# Stability and normal forms (around leaves in Poisson geometry)

Marius Crainic

Mathematics Department Utrecht University

Benasque, September 2009

Marius Crainic Stability and normal forms (around leaves in Poisson geometry)

### Based on joint work with R.L. Fernandes.

Main motivation: Conn's linearization theorem.

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II. Foliations III. Poisson geometry: intro IV. Poisson geometry: the local data and the local model V. Normal forms and stability in Poisson geometry The orbit data Local form Stability

### Actions of Lie groups

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# Lie group actions: the orbit data

Let *G* be a Lie group acting on a manifold *M* and let O be the orbit through a point *x*. Consider:

- the isotropy group at x, call it  $G_x$ .
- the normal space at x to the orbit, call it V.
- the induced (linear) action of  $G_X$  on V (isotropy action).

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### Local form: the slice theorem

### Local model out of the orbit data: the G-manifold

 $G \times_{G_x} V = (G \times V)/G_x.$ 

#### Theorem (the slice theorem)

If G is compact, then  $\mathcal{O}$  admits a G-invariant neighborhood  $\mathcal{U}$ , G-diffeomorphic to  $G \times_{G_x} V$ .

Read: ( $\mathcal{O}$  and  $G_x$  compact) instead of (G-compact).

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# Stability: Hirsch-Stowe stability

#### Theorem (stability theorem, Hirsch 1980, Stowe 1983)

If  $\mathcal{O}$  is compact and  $H^1(G_x; V) = 0$ , then  $\mathcal{O}$  is stable, i.e. any other action of G on M which is close enough to the original one has at least one orbit diffeomorphich to  $\mathcal{O}$ .

Remark: one can also say how many leaves are diff. to  $\mathcal{O}$  (at least as many as  $H^0(G_x, V)$ ).

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### **II. FOLIATIONS**

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II. Foliations II. Foliations III. Poisson geometry: the local data and the local model V. Normal forms and stability in Poisson geometry The leaf data Local form Stability Examples

# Foliations: the leaf data

Let  $\mathcal{F}$  be a foliation on a manifold M and let L be a leaf of the foliation through a point x. Consider

- the fundamental group  $\Gamma_x = \pi(L, x)$ .
- the normal space at x to the leaf L, call it V.
- the (linear) holonomy action of  $\Gamma_x$  on  $\nu_x(L)$ .

Also consider the holonomy group at *x*:

$$\Gamma'_x = \Gamma_x/\text{hol.}$$

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The leaf data Local form Stability Examples

# Local form: Local Reeb stability

Local form out of the local data: the foliated manifold

$$\tilde{L} imes_{\Gamma_x} V = (\tilde{L} imes V) / \Gamma_x$$

with the foliation coming from the product foliation (with leaves  $\tilde{L} \times \{v\}$ ).

#### Theorem (local Reeb stability)

If L is a compact leaf and  $\Gamma'_x$  is finite, then L admits a saturated neighborhood  $\mathcal{U}$  which is diffeomorphic to  $\tilde{L} \times_{\Gamma_x} V$  as foliated manifolds.

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### Foliations: Thurston stability

#### Theorem (Thurston, Langevin & Rosenberg)

If L is compact and  $H^1(\Gamma_x; V) = 0$ , then L is stable i.e. any other foliation on M which is close enough to the original one, has at least one leaf which is diffeomorphic to L.

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### On the torus



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### On the Moebius band



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### **III: POISSON GEOMETRY: INTRODUCTION**

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### Origins



J.L. Lagrange 1736-1813

1809: Lagranian mechanics



S. Poisson 1781-1840



W.R. Hamilton 1805-1865

1833: Hamiltonian mechanics

In 1831-1837, A.L. Cauchy: clear present. of Lagrange's methods, using Hamilton's formalism and Poisson brackets.



Marius Sophus Lie 1842-1899 Lie (pseudo-) groups Linear Poisson structures



Ellie Joseph Cartan 1869-1951 Lie theory, EDS, etc Weil, Chern, etc modern geometry, Lie theory, geometry of PDE's

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Stability and normal forms (around leaves in Poisson geometry)

1809: Poisson brackets

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# Various descriptions

A Poisson structure on *M* is a Lie bracket  $\{-,-\}$  on  $C^{\infty}(M)$  s.t.

$$\{f,-\}: C^{\infty}(M) \longrightarrow C^{\infty}(M)$$

acts as a vector field (the Hamiltonian vector field  $X_f$ ).

Equivalently, it is a bivector  $\pi$  on M with the property that

$$\{f,g\}:=\pi(df,dg)$$

satisfies Jacoby (or  $[\pi, \pi] = 0$ ).

Locally: determined by fcts  $\pi_{i,j} = \{x_i, x_j\}; \pi = \sum \pi_{i,j} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_i}$ .

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- symplectic manifolds.
- duals of Lie algebras.
- *S*/*G* where *S* is symplectic and *G* acts on *S* by symplectomorphisms.

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# Conn's theorem

Let  $(M, \pi)$ -Poisson,  $x \in M$  singular point (i.e.  $\pi_x = 0$ ).

Local data:  $g_x$ - the isotropy Lie algebra at x. Locally:

$$c_{i,j}^k = \frac{\partial \pi_{i,j}}{\partial x_k}(x)$$

Local model out of the local data:  $\mathfrak{g}_{\chi}^{*}$ .

#### Theorem (J. Conn, 1980)

If  $g_x$  is semi-simple of compact type, then x admits a neighborhood  $\mathcal{U}$  which is Poisson diffeomorphic to  $g_x^*$ .

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### The symplectic leaves

In a Poisson manifold  $(M, \pi)$ : the hamiltonians  $X_f$  define an integrable distribution on M. A symplectic leaf is a maximal integral submanifold.

Equivalently:  $x, y \in M$  are in the same leaf iff there exists a piecewise smooth curve joining x to y made of integral curves of hamiltonian vector fields:

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# The symplectic leaves

# In a Poisson manifold $(M, \pi)$ : the hamiltonians $X_f$ define an integrable distribution on M. A symplectic leaf is a maximal integral submanifold.

Equivalently:  $x, y \in M$  are in the same leaf iff there exists a piecewise smooth curve joining x to y made of integral curves of hamiltonian vector fields:

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# The symplectic leaves

### $\implies$ a partition of *M* into symplectic submanifolds.

Actually: a Poisson manifold = a manifold with a (possibly singular) symplectic foliation.

Basic examples:

- $\blacksquare \mathfrak{g}^* \Longrightarrow \mathsf{coadjoint orbits.}$
- *S*/*G*, with *S* a hamiltonian *G*-space ⇒ the symplectic reductions.

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#### Ordinary Geometry

- Tangent bundle TM with usual Lie bracket.
- Curves:  $\gamma : I \longrightarrow M$ (with derivatives  $\dot{\gamma} : I \longrightarrow TM$ ).
- Homotopy of curves.
- *k*-forms:  $\Omega^k(M) = \Gamma(\wedge^k T^*M).$
- DeRham operator and cohomology.

- Cotangent bundle  $T^*M$ : related to TM by  $\pi^{\sharp}: T^*M \longrightarrow TM$ ,  $df \mapsto X_f$ , bracket  $[df, dg] = d\{f, g\}$ .
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# IV. POISSON GEOMETRY: LOCAL DATA AND THE LOCAL MODEL

Marius Crainic Stability and normal forms (around leaves in Poisson geometry)

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A symplectic manifold: the symplectic leaf  $S_x$ A Lie group: the isotropy group  $G_x$ A principal bundle: the Poisson universal cover The local model

## Poisson geometry: the local data and the local model

#### Start with $(M, \pi)$ Poisson manifold, $x \in M$ .

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## The symplectic leaf through x

#### Consider the symplectic leaf through *x*.

 $S_x = \{\gamma(1) : \gamma - \text{cotangent path starting at } x\}.$ 

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Isotropy at x

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The isotropy at *x*, or the Poisson fundamental group at *x*:

 $G_x := \{ \text{cotangent loops at } x \} / \{ \text{cotangent homotopy} \}.$ 

If Hausdorff, it is a Lie group (this is mostly the case, and we understand when this fails).

Infinitesimally,

$$\mathfrak{g}_{\mathbf{X}}=\nu_{\mathbf{X}}^{*}\subset T^{*}M$$

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- a symplectic manifold S.
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- $\Longrightarrow$  a Poisson manifold

$$P[\mathfrak{g}^*] = P imes_G \mathfrak{g}^* = (P imes \mathfrak{g}^*)/G.$$

(the coadjoint bundle of the principal bundle). Here,

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I. Lie group actions II. Foliations III. Foliations III. Poisson geometry: the local data and the local model V. Normal forms and stability in Poisson geometry IV. Poisson geometry: the local data and the local model

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I. Lie group actions II. Foliations V. Normal forms and stability in Poisson geometry

## V. POISSON GEOMETRY: NORMAL FORMS AND STABILITY

Marius Crainic Stability and normal forms (around leaves in Poisson geometry)

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Normal form

Normal form Stability About the proofs

Theorem (together with I. Marcut, based on joint work with L.R. Fernandes and discussions with D.M. Torres)

If  $S_x$  is compact,  $G_x$  is compact and  $P_x$  is 2-connected, then  $S_x$  admits a neighborhood  $\mathcal{U}$  which is Poisson diffeomorphic to  $P_x \times_{G_x} \mathfrak{g}_x^*$ .

Example: coadjoint orbits for semi-simple Lie algebras of compact type.

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Normal form Stability About the proofs

## Stability

#### Theorem (with R.L. Fernandes)

If  $S_x$  is compact and  $H^2_{\pi}(M, S_x) = 0$  then S is stable.

Here:

- $H^2_{\pi}(M, S_x)$  is defined similar to Poisson cohomology, by "restricting to  $S_x$ . It is finite dimensional and computable!
- If  $G_x$  is compact, then

$$H^2_{\pi}(M, S_x) \hookrightarrow H^1(G_x; \nu_x).$$

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## Stability

#### Theorem (with R.L. Fernandes)

## If $S_x$ is compact and $H^2_{\pi}(M, S_x) = 0$ then S is stable.

Here:

- $H^2_{\pi}(M, S_x)$  is defined similar to Poisson cohomology, by "restricting to  $S_x$ . It is finite dimensional and computable!
- If  $G_x$  is compact, then

$$H^2_{\pi}(M, S_x) \hookrightarrow H^1(G_x; \nu_x).$$

Normal form Stability About the proofs

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## About the proof of Conn's theorem

Marius Crainic Stability and normal forms (around leaves in Poisson geometry)

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Normal form Stability About the proofs

### About the proof of the stability theorem

- It is a combination of several ingredients:
  - Geometric: a coupling construction, like in symplectic geometry.
  - Algebraic: the structure that governs complicated equations.

Analytic: the actual proof, based on minimizing a certain functional defined on the Sobolev space of sections and taking advantage of the ellipticity of the operators (complexes) involved.

#### Main idea: work in a tubular neighborhood and look for leaves a see

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