The isoperimetric problem in the plane with a piecewise constant density

Antonio Cañete

Dpto. Matemática Aplicada I

Universidad de Sevilla

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The Isoperimetric Problem

We look for the least perimeter set in $\mathbb{R}^2$ enclosing a prescribed quantity of area

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We look for the least perimeter set in $\mathbb{R}^2$ enclosing a prescribed quantity of area

- Existence is not guaranteed

- In case of existence $\longrightarrow$ isoperimetric region

Density setting

We shall use a density $f : \mathbb{R}^2 \rightarrow \mathbb{R}^+$ to weight the area and the perimeter:

For $\Omega \subset \mathbb{R}^2$,

$$area(\Omega) = \int_\Omega f, \qquad P(\Omega) = \int_{\partial\Omega} f$$

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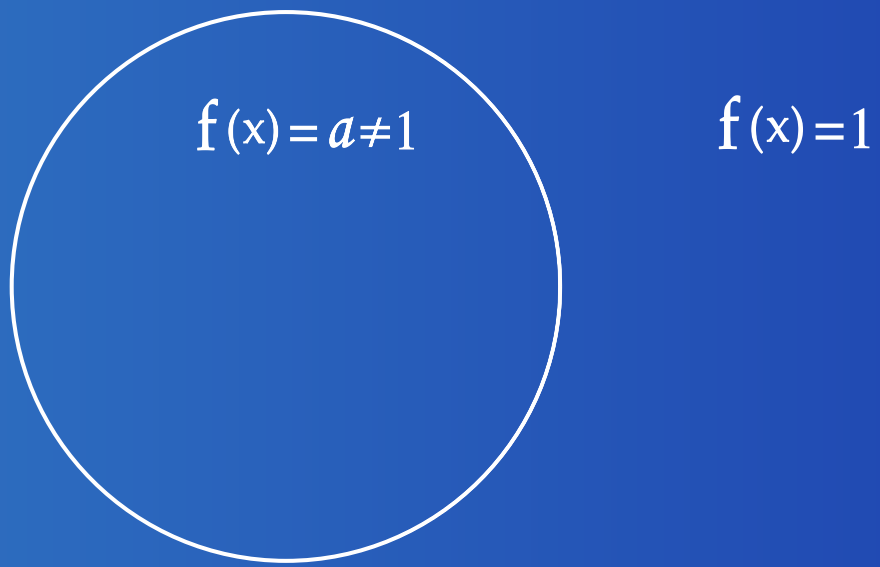
- We will consider piecewise constant densities (examples of discontinuous ones)

Piecewise constant densities

Ball density in $\mathbb{R}^2$:



Piecewise constant densities

Strip density in $\mathbb{R}^2$:

$f(x) = a > 1$

$\{x = 1\}$

$f(x) = 1$

$\{x = -1\}$

$f(x) = a > 1$

Piecewise constant densities

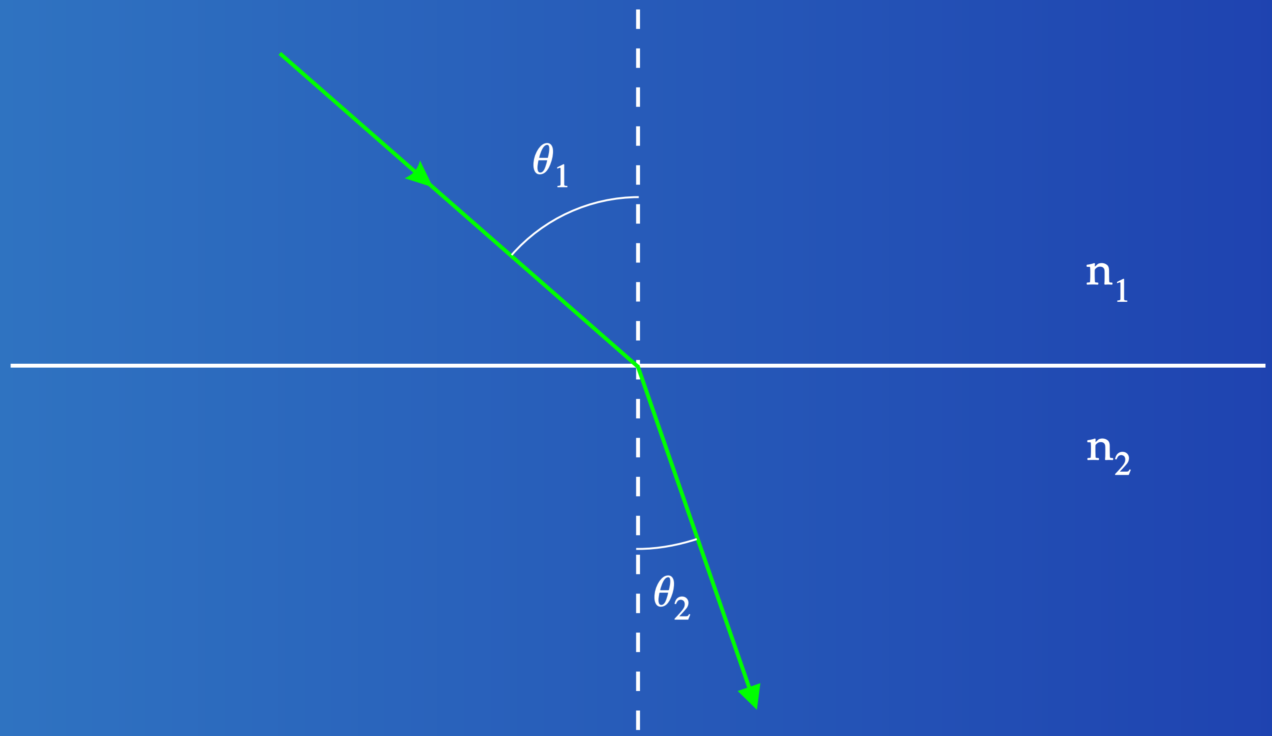
Half-plane density in $\mathbb{R}^2$:

$f(x) = a > 1$

$\{x = 0\}$

$f(x) = 1$

Snell's law in Optics



$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$$

Snell's law in Optics

Experimentally:

Properties of the isoperimetric regions

$\Omega \equiv$ isoperimetric region, $\quad \partial\Omega = \Sigma$

$\Gamma \equiv$ set of discontinuities of f

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Properties of the isoperimetric regions

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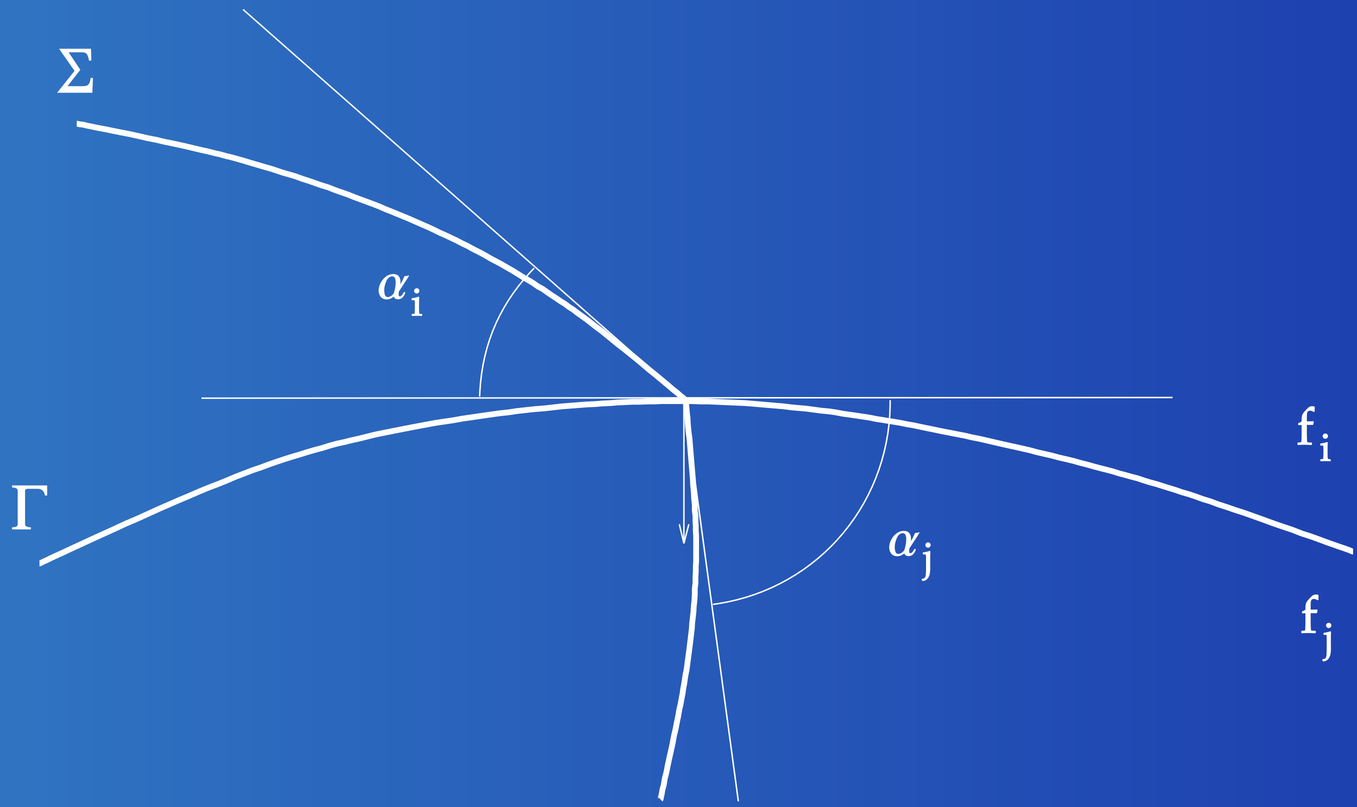
$\Gamma \equiv$ set of discontinuities of f

1) $\Sigma - \Gamma$ is composed by curves with constant geodesic curvature
2) Pieces of Γ may be part of Σ
3) Σ may not be smooth: when it crosses transversally Γ, a corner is formed according to Snell's law

Snell's law



$$f_i \cos(\alpha_i) = f_j \cos(\alpha_j)$$

Snell's law: Proof

$p \in \Sigma \cap \Gamma$, local variation of Σ preserving the area

$X \equiv$ variational field, $\nu \equiv$ normal vector

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$$0 = P'(0) = \int_\Sigma \langle \nabla\psi, \nu_\Sigma \rangle \, u \, f - \int_\Sigma H \, f \, u$$

$$+ \sum_{i=1}^{k} f_i(p) \, \langle X(p), \nu_{\Sigma_i}(p) \rangle \, ,$$

where $f = e^\psi$, $u = \langle X, \nu_\Sigma \rangle$, $H \equiv$ geod. curv.,
$f_i = f|_{\Omega_i}$, $\Sigma_i = \Sigma \cap \Omega_i$

Snell's law: Proof

$p \in \Sigma_i \cap \Sigma_j \cap \Gamma$

$X \equiv$ variational field, $\nu \equiv$ normal vector

$$0 = f_i(p) \, \langle X(p), \nu_{\Sigma_i}(p) \rangle + f_j(p) \, \left\langle X(p), \nu_{\Sigma_j}(p) \right\rangle$$

$$f_i(p) \, \langle X(p), \nu_{\Sigma_i}(p) \rangle = -f_j(p) \, \left\langle X(p), \nu_{\Sigma_j}(p) \right\rangle$$

Snell's law: Proof

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$X \equiv$ variational field, $\nu \equiv$ normal vector

$$0 = f_i(p)\ \langle X(p), \nu_{\Sigma_i}(p)\rangle + f_j(p)\ \left\langle X(p), \nu_{\Sigma_j}(p)\right\rangle$$

$$f_i(p)\ \langle X(p), \nu_{\Sigma_i}(p)\rangle = -f_j(p)\ \left\langle X(p), \nu_{\Sigma_j}(p)\right\rangle$$

In particular, taking $X(p)$ tangent to Γ, $|X(p)| = 1$,

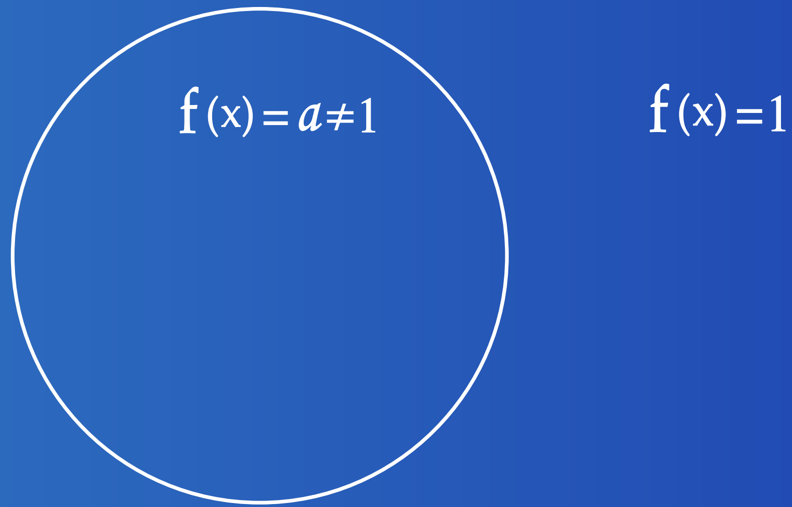
$$f_i(p)\ \cos(\alpha_i) = f_j(p)\ \cos(\alpha_j)$$

The boundary of an isoperimetric region $\rightsquigarrow$

- curves with constant geodesic curvature
- possibly pieces of Γ
- Snell's law is satisfied

$$\text{We focus on } \left\{ \begin{array}{l} \text{ball density} \\ \text{strip density} \\ \text{half-plane density} \end{array} \right.$$

Ball density

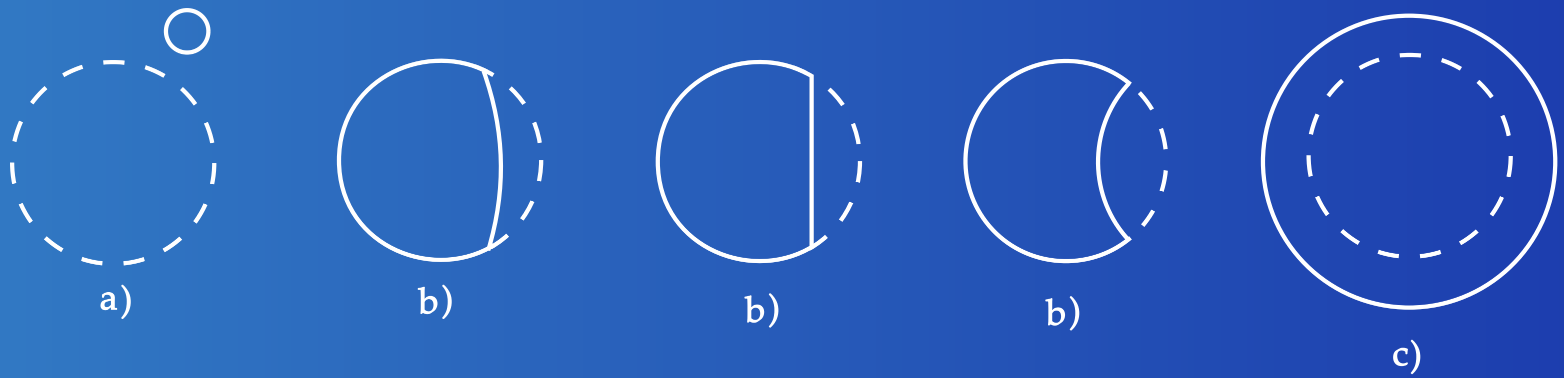


- Existence
- Two different cases $\begin{cases} a>1 \\ a<1 \end{cases}$

Ball density with $a > 1$

Isoperimetric regions are:

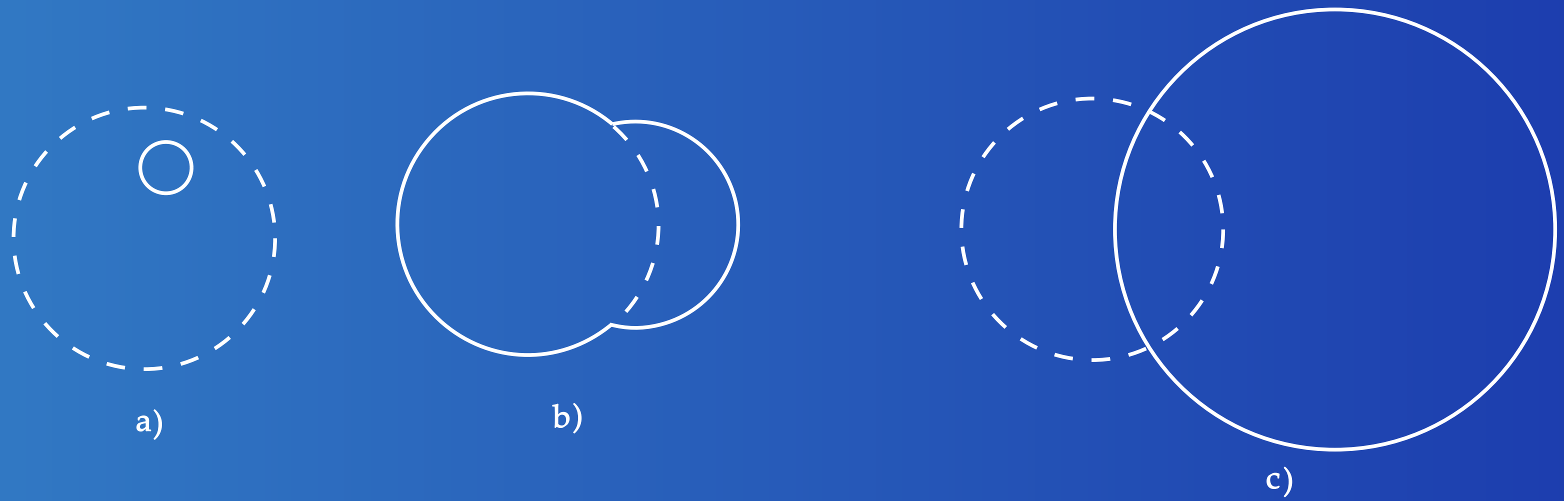
- For areas $v \leq v_0$, balls of type a)
- For areas $v_0 \leq v \leq a\,\pi$, sets of type b)
- For areas $v \geq a\,\pi$, balls of type c)



Ball density with $a < 1$

Isoperimetric regions are:

- For areas $v \leq a\,\pi$, balls of type a)
- For areas $a\,\pi \leq v \leq v_1$, sets of type b)
- For areas $v_1 \leq v \leq v_2$, sets of type b) or c)
- For areas $v \geq v_2$, balls of type c)



Ball density with $a < 1$

Isoperimetric regions are:

- For areas $v \leq a\,\pi$, balls of type a)
- For areas $a\,\pi \leq v \leq v_1$, sets of type b)
- For areas $v_1 \leq v \leq v_2$, sets of type b) or c)
- For areas $v \geq v_2$, balls of type c)

- We believe $v_1 = v_2$, but we have not proved it

Strip density

$f(x) = a > 1$

$\{x = 1\}$

$f(x) = 1$

$\{x = -1\}$

$f(x) = a > 1$

- Existence
- Vertical symmetry

Strip density

Isoperimetric regions are:

- For areas $v \le \pi$, balls of type i)
- For areas $\pi \le v \le v_0$, sets of type ii)
- For areas $v_0 \le v \le v_1$, sets of type iii) or iv)
- For areas $v \ge v_1$, sets of type iii)

Strip density

Isoperimetric regions are:

- For areas $v \leq \pi$, balls of type i)
- For areas $\pi \leq v \leq v_0$, sets of type ii)
- For areas $v_0 \leq v \leq v_1$, sets of type iii) or iv)
- For areas $v \geq v_1$, sets of type iii)

- In most cases, type iv) does not appear

Half-plane density

$$f(x) = a > 1$$

$$\{x = 0\}$$

$$f(x) = 1$$

Half-plane density

Isoperimetric regions are balls in $\{x < 0\}$

- Also true in $\mathbb{R}^n$, $n > 2$

Summary

- Density $\longrightarrow$ New definitions of area and perimeter
- Piecewise Constant Density $\longrightarrow$ Corners may appear
- Particular densities $\longrightarrow$ Different isoperimetric regions