one-forms on the jet groupoid $\mathcal{G}^{(\infty)}$.

The structure equations can be determined immediately from the infinitesimal determining equations.

The Variational Bicomplex

★ The differential one-forms on an infinite jet bundle split into two types:

- horizontal forms
- contact forms

★ Consequently, the exterior derivative

$$d = d_M + d_G$$

on $\mathcal{D}^{(\infty)}$ splits into horizontal (manifold) and contact (group) components, leading to the variational bicomplex structure on the algebra of differential forms on $\mathcal{D}^{(\infty)}$.

For the diffeomorphism jet bundle

$$\mathcal{D}^{(\infty)} \subset \mathrm{J}^{\infty}(M, M)$$

Local coordinates:

$$\underbrace{z^1, \ \ldots \ z^m,}_{\text{source}} \quad \underbrace{Z^1, \ \ldots \ Z^m,}_{\text{target}} \quad \underbrace{\ldots \ Z^b_A, \ \ldots}_{\text{jet}}$$

Horizontal forms:

$$dz^1, \ \ldots \ , dz^m$$

Basis contact forms:

$$\Theta^b_A = d_G\, Z^b_A = dZ^b_A - \sum_{a=1}^{m} Z^a_{A,a}\, dz^a$$

One-dimensional case: $M = \mathbb{R}$

Local coordinates on $\mathcal{D}^{(\infty)}(\mathbb{R})$

$$(x, X, X_x, X_{xx}, X_{xxx}, \ \ldots \ , X_n, \ \ldots \ )$$

Horizontal form:

$$dx$$

Contact forms:

$$\Theta = dX - X_x \, dx$$

$$\Theta_x = dX_x - X_{xx} \, dx$$

$$\Theta_{xx} = dX_{xx} - X_{xxx} \, dx$$

$$\vdots$$

Maurer–Cartan Forms

The Maurer–Cartan forms for the diffeomorphism pseudo-group are the right-invariant one-forms on the diffeomorphism jet groupoid $\mathcal{D}^{(\infty)}$.

Key observation:

The target coordinate functions Z^a are right-invariant.

Thus, when we decompose

$$dZ^a = \underbrace{\sigma^a}_{\text{horizontal}} + \underbrace{\mu^a}_{\text{contact}}$$

the two constituents are also right-invariant.

Invariant horizontal forms:

$$\sigma^a = d_M Z^a = \sum_{b=1}^{m} Z^a_b \, dz^b$$

Invariant total differentiation (dual operators):

$$\mathbb{D}_{Z^a} = \sum_{b=1}^{m} (Z^a_b)^{-1} \, \mathbb{D}_{z^b}$$

Thus, the invariant contact forms are obtained by invariant differentiation of the order zero contact forms:

$$\mu^b = d_G Z^b = \Theta^b = dZ^b - \sum_{a=1}^{m} Z^b_a \, dz^a$$

$$\mu^b_A = \mathbb{D}^A_Z \mu^b = \mathbb{D}_{Z^{a_1}} \cdots \mathbb{D}_{Z^{a_n}} \mu^b$$

$$b = 1, \ldots, m, \;\; \#A \geq 0$$

One-dimensional case: $M = \mathbb{R}$

Contact forms:

$$\Theta = dX - X_x\, dx$$

$$\Theta_x = \mathbb{D}_x \Theta = dX_x - X_{xx}\, dx$$

$$\Theta_{xx} = \mathbb{D}_x^2 \Theta = dX_{xx} - X_{xxx}\, dx$$

Right-invariant horizontal form:

$$\sigma = d_M X = X_x\, dx$$

Invariant differentiation:

$$\mathbb{D}_X = \frac{1}{X_x}\, \mathbb{D}_x$$

Invariant contact forms:

$$\mu = \Theta = dX - X_x\,dx$$

$$\mu_X = \mathbb{D}_X \mu = \frac{\Theta_x}{X_x} = \frac{dX_x - X_{xx}\,dx}{X_x}$$

$$\mu_{XX} = \mathbb{D}_X^2 \mu = \frac{X_x\,\Theta_{xx} - X_{xx}\,\Theta_x}{X_x^3}$$

$$= \frac{X_x\,dX_{xx} - X_{xx}\,dX_x + (X_{xx}^2 - X_x X_{xxx})\,dx}{X_x^3}$$

$$\vdots$$

$$\mu_n = \mathbb{D}_X^n \mu$$

The Structure Equations for the Diffeomorphism Pseudo–group

$$d\mu_A^b = \sum C_{A,c,d}^{b,B,C} \; \mu_B^c \wedge \mu_C^d$$

The Structure Equations for the Diffeomorphism Pseudo–group

$$d\mu_A^b = \sum C_{A,c,d}^{b,B,C}\, \mu_B^c \wedge \mu_C^d$$

Maurer–Cartan series:

$$\mu^b[\![\, H \,]\!] = \sum_A \frac{1}{A!}\, \mu_A^b\, H^A$$

$$H = (H^1, \ldots, H^m) \quad \text{— formal parameters}$$

$$d\mu[\![\, H \,]\!] = \nabla\mu[\![\, H \,]\!] \wedge (\, \mu[\![\, H \,]\!] - dZ\,)$$

$$d\sigma = -\, d\mu[\![\, 0 \,]\!] = \nabla\mu[\![\, 0 \,]\!] \wedge \sigma$$

One-dimensional case: $M = \mathbb{R}$

Structure equations:

$$d\sigma = \mu_X \wedge \sigma \qquad d\mu[\![\, H \,]\!] = \frac{d\mu}{dH}[\![\, H \,]\!] \wedge (\mu[\![\, H \,]\!] - dZ)$$

where

$$\sigma = X_x\, dx = dX - \mu$$

$$\mu[\![\, H \,]\!] = \mu + \mu_X\, H + \tfrac{1}{2}\mu_{XX}\, H^2 + \cdots$$

$$\mu[\![\, H \,]\!] - dZ = -\sigma + \mu_X\, H + \tfrac{1}{2}\mu_{XX}\, H^2 + \cdots$$

$$\frac{d\mu[\![\, H \,]\!]}{dH} = \mu_X + \mu_{XX}\, H + \tfrac{1}{2}\mu_{XXX}\, H^2 + \cdots$$

In components:

$$d\sigma = \mu_1 \wedge \sigma$$

$$d\mu_n = -\,\mu_{n+1} \wedge \sigma \;+\; \sum_{i=0}^{n-1} \binom{n}{i}\, \mu_{i+1} \wedge \mu_{n-i}$$

$$= \sigma \wedge \mu_{n+1} - \sum_{j=1}^{\left[\frac{n+1}{2}\right]} \frac{n-2j+1}{n+1}\binom{n+1}{j}\, \mu_j \wedge \mu_{n+1-j}.$$

$\Longrightarrow$ Cartan

The Maurer–Cartan Forms for a Lie Pseudo-group

The Maurer–Cartan forms for $\mathcal{G}$ are obtained by restricting the diffeomorphism Maurer–Cartan forms σ^a, μ^b_A to $\mathcal{G}^{(\infty)} \subset \mathcal{D}^{(\infty)}$.

The Maurer–Cartan Forms for a Lie Pseudo-group

The Maurer–Cartan forms for $\mathcal{G}$ are obtained by restricting the diffeomorphism Maurer–Cartan forms σ^a, μ^b_A to $\mathcal{G}^{(\infty)} \subset \mathcal{D}^{(\infty)}$.

$\star\star$ The resulting one-forms are no longer linearly independent.

Theorem. The Maurer–Cartan forms on $\mathcal{G}^{(\infty)}$ satisfy the invariant infinitesimal determining equations

$$\mathcal{L}(\;\ldots\; Z^a \;\ldots\; \mu^b_A \;\ldots\;) = 0 \qquad (\star\star)$$

obtained from the infinitesimal determining equations

$$\mathcal{L}(\;\ldots\; z^a \;\ldots\; \zeta^b_A \;\ldots\;) = 0 \qquad (\star)$$

by replacing

- source variables z^a by target variables Z^a
- derivatives of vector field coefficients ζ^b_A by
  right-invariant Maurer–Cartan forms μ^b_A

The Structure Equations for a Lie Pseudo-group

Theorem. The structure equations for the pseudo-group $\mathcal{G}$ are obtained by restricting the universal diffeomorphism structure equations

$$d\mu[\![\, H \,]\!] = \nabla\mu[\![\, H \,]\!] \wedge (\, \mu[\![\, H \,]\!] - dZ \,)$$

to the solution space of the linearized involutive system

$$\mathcal{L}(\,\ldots\, Z^a, \,\ldots\, \mu^b_A, \,\ldots\,) = 0.$$

The Korteweg–deVries Equation

$$\boxed{u_t + u_{xxx} + u u_x = 0}$$

Diffeomorphism Maurer–Cartan forms:

$$\mu^t,\ \mu^x,\ \mu^u,\ \mu^t_T,\ \mu^t_X,\ \mu^t_U,\ \mu^x_T,\ \ldots,\ \mu^u_U,\ \mu^t_{TT},\ \mu^T_{TX},\ \ldots$$

Infinitesimal determining equations:

$$\tau_x = \tau_u = \xi_u = \varphi_t = \varphi_x = 0$$

$$\varphi = \xi_t - \tfrac{2}{3}\, u \tau_t \qquad \varphi_u = -\tfrac{2}{3}\, \tau_t = -\, 2\, \xi_x$$

$$\tau_{tt} = \tau_{tx} = \tau_{xx} = \ \cdots \ = \varphi_{uu} = 0$$

Maurer–Cartan determining equations:

$$\mu^t_X = \mu^t_U = \mu^x_U = \mu^u_T = \mu^u_X = 0,$$

$$\mu^u = \mu^x_T - \tfrac{2}{3}\, U \mu^t_T, \qquad \mu^u_U = -\tfrac{2}{3}\, \mu^t_T = -\, 2\, \mu^x_X,$$

$$\mu^t_{TT} = \mu^t_{TX} = \mu^t_{XX} = \ \cdots \ = \mu^u_{UU} = \ \ldots \ = 0.$$

Basis ($\dim \mathcal{G}_{KdV} = 4$):

$$\mu^1 = \mu^t, \qquad \mu^2 = \mu^x, \qquad \mu^3 = \mu^u, \qquad \mu^4 = \mu^t_T.$$

Substituting into the full diffeomorphism structure equations yields the structure equations for $\mathfrak{g}_{KdV}$:

$$\begin{aligned}
d\mu^1 &= -\,\mu^1 \wedge \mu^4, \\
d\mu^2 &= -\mu^1 \wedge \mu^3 - \tfrac{2}{3} U\, \mu^1 \wedge \mu^4 - \tfrac{1}{3}\, \mu^2 \wedge \mu^4, \\
d\mu^3 &= \tfrac{2}{3}\, \mu^3 \wedge \mu^4, \\
d\mu^4 &= 0.
\end{aligned}$$

$$\boxed{d\mu^i = C^i_{jk}\, \mu^j \wedge \mu^k}$$

$$
\begin{aligned}
d\mu^1 &= -\,\mu^1 \wedge \mu^4,\\
d\mu^2 &= -\mu^1 \wedge \mu^3 - \tfrac{2}{3}\, U\, \mu^1 \wedge \mu^4 - \tfrac{1}{3}\, \mu^2 \wedge \mu^4,\\
d\mu^3 &= \tfrac{2}{3}\, \mu^3 \wedge \mu^4,\\
d\mu^4 &= 0.
\end{aligned}
$$

In general, the pseudo-group structure equations live on the principal bundle $\mathcal{G}^{(\infty)}$; if G is a finite-dimensional Lie group, then $\mathcal{G}^{(\infty)} \simeq M \times G$, and the usual Lie group structure equations are found by restriction to the target fibers $\{\, Z = c \,\} \simeq G$. Note that the constructed basis $\mu^1, \ldots, \mu^r$ of $\mathfrak{g}^*$ might vary from fiber to fiber.

Lie–Kumpera Example

$$\boxed{X = f(x) \qquad\qquad U = \frac{u}{f'(x)}}$$

Linearized determining system

$$\xi_x = -\frac{\varphi}{u} \qquad\qquad \xi_u = 0 \qquad\qquad \varphi_u = \frac{\varphi}{u}$$

Maurer–Cartan forms:

$$\sigma = \frac{u}{U}\,dx = f_x\,dx, \qquad \tau = U_x\,dx + \frac{U}{u}\,du = \frac{-u\,f_{xx}\,dx + f_x\,du}{{f_x}^2}$$

$$\mu = dX - \frac{U}{u}\,dx = df - f_x\,dx, \qquad \nu = dU - U_x\,dx - \frac{U}{u}\,du = -\frac{u}{{f_x}^2}\left(df_x - f_{xx}\,dx\right)$$

$$\mu_X = \frac{du}{u} - \frac{dU - U_x\,dx}{U} = \frac{df_x - f_{xx}\,dx}{f_x}, \qquad \mu_U = 0$$

$$\nu_X = \frac{U}{u}(dU_x - U_{xx}\,dx) - \frac{U_x}{u}(dU - U_x\,dx)$$

$$= -\frac{u}{{f_x}^3}\left(df_{xx} - f_{xxx}\,dx\right) + \frac{u\,f_{xx}}{{f_x}^4}\left(df_x - f_{xx}\,dx\right)$$

$$\nu_U = -\frac{du}{u} + \frac{dU - U_x\,dx}{U} = -\frac{df_x - f_{xx}\,dx}{f_x}$$

First order linearized determining equations:

$$\xi_x = -\frac{\varphi}{u} \qquad \xi_u = 0 \qquad \varphi_u = \frac{\varphi}{u}$$

First order Maurer–Cartan determining equations:

$$\mu_X = -\frac{\nu}{U} \qquad \mu_U = 0 \qquad \nu_U = \frac{\nu}{U}$$

First order structure equations:

$$d\mu = -\,d\sigma = \frac{\nu \wedge \sigma}{U}, \qquad d\nu = -\,\nu_X \wedge \sigma \; - \; \frac{\nu \wedge \tau}{U}$$

$$d\nu_X = -\,\nu_{XX} \wedge \sigma \; - \; \frac{\nu_X \wedge (\tau + 2\,\nu)}{U}$$

Comparison of Structure Equations

If the action is transitive, then our structure equations are isomorphic to Cartan's. However, this is not true for intransitive pseudo-groups.
Which are "right"?

Comparison of Structure Equations

If the action is transitive, then our structure equations are isomorphic to Cartan's. However, this is not true for intransitive pseudo-groups. Which are "right"?

- To find the Cartan structure equations, one first needs to work in an adapted coordinate chart, which requires identification of the invariants on M. Ours can be found in any system of local coordinates.

Comparison of Structure Equations

If the action is transitive, then our structure equations are isomorphic to Cartan's. However, this is not true for intransitive pseudo-groups.
Which are "right"?

- To find the Cartan structure equations, one first needs to work in an adapted coordinate chart, which requires identification of the invariants on M. Ours can be found in any system of local coordinates.
- Cartan's procedure for identifying the invariant forms is recursive, and not easy to implement. Ours follow immediately from the structure equations for the diffeomorphism pseudo-group using merely linear algebra.

Comparison of Structure Equations

If the action is transitive, then our structure equations are isomorphic to Cartan's. However, this is not true for intransitive pseudo-groups. Which are "right"?

- To find the Cartan structure equations, one first needs to work in an adapted coordinate chart, which requires identification of the invariants on M. Ours can be found in any system of local coordinates.
- Cartan's procedure for identifying the invariant forms is recursive, and not easy to implement. Ours follow immediately from the structure equations for the diffeomorphism pseudo-group using merely linear algebra.
- For finite-dimensional intransitive Lie group actions, Cartan's pseudo-group structure equations do not coincide with the standard Maurer–Cartan equations. Ours do (upon restriction to a source fiber).

Comparison of Structure Equations

If the action is transitive, then our structure equations are isomorphic to Cartan's. However, this is not true for intransitive pseudo-groups. Which are "right"?

- To find the Cartan structure equations, one first needs to work in an adapted coordinate chart, which requires identification of the invariants on M. Ours can be found in any system of local coordinates.
- Cartan's procedure for identifying the invariant forms is recursive, and not easy to implement. Ours follow immediately from the structure equations for the diffeomorphism pseudo-group using merely linear algebra.
- For finite-dimensional intransitive Lie group actions, Cartan's pseudo-group structure equations do not coincide with the standard Maurer–Cartan equations. Ours do (upon restriction to a source fiber).
- Cartan's structure equations for isomorphic pseudo-groups can be non-isomorphic. Ours are always isomorphic.

Action of Pseudo-groups on Submanifolds
a.k.a. Solutions of Differential Equations

$\mathcal{G}$ — Lie pseudo-group acting on p-dimensional submanifolds:

$$N = \{ u = f(x) \} \subset M$$

For example, $\mathcal{G}$ may be the symmetry group of a
system of differential equations

$$\Delta(x, u^{(n)}) = 0$$

and the submanifolds the graphs of solutions $u = f(x)$.

Prolongation

$\mathrm{J}^n = \mathrm{J}^n(M,p)$ — n^{th} order submanifold jet bundle

Local coordinates :

$$z^{(n)} = (x, u^{(n)}) = (\ \ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\ )$$

Prolonged action of $\mathcal{G}^{(n)}$ on submanifolds:

$$(x, u^{(n)}) \quad \longmapsto \quad (X, \widehat{U}^{(n)})$$

Coordinate formulae:

$$\widehat{U}^\alpha_J = F^\alpha_J(x, u^{(n)}, g^{(n)})$$

$\Longrightarrow$ Implicit differentiation.

Differential Invariants

A differential invariant is an invariant function $I : \mathrm{J}^n \to \mathbb{R}$ for the prolonged pseudo-group action

$$I(g^{(n)} \cdot (x, u^{(n)})) = I(x, u^{(n)})$$

$$\Longrightarrow \quad \text{curvature, torsion, } \ldots$$

Invariant differential operators:

$$\mathcal{D}_1, \ldots, \mathcal{D}_p \qquad \Longrightarrow \quad \text{arc length derivative}$$

- If I is a differential invariant, so is $\mathcal{D}_j I$.

$\mathbb{I}(\mathcal{G})$ — the algebra of differential invariants

The Basis Theorem

Theorem. The differential invariant algebra $\mathbb{I}(\mathcal{G})$ is locally generated by a finite number of differential invariants

$$I_1, \ \ldots \ , I_\ell$$

and $p = \dim S$ invariant differential operators

$$\mathcal{D}_1, \ \ldots \ , \mathcal{D}_p$$

meaning that *every* differential invariant can be locally expressed as a function of the generating invariants and their invariant derivatives:

$$\mathcal{D}_J I_\kappa = \mathcal{D}_{j_1} \mathcal{D}_{j_2} \cdots \mathcal{D}_{j_n} I_\kappa.$$

$\Longrightarrow$ Lie groups: *Lie, Ovsiannikov*

$\Longrightarrow$ Lie pseudo-groups: *Tresse, Kumpera, Kruglikov–Lychagin, Muñoz–Muriel–Rodríguez, Pohjanpelto–O*

Key Issues

- Minimal basis of generating invariants: $I_1, \ldots, I_\ell$
- Commutation formulae for

  the invariant differential operators:

$$[\,\mathcal{D}_j, \mathcal{D}_k\,] = \sum_{i=1}^{p} Y_{jk}^{i}\,\mathcal{D}_i$$

$\Longrightarrow$ Non-commutative differential algebra

- Syzygies (functional relations) among

  the differentiated invariants:

$$\Phi(\,\ldots\;\mathcal{D}_J I_\kappa\;\ldots\,) \equiv 0$$

$\Longrightarrow$ Codazzi relations

Computing Differential Invariants

♠ The infinitesimal method:

$$\mathbf{v}(I) = 0 \qquad \text{for every infinitesimal generator} \qquad \mathbf{v} \in \mathfrak{g}$$

$\Longrightarrow$ Requires solving differential equations.

♡ Moving frames.

- Completely algebraic.
- Can be adapted to arbitrary group and pseudo-group actions.
- Describes the complete structure of the differential invariant algebra $\mathbb{I}(\mathcal{G})$ — using only linear algebra & differentiation!
- Prescribes differential invariant signatures for equivalence and symmetry detection.

Moving Frames

In the finite-dimensional Lie group case, a moving frame is defined as an equivariant map

$$\rho^{(n)} : \mathrm{J}^n \longrightarrow G$$

However, we do not have an appropriate abstract object to represent our pseudo-group $\mathcal{G}$.

Consequently, the moving frame will be an equivariant section

$$\rho^{(n)} : \mathrm{J}^n \longrightarrow \mathcal{H}^{(n)}$$

of the pulled-back pseudo-group jet groupoid:

$$\begin{array}{ccc} \mathcal{G}^{(n)} & & \mathcal{H}^{(n)} \\ \downarrow & & \downarrow \\ M & \longleftarrow & \mathrm{J}^n. \end{array}$$

Moving Frames for Pseudo–Groups

Definition. A (right) *moving frame* of *order* n is a right-equivariant section $\rho^{(n)} : V^n \to \mathcal{H}^{(n)}$ defined on an open subset $V^n \subset \mathrm{J}^n$.

$\Longrightarrow$ Groupoid action.

Proposition. A moving frame of order n exists if and only if $\mathcal{G}^{(n)}$ acts *freely* and regularly.

Freeness

For Lie group actions, freeness means no isotropy. For infinite-dimensional pseudo-groups, this definition cannot work, and one must restrict to the transformation jets of order n, using the n^{th} order isotropy subgroup:

$$\mathcal{G}^{(n)}_{z^{(n)}} = \left\{ g^{(n)} \in \mathcal{G}^{(n)}_{z} \;\middle|\; g^{(n)} \cdot z^{(n)} = z^{(n)} \right\}$$

Definition. At a jet $z^{(n)} \in \mathrm{J}^n$, the pseudo-group $\mathcal{G}$ acts

- freely if $\mathcal{G}^{(n)}_{z^{(n)}} = \{ \mathbf{1}^{(n)}_z \}$
- locally freely if
  - $\mathcal{G}^{(n)}_{z^{(n)}}$ is a discrete subgroup of $\mathcal{G}^{(n)}_z$
  - the orbits have dim $= r_n = \dim \mathcal{G}^{(n)}_z$

Persistence of Freeness

Theorem. If $n \geq 1$ and $\mathcal{G}^{(n)}$ acts locally freely at $z^{(n)} \in \mathrm{J}^n$, then it acts locally freely at any $z^{(k)} \in \mathrm{J}^k$ with $\widetilde{\pi}^k_n(z^{(k)}) = z^{(n)}$ for all $k > n$.

The Normalization Algorithm

To construct a moving frame :

I. Compute the prolonged pseudo-group action

$$u_K^\alpha \quad \longmapsto \quad U_K^\alpha = F_K^\alpha(x, u^{(n)}, g^{(n)})$$

by implicit differentiation.

II. Choose a cross-section to the pseudo-group orbits:

$$u_{J_\kappa}^{\alpha_\kappa} = c_\kappa, \qquad \kappa = 1, \ldots, r_n = \text{fiber dim } \mathcal{G}^{(n)}$$

III. Solve the normalization equations

$$U^{\alpha_\kappa}_{J_\kappa} = F^{\alpha_\kappa}_{J_\kappa}(x, u^{(n)}, g^{(n)}) = c_\kappa$$

for the n^{th} order pseudo-group parameters

$$g^{(n)} = \rho^{(n)}(x, u^{(n)})$$

IV. Substitute the moving frame formulas into the unnormalized jet coordinates $u^\alpha_K = F^\alpha_K(x, u^{(n)}, g^{(n)})$. The resulting functions form a complete system of n^{th} order differential invariants

$$I^\alpha_K(x, u^{(n)}) = F^\alpha_K(x, u^{(n)}, \rho^{(n)}(x, u^{(n)}))$$

Invariantization

A moving frame induces an invariantization process, denoted ι, that projects functions to invariants, differential operators to invariant differential operators; differential forms to invariant differential forms, etc.

Geometrically, the invariantization of an object is the unique invariant version that has the same cross-section values.

Algebraically, invariantization amounts to replacing the group parameters in the transformed object by their moving frame formulas.

Invariantization

In particular, invariantization of the jet coordinates leads to a complete system of functionally independent differential invariants:

$$\iota(x^i) = H^i \qquad \iota(u^\alpha_J) = I^\alpha_J$$

- Phantom differential invariants: $I^{\alpha_\kappa}_{J_\kappa} = c_\kappa$
- The non-constant invariants form a functionally independent generating set for the differential invariant algebra $\mathcal{I}(\mathcal{G})$

• Replacement Theorem

$$I(\ \ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\ ) = \iota(\,I(\ \ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\ )\,)$$

$$= I(\ \ldots\ H^i\ \ldots\ I^\alpha_J\ \ldots\ )$$

◊ Differential forms $\Longrightarrow$ invariant differential forms

$$\iota(\,dx^i) = \omega^i \qquad\qquad i = 1, \ldots, p$$

◊ Differential operators $\Longrightarrow$

invariant differential operators

$$\iota\,(\,\mathrm{D}_{x^i}\,) = \mathcal{D}_i \qquad\qquad i = 1, \ldots, p$$

Recurrence Formulae

$\star\star$ Invariantization and differentiation do not commute $\star\star$

The *recurrence formulae* connect the differentiated invariants with their invariantized counterparts:

$$\boxed{\boxed{\mathcal{D}_i I^\alpha_J = I^\alpha_{J,i} + M^\alpha_{J,i}}}$$

$\Longrightarrow$ $M^\alpha_{J,i}$ — correction terms

♡ Once established, the recurrence formulae completely prescribe the structure of the differential invariant algebra $\mathbb{I}(\mathcal{G})$ — thanks to the functional independence of the non-phantom normalized differential invariants.

⋆ ⋆ The recurrence formulae can be explicitly determined using only the infinitesimal generators and linear differential algebra!

Korteweg–deVries Equation

Prolonged Symmetry Group Action:

$$T = e^{3\lambda_4}(t + \lambda_1)$$

$$X = e^{\lambda_4}(\lambda_3 t + x + \lambda_1\lambda_3 + \lambda_2)$$

$$U = e^{-2\lambda_4}(u + \lambda_3)$$

$$U_T = e^{-5\lambda_4}(u_t - \lambda_3 u_x)$$

$$U_X = e^{-3\lambda_4} u_x$$

$$U_{TT} = e^{-8\lambda_4}(u_{tt} - 2\lambda_3 u_{tx} + \lambda_3{}^2 u_{xx})$$

$$U_{TX} = D_X D_T U = e^{-6\lambda_4}(u_{tx} - \lambda_3 u_{xx})$$

$$U_{XX} = e^{-4\lambda_4} u_{xx}$$

$$\vdots$$

Cross Section:

$$T = e^{3\lambda_4}(t + \lambda_1) = 0$$

$$X = e^{\lambda_4}(\lambda_3 t + x + \lambda_1\lambda_3 + \lambda_2) = 0$$

$$U = e^{-2\lambda_4}(u + \lambda_3) = 0$$

$$U_T = e^{-5\lambda_4}(u_t - \lambda_3 u_x) = 1$$

Moving Frame:

$$\lambda_1 = -t, \qquad \lambda_2 = -x, \qquad \lambda_3 = -u, \qquad \lambda_4 = \tfrac{1}{5}\log(u_t + u u_x)$$

Moving Frame:

$$\lambda_1 = -t, \qquad \lambda_2 = -x, \qquad \lambda_3 = -u, \qquad \lambda_4 = \frac{1}{5}\log(u_t + uu_x)$$

Invariantization:

$$\iota(u_K) = U_K \,|_{\lambda_1=-t,\lambda_2=-x,\lambda_3=-u,\lambda_4=\log(u_t+uu_x)/5}$$

Phantom Invariants:

$$H^1 = \iota(t) = 0$$

$$H^2 = \iota(x) = 0$$

$$I_{00} = \iota(u) = 0$$

$$I_{10} = \iota(u_t) = 1$$

Normalized differential invariants:

$$I_{01} = \iota(u_x) = \frac{u_x}{(u_t + uu_x)^{3/5}}$$

$$I_{20} = \iota(u_{tt}) = \frac{u_{tt} + 2uu_{tx} + u^2 u_{xx}}{(u_t + uu_x)^{8/5}}$$

$$I_{11} = \iota(u_{tx}) = \frac{u_{tx} + uu_{xx}}{(u_t + uu_x)^{6/5}}$$

$$I_{02} = \iota(u_{xx}) = \frac{u_{xx}}{(u_t + uu_x)^{4/5}}$$

$$I_{03} = \iota(u_{xxx}) = \frac{u_{xxx}}{u_t + uu_x}$$

$$\vdots$$

Invariantization:

$$\iota(\, F(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \ \ldots\,)\,)$$
$$= F(\iota(t), \iota(x), \iota(u), \iota(u_t), \iota(u_x), \iota(u_{tt}), \iota(u_{tx}), \iota(u_{xx}), \ \ldots\,)$$
$$= F(H^1, H^2, I_{00}, I_{10}, I_{01}, I_{20}, I_{11}, I_{02}, \ \ldots\,)$$
$$= F(0, 0, 0, 1, I_{01}, I_{20}, I_{11}, I_{02}, \ \ldots\,)$$

Replacement Theorem:

$$0 = \iota(u_t + u\, u_x + u_{xxx}) = 1 + I_{03} = \frac{u_t + uu_x + u_{xxx}}{u_t + uu_x}.$$

Invariant horizontal one-forms:

$$\omega^1 = \iota(dt) = (u_t + uu_x)^{3/5}\, dt,$$
$$\omega^2 = \iota(dx) = -u(u_t + uu_x)^{1/5}\, dt + (u_t + uu_x)^{1/5}\, dx.$$

Invariant differential operators:

$$\mathcal{D}_1 = \iota(D_t) = (u_t + u u_x)^{-3/5} D_t + u(u_t + u u_x)^{-3/5} D_x,$$
$$\mathcal{D}_2 = \iota(D_x) = (u_t + u u_x)^{-1/5} D_x.$$

Commutation formula:

$$[\,\mathcal{D}_1, \mathcal{D}_2\,] = I_{01}\,\mathcal{D}_1$$

Recurrence formulae:

$$\mathcal{D}_1 I_{01} = I_{11} - \tfrac{3}{5} I_{01}^2 - \tfrac{3}{5} I_{01} I_{20}, \qquad \mathcal{D}_2 I_{01} = I_{02} - \tfrac{3}{5}\, I_{01}^3 - \tfrac{3}{5}\, I_{01} I_{11},$$

$$\mathcal{D}_1 I_{20} = I_{30} + 2 I_{11} - \tfrac{8}{5} I_{01} I_{20} - \tfrac{8}{5} I_{20}^2, \qquad \mathcal{D}_2 I_{20} = I_{21} + 2 I_{01} I_{11} - \tfrac{8}{5} I_{01}^2 I_{20} - \tfrac{8}{5} I_{11} I_{20},$$

$$\mathcal{D}_1 I_{11} = I_{21} + I_{02} - \tfrac{6}{5} I_{01} I_{11} - \tfrac{6}{5} I_{11} I_{20}, \qquad \mathcal{D}_2 I_{11} = I_{12} + I_{01} I_{02} - \tfrac{6}{5} I_{01}^2 I_{11} - \tfrac{6}{5} I_{11}^2,$$

$$\mathcal{D}_1 I_{02} = I_{12} - \tfrac{4}{5} I_{01} I_{02} - \tfrac{4}{5} I_{02} I_{20}, \qquad \mathcal{D}_2 I_{02} = I_{03} - \tfrac{4}{5} I_{01}^2 I_{02} - \tfrac{4}{5} I_{02} I_{11},$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

Generating differential invariants:

$$I_{01} = \iota(u_x) = \frac{u_x}{(u_t + u u_x)^{3/5}}, \qquad I_{20} = \iota(u_{tt}) = \frac{u_{tt} + 2u u_{tx} + u^2 u_{xx}}{(u_t + u u_x)^{8/5}}.$$

Fundamental syzygy:

$$\mathcal{D}_1^2 I_{01} + \tfrac{3}{5}\, I_{01} \mathcal{D}_1 I_{20} - \mathcal{D}_2 I_{20} + \left(\tfrac{1}{5}\, I_{20} + \tfrac{19}{5}\, I_{01}\right) \mathcal{D}_1 I_{01}$$
$$- \mathcal{D}_2 I_{01} - \tfrac{6}{25}\, I_{01} I_{20}^2 - \tfrac{7}{25}\, I_{01}^2 I_{20} + \tfrac{24}{25}\, I_{01}^3 = 0.$$

Lie–Tresse–Kumpera Example

$$\boxed{X = f(x), \qquad Y = y, \qquad U = \frac{u}{f'(x)}}$$

Horizontal coframe

$$d_H X = f_x \, dx, \qquad d_H Y = dy,$$

Implicit differentiations

$$\mathrm{D}_X = \frac{1}{f_x} \mathrm{D}_x, \qquad \mathrm{D}_Y = \mathrm{D}_y.$$

Prolonged pseudo-group transformations on surfaces $S \subset \mathbb{R}^3$

$$X = f \qquad Y = y \qquad U = \frac{u}{f_x}$$

$$U_X = \frac{u_x}{f_x^2} - \frac{u\, f_{xx}}{f_x^3} \qquad U_Y = \frac{u_y}{f_x}$$

$$U_{XX} = \frac{u_{xx}}{f_x^3} - \frac{3u_x\, f_{xx}}{f_x^4} - \frac{u\, f_{xxx}}{f_x^4} + \frac{3u\, f_{xx}^2}{f_x^5}$$

$$U_{XY} = \frac{u_{xy}}{f_x^2} - \frac{u_y\, f_{xx}}{f_x^3} \qquad U_{YY} = \frac{u_{yy}}{f_x}$$

$\Longrightarrow$ action is free at every order.

Coordinate cross-section

$$X = f = 0, \quad U = \frac{u}{f_x} = 1, \quad U_X = \frac{u_x}{f_x^2} - \frac{u\, f_{xx}}{f_x^3} = 0, \quad U_{XX} = \cdots = 0.$$

Moving frame

$$f = 0, \qquad f_x = u, \qquad f_{xx} = u_x, \qquad f_{xxx} = u_{xx}.$$

Differential invariants

$$U_Y \longmapsto J = \frac{u_y}{u}$$

$$U_{XY} \longmapsto J_1 = \frac{u u_{xy} - u_x u_y}{u^3} \qquad U_{YY} \longmapsto J_2 = \frac{u_{yy}}{u}$$

Invariant horizontal forms

$$d_H X = f_x \, dx \longmapsto u \, dx, \qquad d_H Y = dy \longmapsto dy,$$

Invariant differentiations

$$\mathcal{D}_1 = \frac{1}{u} \mathrm{D}_x \qquad \mathcal{D}_2 = \mathrm{D}_y$$

Higher order differential invariants: $\mathcal{D}_1^m \, \mathcal{D}_2^n \, J$

$$J_{,1} = \mathcal{D}_1 J = \frac{u u_{xy} - u_x u_y}{u^3} = J_1,$$

$$J_{,2} = \mathcal{D}_2 J = \frac{u u_{yy} - u_y^2}{u^2} = J_2 - J^2.$$

Recurrence formulae:

$$\mathcal{D}_1 J = J_1, \qquad \mathcal{D}_2 J = J_2 - J^2,$$

$$\mathcal{D}_1 J_1 = J_3, \qquad \mathcal{D}_2 J_1 = J_4 - 3\, J\, J_1,$$

$$\mathcal{D}_1 J_2 = J_4, \qquad \mathcal{D}_2 J_2 = J_5 - J\, J_2,$$

The Master Recurrence Formula

$$d_H I_J^\alpha = \sum_{i=1}^{p} (\mathcal{D}_i I_J^\alpha)\, \omega^i = \sum_{i=1}^{p} I_{J,i}^\alpha\, \omega^i + \widehat{\psi}_J^\alpha$$

where

$$\widehat{\psi}_J^\alpha = \iota(\widehat{\varphi}_J^\alpha) = \Phi_J^\alpha(\ \ldots\ H^i\ \ldots\ I_J^\alpha\ \ldots\ ;\ \ldots\ \gamma_A^b\ \ldots\ )$$

are the invariantized prolonged vector field coefficients, which are particular linear combinations of

$\gamma_A^b \ = \ \iota(\zeta_A^b)$ — invariantized Maurer–Cartan forms prescribed by the invariantized prolongation map.

- The invariantized Maurer–Cartan forms are subject to the *invariantized determining equations*:

$$\mathcal{L}(H^1, \ldots, H^p, I^1, \ldots, I^q,\ \ldots\ , \gamma_A^b,\ \ldots\ ) = 0$$

$$d_H I^\alpha_J = \sum_{i=1}^{p} I^\alpha_{J,i}\,\omega^i + \widehat{\psi}\,^\alpha_J(\ \ldots\ \gamma^b_A\ \ldots\ )$$

Step 1: Solve the phantom recurrence formulas

$$0 = d_H I^\alpha_J = \sum_{i=1}^{p} I^\alpha_{J,i}\,\omega^i + \widehat{\psi}\,^\alpha_J(\ \ldots\ \gamma^b_A\ \ldots\ )$$

for the invariantized Maurer–Cartan forms:

$$\gamma^b_A = \sum_{i=1}^{p} J^b_{A,i}\,\omega^i \qquad (*)$$

Step 2: Substitute $(*)$ into the non-phantom recurrence formulae to obtain the explicit correction terms.

♢ Only uses linear differential algebra based on the specification of cross-section.

♡ Does not require explicit formulas for the moving frame, the differential invariants, the invariant differential operators, or even the Maurer–Cartan forms!

The Korteweg–deVries Equation (continued)

Recurrence formula:

$$dI_{jk} = I_{j+1,k}\,\omega^1 + I_{j,k+1}\,\omega^2 + \iota(\varphi^{jk})$$

Invariantized Maurer–Cartan forms:

$$\iota(\tau) = \lambda, \qquad \iota(\xi) = \mu, \qquad \iota(\varphi) = \psi = \nu, \qquad \iota(\tau_t) = \psi^t = \lambda_t, \qquad \ldots$$

Invariantized determining equations:

$$\lambda_x = \lambda_u = \mu_u = \nu_t = \nu_x = 0$$

$$\nu = \mu_t \qquad \nu_u = -\,2\,\mu_x = -\,\tfrac{2}{3}\,\lambda_t$$

$$\lambda_{tt} = \lambda_{tx} = \lambda_{xx} = \;\cdots\; = \nu_{uu} = \;\cdots\; = 0$$

Invariantizations of prolonged vector field coefficients:

$$\iota(\tau) = \lambda, \qquad \iota(\xi) = \mu, \qquad \iota(\varphi) = \nu, \qquad \iota(\varphi^t) = -I_{01}\nu - \tfrac{5}{3}\,\lambda_t,$$

$$\iota(\varphi^x) = -I_{01}\lambda_t, \qquad \iota(\varphi^{tt}) = -2I_{11}\nu - \tfrac{8}{3}I_{20}\lambda_t, \qquad \ldots$$

Phantom recurrence formulae:

$$0 = d_H H^1 = \omega^1 + \lambda,$$

$$0 = d_H H^2 = \omega^2 + \mu,$$

$$0 = d_H I_{00} = I_{10}\omega^1 + I_{01}\omega^2 + \psi = \omega^1 + I_{01}\omega^2 + \nu,$$

$$0 = d_H I_{10} = I_{20}\omega^1 + I_{11}\omega^2 + \psi^t = I_{20}\omega^1 + I_{11}\omega^2 - I_{01}\nu - \tfrac{5}{3}\,\lambda_t,$$

$$\Longrightarrow \quad \text{Solve for} \qquad \lambda = -\omega^1, \qquad \mu = -\omega^2, \qquad \nu = -\omega^1 - I_{01}\omega^2,$$

$$\lambda_t = \tfrac{3}{5}\,(I_{20} + I_{01})\omega^1 + \tfrac{3}{5}\,(I_{11} + I_{01}^2)\omega^2.$$

Non-phantom recurrence formulae:

$$d_H I_{01} = I_{11}\omega^1 + I_{02}\omega^2 - I_{01}\lambda_t,$$

$$d_H I_{20} = I_{30}\omega^1 + I_{21}\omega^2 - 2I_{11}\nu - \tfrac{8}{3}I_{20}\lambda_t,$$

$$d_H I_{11} = I_{21}\omega^1 + I_{12}\omega^2 - I_{02}\nu - 2I_{11}\lambda_t,$$

$$d_H I_{02} = I_{12}\omega^1 + I_{03}\omega^2 - \tfrac{4}{3}I_{02}\lambda_t,$$

$$\vdots$$

$$
\begin{aligned}
\mathcal{D}_1 I_{01} &= I_{11} - \tfrac{3}{5} I_{01}^2 - \tfrac{3}{5} I_{01} I_{20}, & \mathcal{D}_2 I_{01} &= I_{02} - \tfrac{3}{5}\, I_{01}^3 - \tfrac{3}{5}\, I_{01} I_{11}, \\
\mathcal{D}_1 I_{20} &= I_{30} + 2I_{11} - \tfrac{8}{5} I_{01} I_{20} - \tfrac{8}{5} I_{20}^2, & \mathcal{D}_2 I_{20} &= I_{21} + 2I_{01} I_{11} - \tfrac{8}{5} I_{01}^2 I_{20} - \tfrac{8}{5} I_{11} I_{20}, \\
\mathcal{D}_1 I_{11} &= I_{21} + I_{02} - \tfrac{6}{5} I_{01} I_{11} - \tfrac{6}{5} I_{11} I_{20}, & \mathcal{D}_2 I_{11} &= I_{12} + I_{01} I_{02} - \tfrac{6}{5} I_{01}^2 I_{11} - \tfrac{6}{5} I_{11}^2, \\
\mathcal{D}_1 I_{02} &= I_{12} - \tfrac{4}{5} I_{01} I_{02} - \tfrac{4}{5} I_{02} I_{20}, & \mathcal{D}_2 I_{02} &= I_{03} - \tfrac{4}{5} I_{01}^2 I_{02} - \tfrac{4}{5} I_{02} I_{11}, \\
&\ \vdots & &\ \vdots
\end{aligned}
$$

Lie–Tresse–Kumpera Example (continued)

$$\boxed{X = f(x), \qquad Y = y, \qquad U = \frac{u}{f'(x)}}$$

Phantom recurrence formulae:

$$0 = dH = \varpi^1 + \gamma, \qquad 0 = dI_{10} = J_1\,\varpi^2 + \vartheta_1 - \gamma_2,$$

$$0 = dI_{00} = J\,\varpi^2 + \vartheta - \gamma_1, \qquad 0 = dI_{20} = J_3\,\varpi^2 + \vartheta_3 - \gamma_3,$$

Solve for pulled-back Maurer–Cartan forms:

$$\gamma = -\,\varpi^1, \qquad \gamma_2 = J_1\,\varpi^2 + \vartheta_1,$$

$$\gamma_1 = J\,\varpi^2 + \vartheta, \qquad \gamma_3 = J_3\,\varpi^2 + \vartheta_3,$$

Recurrence formulae: $\quad dy = \varpi^2$

$$dJ = J_1\,\varpi^1 + (J_2 - J^2)\,\varpi^2 + \vartheta_2 - J\,\vartheta,$$

$$dJ_1 = J_3\,\varpi^1 + (J_4 - 3\,J\,J_1)\,\varpi^2 + \vartheta_4 - J\,\vartheta_1 - J_1\,\vartheta,$$

$$dJ_2 = J_4\,\varpi^1 + (J_5 - J\,J_2)\,\varpi^2 + \vartheta_5 - J_2\,\vartheta,$$

Gröbner Basis Approach

Identify the cross-section variables with the complementary monomials to a certain algebraic module $\mathcal{J}$, which is the pull-back of the symbol module of the pseudo-group under a certain explicit linear map.

$\Longrightarrow$ Compatible term ordering.

$\Longrightarrow$ Algebraic specification of compatible moving frames of all orders $n > n^\star$.

Theorem. Suppose $\mathcal{G}$ acts freely at order $n^\star$. Then a system of generating differential invariants is contained in the non-phantom normalized differential invariants of order $n^\star$ and those differential invariants corresponding to a Gröbner basis for the module $\mathcal{J}^{>n^\star}$.

The Symbol Module

Linearized determining equations

$$\mathcal{L}(z, \zeta^{(n)}) = 0$$

$$t = (t_1, \ldots, t_m), \qquad T = (T_1, \ldots, T_m)$$

$$\mathcal{T} = \left\{ P(t,T) = \sum_{a=1}^{m} P_a(t)\, T_a \right\} \simeq \mathbb{R}[t] \otimes \mathbb{R}^m \subset \mathbb{R}[t, T]$$

$$\mathcal{I} \subset \mathcal{T} \qquad \text{— symbol module}$$

$$s = (s_1, \ldots, s_p), \qquad S = (S_1, \ldots, S_q),$$

$$\widehat{\mathcal{S}} = \left\{ T(s,S) = \sum_{\alpha=1}^{q} T_\alpha(s)\, S_\alpha \right\} \simeq \mathbb{R}[s] \otimes \mathbb{R}^q \subset \mathbb{R}[s, S]$$

Define the linear map

$$s_i = \beta_i(t) = t_i + \sum_{\alpha=1}^{q} u_i^\alpha \, t_{p+\alpha}, \qquad i = 1, \ldots, p,$$

$$S_\alpha = B_\alpha(T) = T_{p+\alpha} - \sum_{i=1}^{p} u_i^\alpha \, T_i, \qquad \alpha = 1, \ldots, q.$$

Prolonged symbol module:

$$\boxed{\mathcal{J} = (\boldsymbol{\beta}^*)^{-1}(\mathcal{I})}$$

$\mathcal{N}$ — leading monomials $s^J S_\alpha$

$\Longrightarrow$ normalized differential invariants I_J^α

$\mathcal{K}$ — complementary monomials $s^K S_\beta$

$\Longrightarrow$ phantom differential invariants I_K^β

The Symbol Module

Vector field:

$$\mathbf{v} = \sum_{a=1}^{m} \zeta^b(z) \frac{\partial}{\partial z^b}$$

Vector field jet:

$$\mathrm{j}_\infty \mathbf{v} \quad \Longleftrightarrow \quad \zeta^{(\infty)} = (\; \ldots \; \zeta^b_A \; \ldots \;)$$

$$\zeta^b_A = \frac{\partial^{\#A} \zeta^b}{\partial z^A} = \frac{\partial^k \zeta^b}{\partial z^{a_1} \cdots \partial z^{a_k}}$$

Determining Equations for $\mathbf{v} \in \mathfrak{g}$

$$\mathcal{L}(z; \; \ldots \; \zeta^b_A \; \ldots \;) = 0 \qquad (*)$$

Duality

$$t = (t_1, \ldots, t_m) \qquad\qquad T = (T_1, \ldots, T_m)$$

Polynomial module:

$$\mathcal{T} = \left\{ P(t,T) = \sum_{a=1}^{m} P_a(t)\, T_a \right\} \simeq \mathbb{R}[t] \otimes \mathbb{R}^m \subset \mathbb{R}[t,T]$$

$$\mathcal{T} \;\simeq\; (\mathrm{J}^\infty TM|_z)^*$$

Dual pairing:

$$\left\langle\ \mathrm{j}_\infty \mathbf{v}\,;\, t^A T_b\ \right\rangle = \zeta_A^b.$$

Each polynomial

$$\tau(z;t,T) = \sum_{b=1}^{m} \sum_{\#A \le n} h_A^b(z)\, t^A T_b \ \in \mathcal{T}$$

induces a linear partial differential equation

$$L(z,\zeta^{(n)}) = \left\langle\ \mathrm{j}_\infty \mathbf{v}\,;\ \tau(z;t,T)\ \right\rangle$$

$$= \sum_{b=1}^{m} \sum_{\#A \le n} h_A^b(z)\, \zeta_A^b = 0$$

The Linear Determining Equations

Annihilator:

$$\mathcal{L} = (\mathrm{J}^\infty \mathfrak{g})^\perp$$

Determining Equations

$$\left\langle \; \mathrm{j}_\infty \mathbf{v} \, ; \, \tau \; \right\rangle = 0 \quad \text{for all} \quad \eta \in \mathcal{L} \qquad \Longleftrightarrow \qquad \mathbf{v} \in \mathfrak{g}$$

Symbol $=$ highest degree terms:

$$\mathbf{\Sigma}[\,L(z,\zeta^{(n)})\,] = \mathbf{\Lambda}[\,\tau(z;t,T)\,] = \sum_{b=1}^{m} \; \sum_{\#A=n} \; h^b_A(z)\, t^A T_b.$$

Symbol submodule:

$$\mathcal{I} = \mathbf{\Lambda}(\mathcal{L})$$

$$\Longrightarrow \quad \text{Formal integrability (involutivity)}$$

Prolonged Duality

Prolonged vector field:

$$\mathbf{v}^{(\infty)} = \sum_{i=1}^{p} \xi^i(x,u)\,\frac{\partial}{\partial x^i} + \sum_{\alpha,J} \widehat{\varphi}\,{}_J^\alpha(x,u^{(k)})\,\frac{\partial}{\partial u_J^\alpha}$$

$$\widetilde{s} = (\widetilde{s}_1,\ldots,\widetilde{s}_p), \quad s = (s_1,\ldots,s_p), \quad S = (S_1,\ldots,S_q)$$

"Prolonged" polynomial module:

$$\widehat{\mathcal{S}} = \left\{ \sigma(s,S,\widetilde{s}) = \sum_{i=1}^{p} c_i \widetilde{s}_i + \sum_{\alpha=1}^{q} \widehat{\sigma}_\alpha(s)\,S_\alpha \right\} \simeq \mathbb{R}^p \oplus (\mathbb{R}[s] \otimes \mathbb{R}^q)$$

$$\widehat{\mathcal{S}} \;\simeq\; T^*\mathrm{J}^\infty\big|_{z^{(\infty)}}$$

Dual pairing:

$$\left\langle \; \mathbf{v}^{(\infty)} ; \, \widetilde{s}_i \; \right\rangle = \xi^i$$

$$\left\langle \; \mathbf{v}^{(\infty)} ; \, S^\alpha \; \right\rangle = Q^\alpha = \varphi^\alpha - \sum_{i=1}^{p} u_i^\alpha \, \xi^i$$

$$\left\langle \; \mathbf{v}^{(\infty)} ; \, s^J S_\alpha \; \right\rangle = \widehat{\varphi}\,{}_J^\alpha = \Phi_J^\alpha(u^{(n)}; \zeta^{(n)})$$

Algebraic Prolongation

Prolongation of vector fields:

$$\begin{aligned} \mathbf{p}:\ \mathrm{J}^\infty\mathfrak{g} &\longmapsto \mathfrak{g}^{(\infty)} \\ \mathrm{j}_\infty\mathbf{v} &\longmapsto \mathbf{v}^{(\infty)} \end{aligned}$$

Dual prolongation map:

$$\mathbf{p}^*:\ \mathcal{S} \longrightarrow \mathcal{T}$$

$$\left\langle\ \mathrm{j}_\infty\mathbf{v}\,;\,\mathbf{p}^*(\sigma)\ \right\rangle = \left\langle\ \mathbf{p}(\mathrm{j}_\infty\mathbf{v})\,;\,\sigma\ \right\rangle = \left\langle\ \mathbf{v}^{(\infty)}\,;\,\sigma\ \right\rangle$$

★★ On the symbol level, $\mathbf{p}^*$ is algebraic ★★

Prolongation Symbols

Define the linear map $\boldsymbol{\beta} : \mathbb{R}^{2m} \longrightarrow \mathbb{R}^m$

$$s_i = \beta_i(t) = t_i + \sum_{\alpha=1}^{q} u_i^\alpha \, t_{p+\alpha}, \qquad i = 1, \ldots, p,$$

$$S_\alpha = B_\alpha(T) = T_{p+\alpha} - \sum_{i=1}^{p} u_i^\alpha \, T_i, \qquad \alpha = 1, \ldots, q.$$

Pull-back map

$$\boldsymbol{\beta}^*[\,\sigma(s_1, \ldots, s_p, S_1, \ldots, S_q)\,]$$
$$= \sigma(\,\beta_1(t), \ldots, \beta_p(t), B_1(T) \ldots, B_q(T)\,)$$

Lemma. The symbols of the prolonged vector field coefficients are

$$\boldsymbol{\Sigma}(\xi^i) = T^i \qquad \boldsymbol{\Sigma}(\widehat{\varphi}^{\,\alpha}) = T^{\alpha+p}$$

$$\boldsymbol{\Sigma}(Q^\alpha) = \boldsymbol{\beta}^*(S_\alpha) = B_\alpha(T)$$

$$\begin{aligned}\boldsymbol{\Sigma}(\widehat{\varphi}^{\,\alpha}_{\,J}) =\boldsymbol{\beta}^*(s^J S_\alpha) &= \boldsymbol{\beta}^*(s_{j_1} \cdots s_{j_n} S^\alpha) \\ &= \beta_{j_1}(t) \,\cdots\, \beta_{j_n}(t)\, B_\alpha(T)\end{aligned}$$

Prolonged annihilator:

$$\mathcal{Z} = (\mathbf{p}^*)^{-1}\mathcal{L} = (\mathfrak{g}^{(\infty)})^{\perp}$$

$$\langle\, \mathbf{v}^{(\infty)} \,;\sigma\,\rangle = 0 \qquad \text{for all} \quad \mathbf{v} \in \mathfrak{g} \quad\Longleftrightarrow\quad \sigma \in \mathcal{Z}$$

Prolonged symbol subbundle:

$$\mathcal{U} \;=\; \mathbf{\Lambda}(\mathcal{Z}) \;\subset\; \mathrm{J}^{\infty}(M,p) \times \mathcal{S}$$

Prolonged symbol module:

$$\boxed{\mathcal{J} = (\boldsymbol{\beta}^*)^{-1}(\mathcal{I})}$$

Warning: : $\qquad \mathcal{U} \;\subseteq\; \mathcal{J}$

But

$$\mathcal{U}^n \;=\; \mathcal{J}^n \qquad \text{when} \qquad n > n^\star$$

$n^\star$ — order of freeness.

Algebraic Recurrence

Polynomial:

$$\sigma(\mathbf{I}^{(k)}; s, S) = \sum_{\alpha, J} h_J^a(\mathbf{I}^{(k)}) \; s^J S_\alpha \in \widehat{\mathcal{S}}$$

Differential invariant:

$$I_\sigma = \sum_{\alpha, J} h_J^a(\mathbf{I}^{(k)}) \, I_J^\alpha$$

Recurrence:

$$\boxed{\boxed{\mathcal{D}_i\, I_\sigma = I_{\mathcal{D}_i\sigma} \equiv I_{s_i\sigma} + R_{i,\sigma}}}$$

$$\sigma \in \widetilde{\mathcal{J}}^n,\ n > n^\star \quad \Longrightarrow \quad \begin{aligned} \operatorname{order} I_\sigma \ &= \ n \\ \operatorname{order} I_{\mathcal{D}_i\sigma} \ &= \ n+1 \\ \operatorname{order} R_{i,\sigma} &\leq n \end{aligned}$$

Algebra $\Longrightarrow$ Invariants

$\mathcal{I}$ — symbol module

- determining equations for $\mathfrak{g}$

$\mathcal{M} \simeq \mathcal{T} \,/\, \mathcal{I}$ — complementary monomials $t^A T_b$

- pseudo-group parameters
- Maurer–Cartan forms

$\mathcal{N}$ — leading monomials $s^J S_\alpha$

- normalized differential invariants I_J^α

$\mathcal{K} = \mathcal{S} \,/\, \mathcal{N}$ — complementary monomials $s^K S_\beta$

- cross-section coordinates $u^\beta_K = c^\beta_K$
- phantom differential invariants I^β_K

$$\mathcal{J} = (\boldsymbol{\beta}^*)^{-1}(\mathcal{I})$$

Freeness: $\boldsymbol{\beta}^* \colon \ \mathcal{K} \ \stackrel{\sim}{\longrightarrow} \ \mathcal{M}$

Generating Differential Invariants

Theorem. The differential invariant algebra is generated by differential invariants that are in one-to-one correspondence with the Gröbner basis elements of the prolonged symbol module plus, possibly, a finite number of differential invariants of order $\leq n^{\star}$.

Syzygies

Theorem. Every differential syzygy among the generating differential invariants is either a syzygy among those of order $\leq n^\star$, or arises from an algebraic syzygy among the Gröbner basis polynomials in $\widetilde{\mathcal{J}}$.