Branes in Poisson sigma models

Fernando Falceto

Depto. Física Teórica

Universidad de Zaragoza

Branes in Poisson sigma models

Fernando Falceto

Depto. Física Teórica

Universidad de Zaragoza

Poisson sigma model.

- N. Ikeda: Non linear gauge theories, applications to 2d gravity.

Poisson sigma model.

- N. Ikeda: Non linear gauge theories, applications to 2d gravity.
- P. Schaller and T. Strobl: put the model into the right geometric setup and coined its name, applications to gravity, BF theory, Yang-Mills...

Poisson sigma model.

- N. Ikeda: Non linear gauge theories, applications to 2d gravity.
- P. Schaller and T. Strobl: put the model into the right geometric setup and coined its name, applications to gravity, BF theory, Yang-Mills...
- A. Cattaneo and G. Felder: relation to Kontsevich's deformation quantization, to symplectic groupoids, AKSZ formalism...

Poisson sigma model.

- N. Ikeda: Non linear gauge theories, applications to 2d gravity.
- P. Schaller and T. Strobl: put the model into the right geometric setup and coined its name, applications to gravity, BF theory, Yang-Mills...
- A. Cattaneo and G. Felder: relation to Kontsevich's deformation quantization, to symplectic groupoids, AKSZ formalism...
- Cattaneo and Felder: coisotropic branes.

Poisson sigma model.

- N. Ikeda: Non linear gauge theories, applications to 2d gravity.
- P. Schaller and T. Strobl: put the model into the right geometric setup and coined its name, applications to gravity, BF theory, Yang-Mills...
- A. Cattaneo and G. Felder: relation to Kontsevich's deformation quantization, to symplectic groupoids, AKSZ formalism...
- Cattaneo and Felder: coisotropic branes.
- More general branes?

Plan

- Poisson sigma model
- Examples
- Boundary conditions
- Quantization
- Final Remarks

Plan

- Poisson sigma model
- Examples
- Boundary conditions
- Quantization
- Final Remarks

With: Iván Calvo
David García-Álvarez
Krzysztof Gawędzki

Poisson sigma model

- The target: $(M, \{\ ,\ \})$

- $\{\ ,\ \}$ a Poisson bracket in $C^\infty(M)$.

Poisson sigma model

- The target: $(M, \{\ ,\ \})$

- $\{\ ,\ \}$ a Poisson bracket in $C^\infty(M)$.

- In coordinates $X = (X^1, \ldots, X^n)$ for M

$$\Pi^{ij}(X) = \{X^i, X^j\}(X)$$

Poisson sigma model

- The target: $(M, \{\ ,\ \})$ $\Pi^{ij}(X) = \{X^i, X^j\}(X)$
- The fields:

- Σ two dimensional space-time (worldsheet).

- The fields are given by the bundle map

$$(X, \eta) : T\Sigma \to T^*M$$

i.e. $X : \Sigma \to M$, $\eta \in \Omega^1(\Sigma, X^*T^*M)$

with coordinates $\sigma = (\sigma^1, \sigma^2)$ for Σ

$$\eta = \eta_{\kappa i}(\sigma)\mathrm{d}\sigma^\kappa \mathrm{d}X^i = \eta_i \mathrm{d}X^i$$

Poisson sigma model

- The target: $(M, \{\ ,\ \})$ $\qquad \Pi^{ij}(X) = \{X^i, X^j\}(X)$
- The fields: $X : \Sigma \to M$, $\qquad \eta = \eta_{\kappa i}(\sigma)\mathrm{d}\sigma^\kappa \mathrm{d}X^i = \eta_i \mathrm{d}X^i$
- The action:

$$S(X,\eta) = \int_\Sigma \eta_i \wedge \mathrm{d}X^i + \frac{1}{2}\Pi^{ij}(X)\eta_i \wedge \eta_j$$

Poisson sigma model

- The target: $(M, \{\ ,\ \})$ $\Pi^{ij}(X) = \{X^i, X^j\}(X)$
- The fields: $X : \Sigma \to M$, $\eta = \eta_{\kappa i}(\sigma)\mathrm{d}\sigma^\kappa \mathrm{d}X^i = \eta_i \mathrm{d}X^i$
- The action:

$$S(X,\eta) = \int_\Sigma \eta_i \wedge \mathrm{d}X^i + \frac{1}{2}\Pi^{ij}(X)\eta_i \wedge \eta_j$$

- The equations of motion:

$$\mathrm{d}X - \Pi^\sharp(X)\eta = 0 \qquad (\Pi^\sharp \eta)^j = \Pi^{ij}\eta_i$$

$$\mathrm{d}\eta_i + \frac{1}{2}\partial_i \Pi^{jk}(X)\eta_j \wedge \eta_k = 0$$

i. e. $(X,\eta) : T\Sigma \to T^*M$ Lie algebroid homomorphism.

The Gauge symmetry:

Under the transformations

$$\delta_\beta X = \Pi^\sharp(X)\beta \qquad \beta = \beta_i(\sigma)\mathrm{d}X^i \in \Gamma(X^*T^*M)$$

$$\delta_\beta \eta_i = \mathrm{d}\beta_i + \partial_i \Pi^{jk}\eta_j\beta_k,$$

The Gauge symmetry:

Under the transformations

$$\delta_\beta X = \Pi^\sharp(X)\beta \qquad \beta = \beta_i(\sigma)\mathrm{d}X^i \in \Gamma(X^*T^*M)$$
$$\delta_\beta \eta_i = \mathrm{d}\beta_i + \partial_i \Pi^{jk}\eta_j\beta_k,$$

$$\delta_\beta S = \int_\Sigma \mathrm{d}(\mathrm{d}X^i\beta_i).$$

The Gauge symmetry:

Under the transformations

$$\delta_\beta X = \Pi^\sharp(X)\beta \qquad \beta = \beta_i(\sigma)\mathrm{d}X^i \in \Gamma(X^*T^*M)$$
$$\delta_\beta \eta_i = \mathrm{d}\beta_i + \partial_i \Pi^{jk}\eta_j\beta_k,$$

$$\delta_\beta S = \int_\Sigma \mathrm{d}(\mathrm{d}X^i\beta_i).$$

$$[\delta_\beta, \delta_{\beta'}]X^i = \delta_{[\beta,\beta']}X^i$$
$$[\delta_\beta, \delta_{\beta'}]\eta_i = \delta_{[\beta,\beta']}\eta_i - \beta_k\beta'_l\partial_i\partial_j\Pi^{kl}(dX^j - \Pi^{sj}\eta_s)$$

With $[\beta, \beta']_k = \beta_i\beta'_j\partial_k\Pi^{ij}(X)$

Examples

- R^2 gravity in two dimensions

- dim(M)=3

- $(\eta_1, \eta_2, \eta_3) \equiv (e_1, e_2, \omega)$ (zweibein and connection)

Examples

- R^2 gravity in two dimensions

- dim(M)=3

- $(\eta_1, \eta_2, \eta_3) \equiv (e_1, e_2, \omega)$ (zweibein and connection)

$$\Pi^{23}(X) = X^1, \qquad \Pi^{31}(X) = X^2$$
$$\Pi^{12}(X) = -(X^3)^2 + \Lambda$$

Examples

- R^2 gravity in two dimensions

- dim(M)=3

- $(\eta_1, \eta_2, \eta_3) \equiv (e_1, e_2, \omega)$ (zweibein and connection)

$$\Pi^{23}(X) = X^1, \qquad \Pi^{31}(X) = X^2$$
$$\Pi^{12}(X) = -(X^3)^2 + \Lambda$$

- Then the Poisson sigma model in $(M, \{.,.\})$ upon integration of X-fields, leads to 2-d R^2 gravity.

$$S_{R^2} = \int_\Sigma \left(\frac{1}{4}R^2 + \Lambda\right)\sqrt{g}\,\mathrm{d}^2\sigma$$

Examples

- BF theories.

- $(\mathfrak{g}, [.,.])$ any Lie algebra, $M = \mathfrak{g}^*$.

Examples

- BF theories.

- $(\mathfrak{g}, [.,.])$ any Lie algebra, $M = \mathfrak{g}^*$.

- $A, B \in \mathfrak{g}$ viewed as linear functions on $\mathfrak{g}^*$.

- Then $\{A, B\} = [A, B]$ defines the Kostant-Kirillov-Souriau Poisson bracket in $\mathfrak{g}^*$

Examples

- BF theories.

- $(\mathfrak{g}, [.,.])$ any Lie algebra, $M = \mathfrak{g}^*$.

- $A, B \in \mathfrak{g}$ viewed as linear functions on $\mathfrak{g}^*$.

- Then $\{A, B\} = [A, B]$ defines the Kostant-Kirillov-Souriau Poisson bracket in $\mathfrak{g}^*$

The action in this case is equivalent to

$$S_{BF} = \int_\Sigma X^i F_i$$

with $X(\sigma) \in \mathfrak{g}^*$ and $F = d\eta + [\eta, \eta] \in \Omega^2(M) \otimes \mathfrak{g}$

Examples

- Poisson-Lie sigma models.
  For any Poisson-Lie group $(G, \{.,.\})$ we can define its Poisson-sigma model.
  It has several interesting properties:

Examples

- Poisson-Lie sigma models.
  For any Poisson-Lie group $(G, \{.,.\})$ we can define its Poisson-sigma model.
  It has several interesting properties:
  - Generalizes the BF-theory.

Examples

- Poisson-Lie sigma models.
  For any Poisson-Lie group $(G, \{.,.\})$ we can define its Poisson-sigma model.
  It has several interesting properties:
  - Generalizes the BF-theory.
  - The gauge group is the dual Poisson-Lie group $(G^*, \{.,.\}^*)$, acting by dressing transformation.

Examples

- Poisson-Lie sigma models.
  For any Poisson-Lie group $(G, \{.,.\})$ we can define its Poisson-sigma model.
  It has several interesting properties:
  - Generalizes the BF-theory.
  - The gauge group is the dual Poisson-Lie group $(G^*, \{.,.\}^*)$, acting by dressing transformation.
  - Duality in the Hamiltonian formulation.

Examples

- Poisson-Lie sigma models.
  For any Poisson-Lie group $(G, \{.,.\})$ we can define its Poisson-sigma model.
  It has several interesting properties:
  - Generalizes the BF-theory.
  - The gauge group is the dual Poisson-Lie group $(G^*, \{.,.\}^*)$, acting by dressing transformation.
  - Duality in the Hamiltonian formulation.
  - With group $(G^*, \{.,.\}^*)$ it is equivalent to G/G WZW model.

Boundary conditions

Take a worldsheet with boundary. $\iota : \partial\Sigma \hookrightarrow \Sigma$

Put a brane $N \subset M$. i.e. $X : \Sigma \to M$ s.t. $\iota^* X : \partial\Sigma \to N$

Boundary conditions

Take a worldsheet with boundary. $\iota : \partial\Sigma \hookrightarrow \Sigma$

Put a brane $N \subset M$. i.e. $X : \Sigma \to M$ s.t. $\iota^* X : \partial\Sigma \to N$

$$\delta_X S = -\int_{\partial\Sigma} \delta X^i \eta_i + \int_{\Sigma} \delta X^i (d\eta_i + \frac{1}{2}\partial_i \Pi^{jk} \eta_j \wedge \eta_k)$$

We must have $(\iota^* X, \iota^* \eta) : T\partial\Sigma \to TN^\circ$ $\quad TN^\circ \subset T_N^* M$

$$TN_p^\circ = \{\xi \in T_p^* M | \xi(v) = 0 \; \forall v \in T_p N\}$$

Boundary conditions

Take a worldsheet with boundary. $\iota : \partial\Sigma \hookrightarrow \Sigma$

Put a brane $N \subset M$. i.e. $X : \Sigma \to M$ s.t. $\iota^* X : \partial\Sigma \to N$

$$\delta_X S = -\int_{\partial\Sigma} \delta X^i \eta_i + \int_{\Sigma} \delta X^i (d\eta_i + \frac{1}{2}\partial_i \Pi^{jk} \eta_j \wedge \eta_k)$$

We must have $(\iota^* X, \iota^* \eta) : T\partial\Sigma \to TN^\circ \quad TN^\circ \subset T^*_N M$

From the equations of motion, ($\mathrm{d}X - \Pi^\sharp \eta = 0$)

$$\iota^* \mathrm{d}X = \Pi^\sharp(\iota^* X)\iota^* \eta \quad \Rightarrow$$

$$\Rightarrow \Pi^\sharp(\iota^* X)\iota^* \eta \in \Omega^1(\partial\Sigma, \iota^* X^* TN)$$

Boundary conditions

The boundary conditions (B.C.) for a brane $N \subset M$ are:

$(\iota^* X, \iota^* \eta) : T\partial\Sigma \to AN \qquad AN := TN^\circ \cap \Pi^{\sharp^{-1}}(TN) \subset T^*_N M$

Boundary conditions

The boundary conditions (B.C.) for a brane $N \subset M$ are:

$$(\iota^* X, \iota^* \eta) : T\partial\Sigma \to AN \qquad AN := TN^\circ \cap \Pi^{\sharp^{-1}}(TN) \subset T_N^* M$$

We assume AN of constant rank.

(regular brane or pre-Poisson)

Boundary conditions

The boundary conditions (B.C.) for a brane $N \subset M$ are:

$$(\iota^* X, \iota^* \eta) : T\partial\Sigma \to AN \qquad AN := TN^\circ \cap \Pi^{\sharp^{-1}}(TN) \subset T_N^* M$$

We assume AN of constant rank.

(regular brane or pre-Poisson)

Then:

- AN is a Lie subalgebroid of T^*M.

Boundary conditions

The boundary conditions (B.C.) for a brane $N \subset M$ are:

$$(\iota^* X, \iota^* \eta) : T\partial\Sigma \to AN \qquad AN := TN^\circ \cap {\Pi^\sharp}^{-1}(TN) \subset T^*_N M$$

We assume AN of constant rank.

(regular brane or pre-Poisson)

Then:

- AN is a Lie subalgebroid of T^*M.
- The gauge transformation δ_β subject to the same B. C.

$$\iota^* \beta \in \Gamma(\iota^* X^* AN)$$

preserves B.C. and is a symmetry.

Pre-Poisson branes (examples)

$AN := TN^\circ \cap \Pi^{\sharp^{-1}}(TN)$

- Free Boundary Conditions:
  $N = M$ then $AN = 0$.

Pre-Poisson branes (examples)

$AN := TN^\circ \cap \Pi^{\sharp^{-1}}(TN)$

- Free Boundary Conditions:
  $N = M$ then $AN = 0$.
- Coisotropic brane: Dirac's first class constraints.
  $\Pi^\sharp(TN^\circ) \subset TN \Leftrightarrow AN = TN^\circ$.

Pre-Poisson branes (examples)

$$AN := TN^\circ \cap \Pi^{\sharp^{-1}}(TN)$$

- Free Boundary Conditions:
  $N = M$ then $AN = 0$.
- Coisotropic brane: Dirac's first class constraints.
  $\Pi^\sharp(TN^\circ) \subset TN \Leftrightarrow AN = TN^\circ$.
- Constant rank Poisson-Dirac: $AN \subset \mathrm{Ker}(\Pi^\sharp)$.
  Reduction of symplectic groupoids (M. Crainic, R. L. Fernandes)

Pre-Poisson branes (examples)

$AN := TN^\circ \cap \Pi^{\sharp^{-1}}(TN)$

- Free Boundary Conditions:
  $N = M$ then $AN = 0$.
- Coisotropic brane: Dirac's first class constraints.
  $\Pi^\sharp(TN^\circ) \subset TN \Leftrightarrow AN = TN^\circ$.
- Constant rank Poisson-Dirac: $AN \subset \mathrm{Ker}(\Pi^\sharp)$.
  Reduction of symplectic groupoids (M. Crainic, R. L. Fernandes)
- Cosymplectic brane: Dirac's second class constraints.
  $AN = 0$.

Pre-Poisson branes (examples)

$AN := TN^\circ \cap \Pi^{\sharp^{-1}}(TN)$

- Free Boundary Conditions:
  $N = M$ then $AN = 0$.
- Coisotropic brane: Dirac's first class constraints.
  $\Pi^\sharp(TN^\circ) \subset TN \Leftrightarrow AN = TN^\circ$.
- Constant rank Poisson-Dirac: $AN \subset \mathrm{Ker}(\Pi^\sharp)$.
  Reduction of symplectic groupoids (M. Crainic, R. L. Fernandes)
- Cosymplectic brane: Dirac's second class constraints.
  $AN = 0$.

Theorem: Every pre-Poisson submanifold can be embedded coisotropically in a cosymplectic submanifold.

Quantization

- Batalin-Vilkoviski quantization
  Poisson sigma model has a gauge symmetry of the open type (its algebra closes only on-shell).

Quantization

- The fields
  - X^i, η_i the original fields.
  - β_i, γ^i the ghost and antighosts.
  - λ^i the auxiliary field (Lagrange multiplier)

Lorenz gauge $\mathrm{d} * \eta_i = 0$ $*$ Hodge star operator

Quantization

- The fields
  - X^i, η_i the original fields.
  - β_i, γ^i the ghost and antighosts.
  - λ^i the auxiliary field (Lagrange multiplier)

Lorenz gauge $\mathrm{d} * \eta_i = 0$ $\quad$ $*$ Hodge star operator

- The gauge fixed action

$$
\begin{aligned}
S_{\mathrm{gf}} \;=\; & \int_\Sigma \eta_i \wedge \mathrm{d}X^i + \frac{1}{2}\Pi^{ij}(X)\eta_i \wedge \eta_j - \lambda^i \mathrm{d} * \eta_i - \\
& - * \,\mathrm{d}\gamma^i \wedge (\mathrm{d}\beta_i + \partial_i \Pi^{kl}(X)\eta_k \beta_l) - \\
& - \frac{1}{4} * \mathrm{d}\gamma^i \wedge *\mathrm{d}\gamma^j \partial_i \partial_j \Pi^{kl}(X)\beta_k \beta_l
\end{aligned}
$$

Quantization

- The fields
  - X^i, η_i the original fields.
  - β_i, γ^i the ghost and antighosts.
  - λ^i the auxiliary field (Lagrange multiplier)

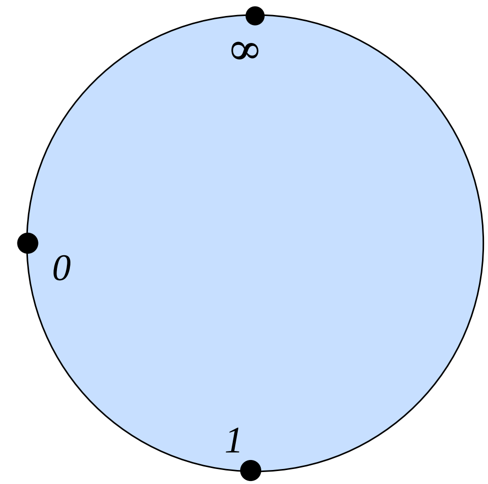
  Lorenz gauge $\mathrm{d} * \eta_i = 0$ $\qquad$ $*$ Hodge star operator
- The gauge fixed action

$$
\begin{aligned}
S_{\mathrm{gf}} \quad = \quad & \int_\Sigma \eta_i \wedge \mathrm{d}X^i + \frac{1}{2}\Pi^{ij}(X)\eta_i \wedge \eta_j - \lambda^i \mathrm{d} * \eta_i - \\
& - * \mathrm{d}\gamma^i \wedge (\mathrm{d}\beta_i + \partial_i \Pi^{kl}(X)\eta_k\beta_l) - \\
& - \frac{1}{4} * \mathrm{d}\gamma^i \wedge *\mathrm{d}\gamma^j \partial_i\partial_j \Pi^{kl}(X)\beta_k\beta_l
\end{aligned}
$$

- Perturbative expansion.

Free B. C. $N = M$

$\Sigma = D$ the unit disk. Pick three points at the boundary $0, 1, \infty$.



Free B. C. $N = M$

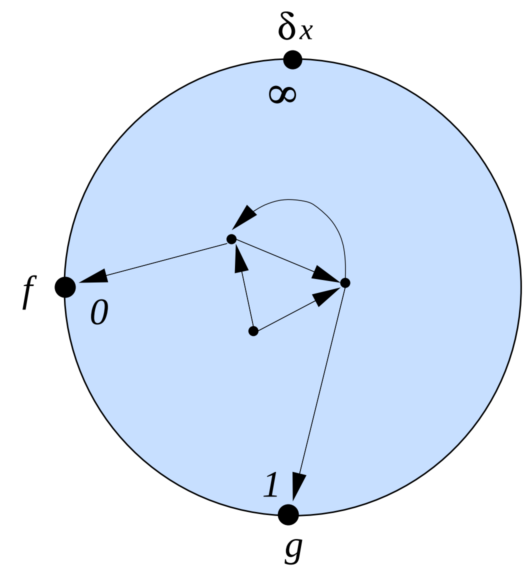
$\Sigma = D$ the unit disk. Pick three points at the boundary $0, 1, \infty$.

Then the perturbative expansion of

$$\int \mathrm{e}^{\frac{i}{\hbar} S_{\mathrm{gf}}} f(X(0)) g(X(1)) \delta(X(\infty) - x)$$

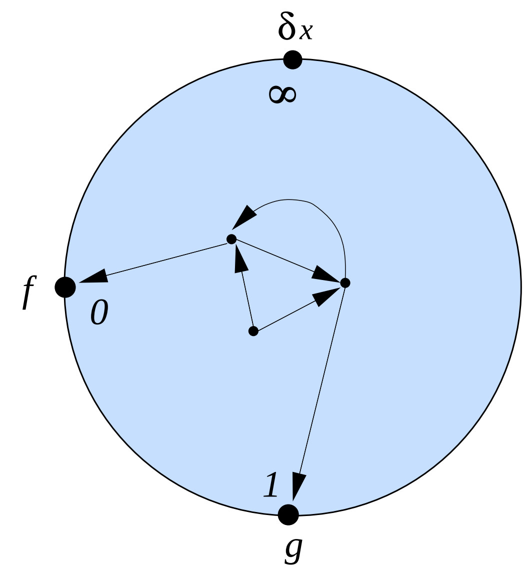
gives the Kontsevich's star product.

$$f \star g(x) = f(x)g(x) + i\frac{\hbar}{2}\{f, g\}(x) + \dots$$



Free B. C. $N = M$

$\Sigma = D$ the unit disk. Pick three points at the boundary $0, 1, \infty$.

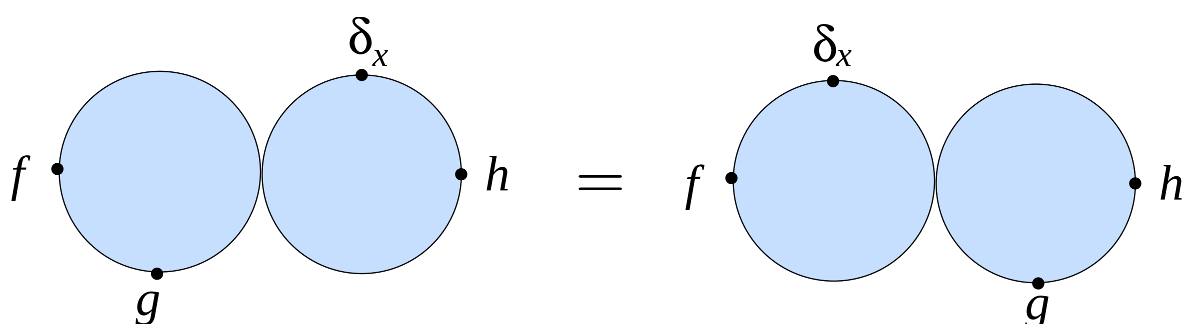


Then the perturbative expansion of

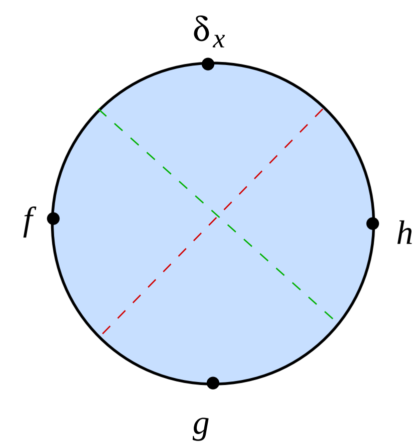
$$\int \mathrm{e}^{\frac{i}{\hbar} S_{\mathrm{gf}}} f(X(0)) g(X(1)) \delta(X(\infty) - x)$$

gives the Kontsevich's star product.

$$f \star g(x) = f(x)g(x) + i\frac{\hbar}{2}\{f, g\}(x) + \ldots$$



$$(f \star g) \star h = f \star (g \star h)$$

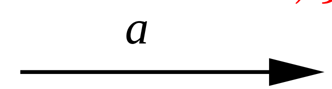


Coisotropic brane.

Adapted coordinates $X = (X^a, X^\mu)$, $N = \{(X^a, X^\mu = 0)\}$

Coisotropic brane.

Adapted coordinates $X = (X^a, X^\mu)$, $N = \{(X^a, X^\mu = 0)\}$

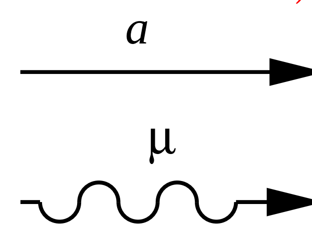


B. C. $\begin{cases} \iota^* X^a = free, \quad \iota^* \eta_a = 0 \\ \end{cases}$

Coisotropic brane.

Adapted coordinates $X = (X^a, X^\mu)$, $N = \{(X^a, X^\mu = 0)\}$

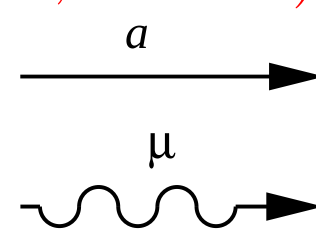
B. C. $\begin{cases} \iota^* X^a = free, \quad \iota^* \eta_a = 0 \\ \iota^* X^\mu = 0, \quad \iota^* \eta_\mu = free \end{cases}$



Coisotropic brane.

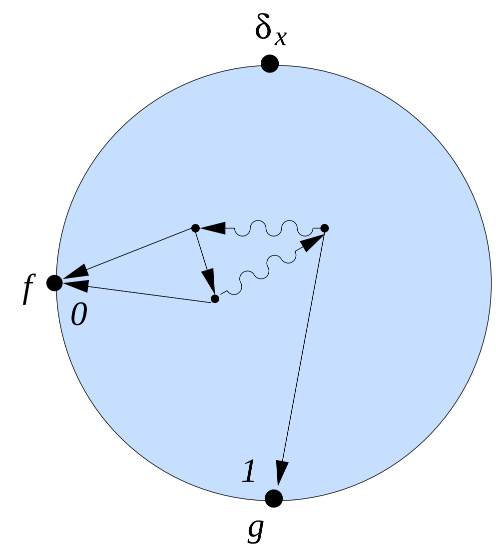
Adapted coordinates $X = (X^a, X^\mu)$, $N = \{(X^a, X^\mu = 0)\}$

B. C. $\begin{cases} \iota^* X^a = free, & \iota^* \eta_a = 0 \\ \iota^* X^\mu = 0, & \iota^* \eta_\mu = free \end{cases}$



The perturbative expansion of

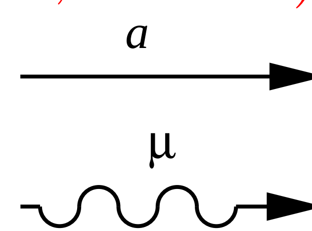
$$\int_{\iota^* X \in N} \mathrm{e}^{\frac{i}{\hbar} S_{\mathrm{gf}}} f(X(0)) g(X(1)) \delta(X(\infty) - x)$$



Coisotropic brane.

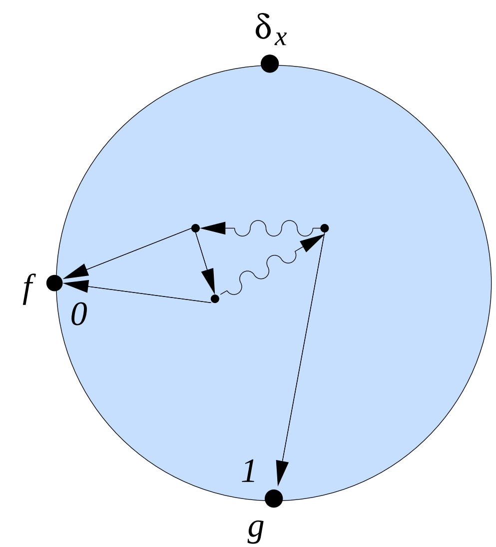
Adapted coordinates $X = (X^a, X^\mu)$, $\quad N = \{(X^a, X^\mu = 0)\}$

B. C. $\begin{cases} \iota^* X^a = free, & \iota^* \eta_a = 0 \\ \iota^* X^\mu = 0, & \iota^* \eta_\mu = free \end{cases}$



The perturbative expansion of

$$\int_{\iota^* X \in N} \mathrm{e}^{\frac{i}{\hbar} S_{\mathrm{gf}}} f(X(0)) g(X(1)) \delta(X(\infty) - x)$$



defines an associative $\star$ product in

$\mathcal{A}^\hbar_N \equiv \{ f \in C^\infty(N)[[\hbar]] \ \text{ s.t. } \delta^\hbar(N) f = 0 \}$,

if anomaly vanishes. $\quad \delta^\hbar(N) X^i = \Pi^{\mu i}(X) \beta_\mu + ...$

Coisotropic branes. Bimodules.

N_0, N_1 coisotropic branes with vanishing anomaly.

$\delta^\hbar(N_0, N_1)X^i = \Pi^{i\nu}\beta_\nu + \dots$ $\qquad$ (dX^ν) a basis of $TN_0^\circ \cap TN_1^\circ$.

$\mathcal{A}^\hbar_{N_0 N_1} \equiv \{f \in C^\infty(N_0 \cap N_1)[[\hbar]] \text{ s.t. } \delta^\hbar(N_0, N_1)f = 0\}$

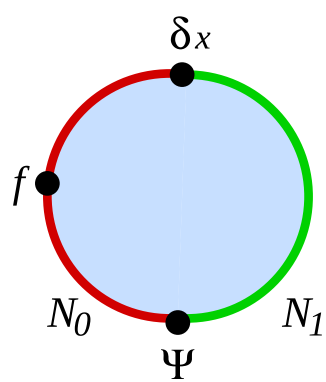
Coisotropic branes. Bimodules.

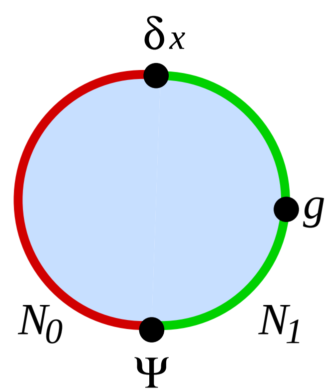
N_0, N_1 coisotropic branes with vanishing anomaly.

$\delta^\hbar(N_0,N_1)X^i = \Pi^{i\nu}\beta_\nu + \ldots$ $\quad$ (dX^ν) a basis of $TN_0^\circ \cap TN_1^\circ$.

$$\mathcal{A}^\hbar_{N_0N_1} \equiv \{f \in C^\infty(N_0 \cap N_1)[[\hbar]] \text{ s.t. } \delta^\hbar(N_0,N_1)f = 0\}$$

We define the action of $\mathcal{A}^\hbar_{N_0}$ and $\mathcal{A}^\hbar_{N_1}$ on $\mathcal{A}^\hbar_{N_0N_1}$

$f \blacktriangleright \Psi(x) =$ 

$\Psi \blacktriangleleft g(x) =$ 

Which makes $\mathcal{A}^\hbar_{N_0N_1}$ a $\mathcal{A}^\hbar_{N_0}$-bimodule-$\mathcal{A}^\hbar_{N_1}$.

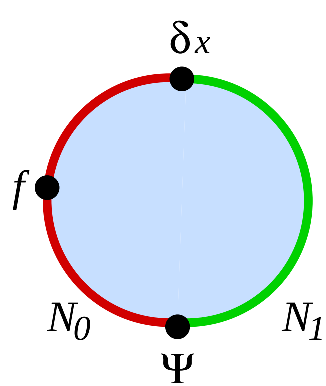
Coisotropic branes. Bimodules.

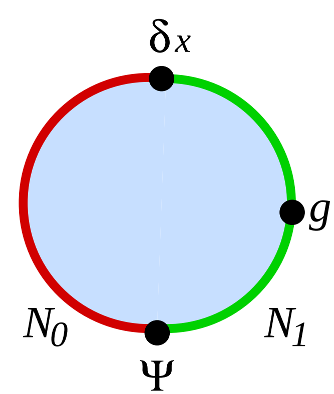
N_0, N_1 coisotropic branes with vanishing anomaly.

$\delta^\hbar(N_0, N_1)X^i = \Pi^{i\nu}\beta_\nu + \ldots$ $\quad$ (dX^ν) a basis of $TN_0^\circ \cap TN_1^\circ$.

$$\mathcal{A}^\hbar_{N_0 N_1} \equiv \{f \in C^\infty(N_0 \cap N_1)[[\hbar]] \text{ s.t. } \delta^\hbar(N_0, N_1) f = 0\}$$

We define the action of $\mathcal{A}^\hbar_{N_0}$ and $\mathcal{A}^\hbar_{N_1}$ on $\mathcal{A}^\hbar_{N_0 N_1}$

$f \blacktriangleright \Psi(x) =$ 

$\Psi \blacktriangleleft g(x) =$ 

Which makes $\mathcal{A}^\hbar_{N_0 N_1}$ a $\mathcal{A}^\hbar_{N_0}$-bimodule-$\mathcal{A}^\hbar_{N_1}$.

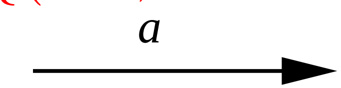
Quantization of Poisson maps.

Cosymplectic brane

Adapted coordinates $X = (X^a, X^A)$, $N = \{(X^a, X^A = 0)\}$

Cosymplectic brane

Adapted coordinates $X = (X^a, X^A)$, $\quad$ $N = \{(X^a, X^A = 0)\}$

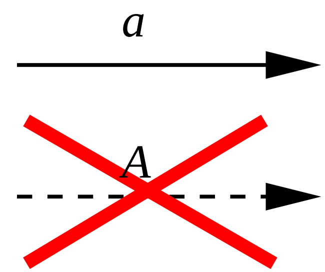


B. C. $\begin{cases} \iota^* X^a = free, \quad \iota^* \eta_a = 0 \\ \\ \end{cases}$

Cosymplectic brane

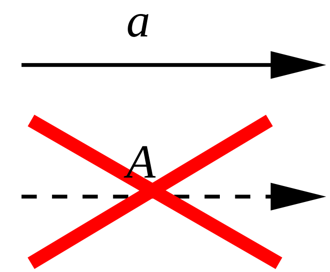
Adapted coordinates $X = (X^a, X^A)$, $N = \{(X^a, X^A = 0)\}$

B. C. $\begin{cases} \iota^* X^a = free, & \iota^* \eta_a = 0 \\ \iota^* X^A = 0, & \iota^* \eta_A = 0 \end{cases}$



Cosymplectic brane

Adapted coordinates $X = (X^a, X^A)$, $N = \{(X^a, X^A = 0)\}$



B. C. $\begin{cases} \iota^* X^a = free, & \iota^* \eta_a = 0 \\ \iota^* X^A = 0, & \iota^* \eta_A = 0 \end{cases}$

1st idea.

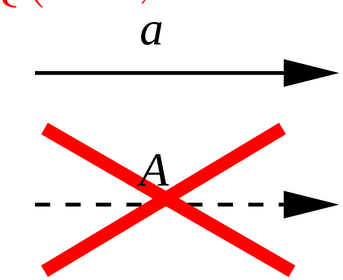
$$S_{\mathrm{gf}} = \int_\Sigma \eta_i \wedge \mathrm{d}X^i + \frac{1}{2}\Pi^{ij}(X)\eta_i \wedge \eta_j - \lambda^i \mathrm{d} * \eta_i - \\ - * \mathrm{d}\gamma^i \wedge (\mathrm{d}\beta_i + \partial_i \Pi^{kl}(X)\eta_k \beta_l) - \\ - \frac{1}{4} * \mathrm{d}\gamma^i \wedge * \mathrm{d}\gamma^j \partial_i \partial_j \Pi^{kl}(X)\beta_k \beta_l$$

$\det \Pi^{AB}(x) \neq 0$, perform the Gaussian integration in η_A

Cosymplectic brane

Adapted coordinates $X=(X^a,X^A)$, $N=\{(X^a,X^A=0)\}$

B. C. $\begin{cases} \iota^* X^a = free, & \iota^*\eta_a = 0 \\ \iota^* X^A = 0, & \iota^*\eta_A = 0 \end{cases}$



1[st] idea.

$$
\begin{aligned}
S_{\mathrm{gf}} &= \int_\Sigma \eta_i \wedge \mathrm{d}X^i + \frac{1}{2}\Pi^{ij}(X)\eta_i \wedge \eta_j - \lambda^i \mathrm{d} * \eta_i - \\
&- * \,\mathrm{d}\gamma^i \wedge (\mathrm{d}\beta_i + \partial_i \Pi^{kl}(X)\eta_k \beta_l) - \\
&- \frac{1}{4} * \mathrm{d}\gamma^i \wedge * \mathrm{d}\gamma^j \partial_i \partial_j \Pi^{kl}(X)\beta_k \beta_l
\end{aligned}
$$

$\det \Pi^{AB}(x) \neq 0$, perform the Gaussian integration in η_A

Effective action has a well defined pert. expansion.

It is hard to compute and relate to star product.

Cosymplectic brane

2$^{\mathrm{nd}}$ idea

Change gauge fixing: $\mathrm{d} * \eta_a = 0, X^A = 0$

for cosymplectic branes only: $\delta_\beta X^A = \Pi^{AB}\beta_B + \Pi^{Aa}\beta_a$.

λ^a and λ_A new Lagrange multipliers.

Cosymplectic brane

2^{nd} idea

Change gauge fixing: $\mathrm{d} * \eta_a = 0, X^A = 0$

for cosymplectic branes only: $\delta_\beta X^A = \Pi^{AB}\beta_B + \Pi^{Aa}\beta_a$.

λ^a and λ_A new Lagrange multipliers.

$$
\begin{aligned}
S_{\mathrm{gf}} \;\; = \;\; & \int_\Sigma \eta_i \wedge \mathrm{d}X^i + \frac{1}{2}\Pi^{ij}(X)\eta_i \wedge \eta_j - \lambda^a \mathrm{d} * \eta_a - \lambda_A X^A - \\
& - * d\gamma^a \wedge (\mathrm{d}\beta_a + \partial_a \Pi^{ij}(X)\eta_i\beta_j) - \gamma_A \Pi^{Ai}(X)\beta_i - \\
& - \frac{1}{4} * \mathrm{d}\gamma^a \wedge *\mathrm{d}\gamma^b \partial_a\partial_b \Pi^{ij}(X)\beta_i\beta_j
\end{aligned}
$$

Cosymplectic brane

2^{nd} idea

Change gauge fixing: $\mathrm{d} * \eta_a = 0, X^A = 0$

for cosymplectic branes only: $\delta_\beta X^A = \Pi^{AB}\beta_B + \Pi^{Aa}\beta_a$.

λ^a and λ_A new Lagrange multipliers.

$$\begin{aligned} S_{\mathrm{gf}} &= \int_\Sigma \eta_i \wedge \mathrm{d}X^i + \frac{1}{2}\Pi^{ij}(X)\eta_i \wedge \eta_j - \lambda^a \mathrm{d} * \eta_a - \lambda_A X^A - \\ &\quad - * d\gamma^a \wedge (\mathrm{d}\beta_a + \partial_a \Pi^{ij}(X)\eta_i\beta_j) - \gamma_A \Pi^{Ai}(X)\beta_i - \\ &\quad - \frac{1}{4} * \mathrm{d}\gamma^a \wedge *\mathrm{d}\gamma^b \partial_a\partial_b \Pi^{ij}(X)\beta_i\beta_j \end{aligned}$$

Integrating in λ_A, γ_A (linear) and in η_A (quadratic). One obtains the effective action

Cosymplectic brane

$$
\begin{aligned}
S^{\text{eff}}_{\text{gf}} \quad = \quad & \int_\Sigma \eta_a \wedge \mathrm{d}X^a + \frac{1}{2}\Pi^{ab}_{\mathcal{D}}(X)\eta_a \wedge \eta_b - \lambda^a \mathrm{d} * \eta_a \\
& - * \,\mathrm{d}\gamma^a \wedge (\mathrm{d}\beta_a + \partial_a \Pi^{cd}_{\mathcal{D}}(X)\eta_c\beta_d) - \\
& - \frac{1}{4} * \mathrm{d}\gamma^a \wedge *\mathrm{d}\gamma^b \partial_a \partial_b \Pi^{cd}_{\mathcal{D}}(X)\beta_c\beta_d
\end{aligned}
$$

$\Pi^{ab}_{\mathcal{D}} = \Pi^{ab} - \Pi^{aA}\Pi_{AB}\Pi^{Bb}$, the Dirac bracket in N.

Cosymplectic brane

$$\begin{aligned} S^{\mathrm{eff}}_{\mathrm{gf}} &= \int_\Sigma \eta_a \wedge \mathrm{d}X^a + \frac{1}{2}\Pi_{\mathcal{D}}^{ab}(X)\eta_a \wedge \eta_b - \lambda^a \mathrm{d} * \eta_a \\ &- * \,\mathrm{d}\gamma^a \wedge (\mathrm{d}\beta_a + \partial_a \Pi_{\mathcal{D}}^{cd}(X)\eta_c\beta_d) - \\ &- \frac{1}{4} * \mathrm{d}\gamma^a \wedge *\mathrm{d}\gamma^b \partial_a\partial_b \Pi_{\mathcal{D}}^{cd}(X)\beta_c\beta_d \end{aligned}$$

$\Pi_{\mathcal{D}}^{ab} = \Pi^{ab} - \Pi^{aA}\Pi_{AB}\Pi^{Bb}$, the Dirac bracket in N.

$$\int_{\iota^* X \in N} \mathrm{e}^{\frac{i}{\hbar} S_{\mathrm{gf}}} f(X(0)) g(X(1)) \delta(X(\infty) - x) = f \star_{\mathcal{D}} g$$

defines an associative product in $\mathcal{A}^{\hbar}_N = C^\infty(N)[[h]]$.

Pre-Poisson brane

Adapted coordinates $X = (X^a, X^\mu, X^A) = (X^p, X^A)$.

B. C. $$\begin{cases} \iota^* X^a = free, & \iota^* \eta_a = 0. & \text{brane} \\ \iota^* X^\mu = 0, & \iota^* \eta_\mu = free. & 1^{\text{st}} \text{ class} \\ \iota^* X^A = 0, & \iota^* \eta_A = 0. & 2^{\text{nd}} \text{ class} \end{cases}$$

Gauge fixing: $\mathrm{d} * \eta_p = 0, X^A = 0$.

Pre-Poisson brane

Adapted coordinates $X = (X^a, X^\mu, X^A) = (X^p, X^A)$.

B. C. $\begin{cases} \iota^* X^a = free, & \iota^* \eta_a = 0. & \text{brane} \\ \iota^* X^\mu = 0, & \iota^* \eta_\mu = free. & 1^{\text{st}} \text{ class} \\ \iota^* X^A = 0, & \iota^* \eta_A = 0. & 2^{\text{nd}} \text{ class} \end{cases}$

Gauge fixing: $\mathrm{d} * \eta_p = 0, X^A = 0$.

$$\begin{aligned} S_{\mathrm{gf}}^{\mathrm{eff}} &= \int_\Sigma \eta_p \wedge \mathrm{d}X^p + \frac{1}{2}\Pi_{\mathcal{D}}^{pq}(X)\eta_p \wedge \eta_q - \lambda^p \mathrm{d} * \eta_p \\ &\quad - * \,\mathrm{d}\gamma^p \wedge (\mathrm{d}\beta_p + \partial_p \Pi_{\mathcal{D}}^{qr}(X)\eta_q\beta_r) - \\ &\quad -\frac{1}{4} * \mathrm{d}\gamma^p \wedge *\mathrm{d}\gamma^q \partial_p\partial_q \Pi_{\mathcal{D}}^{rs}(X)\beta_r\beta_s \end{aligned}$$

Pre-Poisson brane

Adapted coordinates $X = (X^a, X^\mu, X^A) = (X^p, X^A)$.

B. C. $\begin{cases} \iota^* X^a = free, & \iota^* \eta_a = 0. & \text{brane} \\ \iota^* X^\mu = 0, & \iota^* \eta_\mu = free. & 1^{\text{st}} \text{ class} \\ \iota^* X^A = 0, & \iota^* \eta_A = 0. & 2^{\text{nd}} \text{ class} \end{cases}$

Gauge fixing: $\mathrm{d} * \eta_p = 0, X^A = 0$.

$$\begin{aligned} S_{\text{gf}}^{\text{eff}} &= \int_\Sigma \eta_p \wedge \mathrm{d}X^p + \frac{1}{2}\Pi_{\mathcal{D}}^{pq}(X)\eta_p \wedge \eta_q - \lambda^p \mathrm{d} * \eta_p \\ &\quad - * \mathrm{d}\gamma^p \wedge (\mathrm{d}\beta_p + \partial_p \Pi_{\mathcal{D}}^{qr}(X)\eta_q \beta_r) - \\ &\quad - \frac{1}{4} * \mathrm{d}\gamma^p \wedge *\mathrm{d}\gamma^q \partial_p \partial_q \Pi_{\mathcal{D}}^{rs}(X)\beta_r \beta_s \end{aligned}$$

i. e. it defines an effective Poisson sigma model

in $M' = \{(X^a, X^\mu, X^A = 0)\}$ with brane $N' = \{(X^a, X^\mu = 0)\}$.

Final remarks

- Unlike the coisotropic branes, the cosymplectic ones do always lead to associative star products.

Final remarks

- Unlike the coisotropic branes, the cosymplectic ones do always lead to associative star products.
- Cosymplectic branes redefine the perturbative expansion.

Final remarks

- Unlike the coisotropic branes, the cosymplectic ones do always lead to associative star products.
- Cosymplectic branes redefine the perturbative expansion.
- Deformation of branes, from cosymplectic to coisotropic.

Final remarks

- Unlike the coisotropic branes, the cosymplectic ones do always lead to associative star products.
- Cosymplectic branes redefine the perturbative expansion.
- Deformation of branes, from cosymplectic to coisotropic.
- Quantization with several branes?

Final remarks

- Unlike the coisotropic branes, the cosymplectic ones do always lead to associative star products.
- Cosymplectic branes redefine the perturbative expansion.
- Deformation of branes, from cosymplectic to coisotropic.
- Quantization with several branes?

Final remarks

- Unlike the coisotropic branes, the cosymplectic ones do always lead to associative star products.
- Cosymplectic branes redefine the perturbative expansion.
- Deformation of branes, from cosymplectic to coisotropic.
- Quantization with several branes?