# Reduction of Poisson-Nijenhuis Lie algebroids

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joint work with Juan Carlos Marrero and Edith Padrón**

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# Outline

# Overview

Aim of our study:

- Given a Poisson-Nijenhuis Lie algebroid $(A, P, N)$ we want ro reduce it to a symplectic-Nijenhuis Lie algebroid $(A, \widetilde{\Omega}, \widetilde{N})$ with $\widetilde{\Omega}$ symplectic and also $\widetilde{N}$ nondegenerate.

Motivation:

- Reduction in a more general and flexible framework than manifolds (Magri-Morosi).
- Concrete physical examples may be reducible just if seen as Lie algebroids (e.g. Toda Lattice).

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# Lie algebroids

**Definition (Pradines)**

A Lie algebroid is a vector bundle $\tau_A\colon A \longrightarrow M$ endowed with

(i) an *anchor*, i.e., a vector bundle morphism $\rho_A\colon A \longrightarrow TM$

(ii) a Lie algebra bracket on $\Gamma(A)$, $[\,,\,]_A$, such that

$$[X, fY]_A = f\,[X, Y]_A + \rho_A(X)(f)Y,$$

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It follows that

$$\rho_A\left([X, Y]_A\right) = [\rho_A(X), \rho_A(Y)]_M\,.$$

# Examples of Lie algebroids

- The tangent bundle $A = TM$ of a smooth manifold $M$, with $\rho_A = id_{TM}$ and the usual Lie bracket of vector fields.
- An involutive distribution $A = D \subset TM$ with the inclusion map as anchor and the usual Lie bracket of vector fields.
- A Lie algebra $A = \mathfrak{g}$ considered as a vector bundle over the singleton $M = \{pt\}$, with trivial anchor $\rho_A = 0$.

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# Cartan calculus

Associated to a given Lie algebroid $(A, [\,,\,]_A, \rho_A)$ there is a *Lie algebroid differential* $\mathrm{d}^A\colon \Gamma(\wedge^\bullet A^*) \to \Gamma(\wedge^{\bullet+1} A^*)$ defined by

$$(\mathrm{d}^A\omega)(X_0,\dots,X_k) = \sum_{i=0}^{k}(-1)^i \rho_A(X_i)\left(\omega(X_0,\dots,\hat{X}_i,\dots,X_k)\right) + \sum_{0\le i<j\le k}(-1)^{i+j}\omega([X_i,X_j]_A, X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_k),$$

for $\omega \in \Gamma(\wedge^k A^*)$, $X_0,\dots,X_k \in \Gamma(A)$.

- For $X \in \Gamma(A)$, $\mathcal{L}^A_X = i_X d^A + d^A i_X$.

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# Properties of the Lie algebroid differential

- $\mathrm{d}^A$ is a graded derivation of degree 1, i.e.,

$$\mathrm{d}^A(\theta \wedge \omega) = \mathrm{d}^A\theta \wedge \omega + (-1)^{deg(\theta)}\theta \wedge \mathrm{d}^A\omega,$$

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Actually, given a vector bundle $\tau_A \colon A \to M$ and a $\mathbb{R}$-linear map $\mathrm{d}^A \colon \Gamma(\wedge^\bullet A^*) \to \Gamma(\wedge^{\bullet+1} A^*)$ with these properties, the anchor map and the Lie bracket can be recovered.

# Schouten-Gerstenhaber algebra

The Lie algebra bracket on $\Gamma(A)$ can be extended to the exterior algebra $(\Gamma(\wedge^\bullet A), \wedge)$. For $X \in \Gamma(A)$ and $P \in \Gamma(\wedge^p A)$,

$$[X, P]_A(\alpha_1, \dots, \alpha_p) = \rho_A(X)(P(\alpha_1, \dots, \alpha_p)) - \sum_{i=1}^{p} P(\alpha_1, \dots, \mathcal{L}^A_X \alpha_i, \dots \alpha_p),$$

If $P \in \Gamma(\wedge^p A)$, $Q \in \Gamma(\wedge^q A)$ and $R \in \Gamma(\wedge^r A)$, then $[P, Q]_A \in \Gamma(\wedge^{p+q-1} A)$ and

- $[P, Q]_A = -(-1)^{(p-1)(q-1)} [Q, P]_A$
- $[P, Q \wedge R]_A = [P, Q]_A \wedge R + (-1)^{(p-1)q} Q \wedge [P, R]_A$
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# Poisson structures on Lie algebroids

Let $A$ be a Lie algebroid and $P$ a section of the vector bundle $\wedge^2 A \to M$. We denote by $P^\sharp$ the usual bundle map

$$P^\sharp\colon A^* \longrightarrow A\colon \alpha \longmapsto P^\sharp(\alpha) = i_\alpha P.$$

**Definition**

A Poisson structure on $A$ is a section $P \in \Gamma(\wedge^2 A)$, such that

$$[P, P]_A = 0.$$

In this case, the bracket

$$[\alpha, \beta]_P = \mathcal{L}^A_{P^\sharp\alpha}\beta - \mathcal{L}^A_{P^\sharp\beta}\alpha - \mathrm{d}^A\left(P(\alpha,\beta)\right), \quad \alpha, \beta \in \Gamma(A^*),$$

is a Lie bracket and $A^*_P = (A^*, [\,,\,]_P, \rho_A \circ P^\sharp)$ is a Lie algebroid.

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# Nijenhuis operators

Let $(A, [\,,\,], \rho_A)$ be a Lie algebroid and $N : A \to A$ a bundle map. The torsion of $N$ is defined by

$$\mathcal{T}_N(X, Y) := [NX, NY]_A - N[X, Y]_N, \quad X, Y \in \Gamma(A),$$

where
$[X, Y]_N := [NX, Y]_A + [X, NY]_A - N[X, Y]_A, \quad X, Y \in \Gamma(A).$

When $\mathcal{T}_N = 0$, $N$ is called a Nijenhuis operator,
$A_N = (A, [\,,\,]_N, \rho_N = \rho_A \circ N)$ is a new Lie algebroid and

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# Poisson-Nijenhuis Lie algebroids

On a Lie algebroid $A$ with a Poisson structure $P \in \Gamma(\wedge^2 A)$, we say that a bundle map $N : A \longrightarrow A$ is compatible with $P$ if

(i) $NP^\sharp = P^\sharp N^*$

(ii) $\mathcal{C}(P, N)(\alpha, \beta) = [\alpha, \beta]_{NP} - [\alpha, \beta]_P^{N^*} = 0$

**Definition (Grabowski-Urbanski)**

A Poisson-Nijenhuis Lie algebroid $(A, P, N)$ is a Lie algebroid $A$ equipped with a Poisson structure $P$ and a Nijenhuis operator $N : A \longrightarrow A$ compatible with $P$.

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# $1^{st}$ step: Reduction by restriction

$(A, [\,,\,]_A\,, \rho_A, P)$ Poisson Lie algebroid.

$$\downarrow$$

$$D(x) := \rho_A(P^\sharp(A_x^*)) \subset T_xM \quad \text{for } x \in M$$

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$$\left[\rho_A(P^\sharp\alpha), \rho_A(P^\sharp\beta)\right] = \rho_A(P^\sharp\,[\alpha,\beta]_P),$$

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Let

- $L \subset M$ be a leaf of the foliation $D$
- $A_L := P^\sharp(A^*)_{|L} \subset A$

Hypothesis: $P^\sharp : A^* \longrightarrow A$ has constant rank on each leaf $L$.

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$A_L \longrightarrow L$ is a Lie subalgebroid of $A \longrightarrow M$

with $\left([\,,\,]_{A_L}, \rho_{A_L}\right)$ given by

- $\left[P^\sharp\alpha_{|L}, P^\sharp\beta_{|L}\right]_{A_L} = P^\sharp\left[\alpha,\beta\right]_{P|L} \in \Gamma(A_L)$
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# $1^{st}$ step: Reduction by restriction

For any $X_L \in \Gamma(A_L)$ there exists $\alpha \in \Gamma(A^*)$ such that $I \circ X_L = P^\sharp \alpha \circ \iota$. So, we define a section $\Omega_L \colon L \longrightarrow \wedge^2 A_L^*$ by

$$\Omega_L(X_L, Y_L) = P(\alpha, \beta) \circ \iota, \qquad \text{for any } X_L, Y_L \in \Gamma(A_L)$$

Now, consider a Nijenhuis operator $N : A \longrightarrow A$ compatible with $P$. Then, we may induce $N_L : A_L \longrightarrow A_L$ such that

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**Theorem (A)**

*Let $(A, P, N)$ be a Poisson-Nijenhuis Lie algebroid such that the Poisson structure has constant rank in the leaves of the foliation $D = \rho_A(P^\sharp(A^*))$. Then, we have a symplectic-Nijenhuis Lie algebroid $(A_L, \Omega_L, N_L)$ on each leaf $L$ of $D$.*

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# Reduction by epimorphisms of Lie algebroids

Let $\tau_A \colon A \to M$ and $\tau_{\widetilde{A}} \colon \widetilde{A} \to \widetilde{M}$ be Lie algebroids and let

$$\begin{tikzcd} A \arrow[r, "\Pi"] \arrow[d, "\tau_A"'] & \widetilde{A} \arrow[d, "\tau_{\widetilde{A}}"] \\ M \arrow[r, "\pi"] & \widetilde{M} \end{tikzcd}$$

be an epimorphism of Lie algebroids from $A$ to $\widetilde{A}$, i.e.,

- $\pi \colon M \to \widetilde{M}$ is a submersion,
- for each $x \in M$, $\Pi_x \colon A_x \to \widetilde{A}_{\pi(x)}$ is an epimorphism of vector spaces
  and
- $\mathrm{d}^A(\Pi^*\widetilde{\alpha}) = \Pi^*(\mathrm{d}^{\widetilde{A}}\widetilde{\alpha}) \qquad \text{for all } \widetilde{\alpha} \in \Gamma(\wedge^k \widetilde{A}^*).$

# Projectability

- Let $\Gamma_p(A)$ be the set of the $\Pi$-*projectable* sections $X : M \to A$ of $A$, i.e., such that there exists $\widetilde{X} \in \Gamma(\widetilde{A})$ such that $\Pi \circ X = \widetilde{X} \circ \pi$.
- A Poisson structure $P$ on $A$ is said to be $\Pi$-*projectable* if for each $\widetilde{\alpha} \in \Gamma(\widetilde{A}^*)$ we have $P^\sharp \Pi^* \widetilde{\alpha} \in \Gamma_p(A)$. In that case, we can construct the 2-section $\widetilde{P} \in \Gamma(\wedge^2 \widetilde{A})$ of $\widetilde{A}$ characterized by

$$(\widetilde{P}^\sharp \widetilde{\alpha}) \circ \pi = \Pi(P^\sharp(\Pi^* \widetilde{\alpha})), \qquad \text{for any } \widetilde{\alpha} \in \Gamma(\widetilde{A}^*).$$

- Assume that $N \colon A \to A$ is a Nijenhuis structure on $A$. We will say that $N$ is $\Pi$-*projectable* if

$$N(\Gamma_p(A)) \subseteq \Gamma_p(A) \quad \text{and} \quad N(\Gamma(Ker\Pi)) \subseteq \Gamma(Ker\Pi).$$

# Reduction by epimorphisms of Lie algebroids

If $N$ is a $\Pi$-projectable Nijenhuis operator on $A$, then we can construct a new operator $\widetilde{N}\colon \widetilde{A} \longrightarrow \widetilde{A}$ as follows.

$$(\widetilde{N}\widetilde{X}) \circ \pi = \Pi(NX) \qquad \text{for any } \widetilde{X} \in \Gamma(\widetilde{A}),$$

where $X \in \Gamma_p(A)$ is a projectable section such that $\Pi X = \widetilde{X} \circ \pi$.

**Theorem**

*Let $(\Pi, \pi) : A \longrightarrow \widetilde{A}$ be a Lie algebroid epimorphism. Assume that $(P, N)$ is a Poisson-Nijenhuis structure on $A$ such that $P$ and $N$ are $\Pi$-projectable. Then, $(\widetilde{P}, \widetilde{N})$ is a Poisson-Nijenhuis structure on $\widetilde{A}$.*

# Complete and vertical lifts

- $(A, [\,,\,]_A, \rho_A)$ a Lie algebroid
- $X \in \Gamma(A)$

**The vertical lift of $X$: $X^v \in \mathfrak{X}(A)$**

(i) $X^v(f \circ \tau_A) = 0, \quad f \in \mathcal{C}^\infty(M),$

(ii) $X^v(\hat{\alpha}) = \alpha(X) \circ \tau_A, \quad \alpha \in \Gamma(A^*).$

Here, if $\alpha \in \Gamma(A^*)$ then $\hat{\alpha} \colon A \to \mathbb{R}$ is defined by

$$\hat{\alpha}(a) = \alpha(\tau_A(a))(a), \quad \text{for all } a \in A.$$

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**The complete lift of $X$: $X^c \in \mathfrak{X}(A)$**

(i) $X^c(f \circ \tau_A) = \rho_A(X)(f) \circ \tau_A, \quad f \in C^\infty(M),$

(ii) $X^c(\hat{\alpha}) = \widehat{\mathcal{L}^A_X \alpha}, \quad \alpha \in \Gamma(A^*).$

Here, if $\alpha \in \Gamma(A^*)$ then $\hat{\alpha} \colon A \to \mathbb{R}$ is defined by

$$\hat{\alpha}(a) = \alpha(\tau_A(a))(a), \quad \text{for all } a \in A.$$

# Reduction by lifts of sections of a Lie subalgebroid

Let $\tau_A \colon A \to M$ a vector bundle and $(A, [\,,\,]_A, \rho_A)$ a Lie algebroid. Consider a Lie subalgebroid $\tau_B \colon B \to M$ of $A$.

## Key Fact

The distributions $\rho_A(B)$ and $\mathcal{F}$ defined by

$$\mathcal{F}_a = \{X^c(a) + Y^v(a) \mid X, Y \in \Gamma(B)\} \subseteq T_aA, \quad \text{for all } a \in A$$

are generalized foliations.

Now assume that $\rho_A(B)$ and $\mathcal{F}$ are regular foliations.

# Reduction by lifts of sections of a Lie subalgebroid

We define $\tau_{\widetilde{A}}\colon \widetilde{A} = A/\mathcal{F} \to \widetilde{M} = M/\rho_A(B)$ such that the following diagram is commutative

$$\begin{tikzcd} A \arrow[r, "\Pi"] \arrow[d, "\tau_A"] & \widetilde{A} = A/\mathcal{F} \arrow[d, "\tau_{\widetilde{A}}"] \\ M \arrow[r, "\pi"] & \widetilde{M} = M/\rho_A(B) \end{tikzcd}$$

**Proposition**

*In the above conditions we can define a Lie algebroid structure on*

$$\tau_{\widetilde{A}}\colon \widetilde{A} = A/\mathcal{F} \to \widetilde{M} = M/\rho_A(B)$$

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# Reduction by lifts of sections of a Lie subalgebroid

The structure of Lie algebroid over $\widetilde{A} = A/\mathcal{F}$ is characterized by

$$[\widetilde{X}, \widetilde{Y}]_{\widetilde{A}} \circ \pi = \Pi\,[X, Y]_A\,, \qquad \rho_{\widetilde{A}}(\widetilde{X})(\widetilde{f}) = \rho_A(X)(\widetilde{f} \circ \pi),$$

# The Riesz index

Let $(A, P, N)$ a Poisson-Nijenhuis Lie algebroid. For any $x \in M$ consider the map $N_x \colon A_x \to A_x$. Recall that there exists a smallest integer $k > 0$ such that the sequences

$$\operatorname{Im} N_x \supseteq \operatorname{Im} N_x^2 \supseteq \dots$$

and

$$\ker N_x \subseteq \ker N_x^2 \subseteq \dots$$

both stabilize at rank $k$. That is,

$$\operatorname{Im} N_x^k = \operatorname{Im} N_x^{k+1} = \dots, \qquad \text{while } \operatorname{Im} N_x^{k-1} \neq \operatorname{Im} N_x^k,$$

and

$$\ker N_x^k = \ker N_x^{k+1} = \dots, \qquad \text{while } \ker N_x^{k-1} \neq \ker N_x^k.$$

The integer $k$ is called the Riesz index of $N$ at $x$.

# The Reduced nondegenerate PN Lie algebroid

## Theorem (B)

*Let $(A, [\,,\,]_A, \rho_A, P, N)$ be a Poisson-Nijenhuis Lie algebroid such that*

1) *$N$ has constant Riesz index $k$.*
2) *$\rho_A(B)$ and $\mathcal{F}$ are regular foliations for $B = \ker N^k$.*
3) *For all $x \in M$, $a_x - a'_x \in \ker(N_x^k)$ if $a_x$ and $a'_x$ belong to the same leaf of the foliation $\mathcal{F}$.*
4) *$P$ is nondegenerate.*

*Then, we can induce a Poisson-Nijenhuis Lie algebroid structure $([\,,\,]_{\widetilde{A}}, \rho_{\widetilde{A}}, \widetilde{P}, \widetilde{N})$ on $\widetilde{A}$ with $\widetilde{P}$ and $\widetilde{N}$ nondegenerate.*

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# Main Theorem

**(Thm A + Thm B)**

Let $(A, [\,,\,]_A, \rho_A, P, N)$ be a Poisson-Nijenhuis Lie algebroid s.t.

1) $P$ has constant rank in each leaf $L$ of $D = \rho_A(P^\sharp(A^*))$.
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3) $\rho_{A_L}(\ker N_L^k)$ and $(\mathcal{F}_L)_a = \{X^c(a) + Y^v(a) \mid X, Y \in \Gamma(\ker N_L^k)\}$ for $a \in A_L$ are regular foliations.
4) For all $x \in M$, $a_x - a'_x \in \ker(N_L^k)$ if $a_x$ and $a'_x$ are in the same leaf of $\mathcal{F}_L$.

Then, we obtain a symplectic-Nijenhuis Lie algebroid structure $([\,,\,]_{\widetilde{A_L}}, \rho_{\widetilde{A_L}}, \widetilde{\Omega_L}, \widetilde{N_L})$ on $\widetilde{A_L} = A_L/\mathcal{F}_L \to \widetilde{L} = L/\mathcal{F}$ with $\widetilde{N_L}$ nondegenerate.

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