Discrete Dirac Structures and Variational Discrete Dirac Mechanics

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UCSD

Introduction

■ Dirac Structures

- Dirac structures can be viewed as simultaneous generalizations of symplectic and Poisson structures.

- Implicit Lagrangian and Hamiltonian systems[1] provide a unified geometric framework for studying degenerate, interconnected, and nonholonomic Lagrangian and Hamiltonian mechanics.

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[1]H. Yoshimura, J.E. Marsden, Dirac structures in Lagrangian mechanics. Part I: Implicit Lagrangian systems, *J. of Geometry and Physics*, **57**, 133–156, 2006.

Introduction

■ Variational Principles

- The Hamilton–Pontryagin principle[2] on the Pontryagin bundle $TQ \oplus T^*Q$, unifies Hamilton's principle, Hamilton's phase space principle, and the Lagrange–d'Alembert principle.

- Provides a variational characterization of implicit Lagrangian and Hamiltonian systems.

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[2]H. Yoshimura, J.E. Marsden, Dirac structures in Lagrangian mechanics. Part II: Variational structures, *J. of Geometry and Physics*, **57**, 209–250, 2006.

Introduction

■ **Discrete Dirac Structures**

- Continuous Dirac structures are constructed by considering the geometry of symplectic vector fields and their associated Hamiltonians.

- By analogy, we construct discrete Dirac structures by considering the geometry of symplectic maps and their associated generating functions.

- Provides a unified treatment of implicit discrete Lagrangian and Hamiltonian mechanics in the presence of discrete Dirac constraints.

Introduction

■ Discrete Hamilton–Pontryagin principle

- We define a discrete Hamilton–Pontryagin principle on the discrete Pontryagin bundle $(Q \times Q) \oplus T^*Q$.

- Obtained from the discrete Hamilton's principle by imposing the discrete second-order curve condition using Lagrange multipliers.

- Provides an alternative derivation of implicit discrete Lagrangian and Hamiltonian mechanics.

- In the absence of constraints, implicit discrete Hamiltonian mechanics reduce to the usual definition of discrete Hamiltonian mechanics[3] obtained using duality in the sense of optimization.

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[3]S. Lall, M. West, Discrete variational Hamiltonian mechanics, *J. Phys. A* **39**(19), 5509–5519, 2006.

Dirac Structures on Vector Spaces

■ Properties

- Given a n-dimensional vector space V, consider the pairing $\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle$ on $V \oplus V^*$ given by

$$\langle\langle (v, \alpha), (\tilde{v}, \tilde{\alpha}) \rangle\rangle = \langle \alpha, \tilde{v} \rangle + \langle \tilde{\alpha}, v \rangle,$$

  where $\langle\,\cdot\,,\,\cdot\,\rangle$ is the natural pairing between covectors and vectors.

- A **Dirac Structure** is a subspace $D \subset V \oplus V^*$, such that

$$D = D^{\perp}.$$

- In particular, $D \subset V \oplus V^*$ is a Dirac structure iff

$$\dim D = n$$

  and

$$\langle \alpha, \tilde{v} \rangle + \langle \tilde{\alpha}, v \rangle = 0,$$

  for all $(v, \alpha), (\tilde{v}, \tilde{\alpha}) \in D$.

Dirac Structures on Manifolds

■ Properties

- An **almost Dirac Structure** on a manifold M is a subbundle $D \subset TM \oplus T^*M$ such that $D_q \subset T_qM \oplus T_q^*M$ is a Dirac structure.

- A **Dirac structure** on a manifold is an almost Dirac structure such that

$$\langle \pounds_{X_1} \alpha_2, X_3 \rangle + \langle \pounds_{X_2} \alpha_3, X_1 \rangle + \langle \pounds_{X_3} \alpha_1, X_2 \rangle = 0,$$

for all pairs of vector fields and one-forms

$$(X_1, \alpha_1), (X_2, \alpha_2), (X_3, \alpha_3) \in D,$$

and where $\pounds_X$ is the Lie derivative along the vector field X.

- This is a generalization of the condition that the symplectic two-form is closed, or that the Poisson bracket satisfies Jacobi's identity.

Dirac Structures on Manifolds

■ Generalizing Symplectic and Poisson Structures

- Let $M = T^*Q$.

- The graph of the symplectic two-form $\Omega : TM \times TM \to \mathbb{R}$, viewed as a map $TM \to T^*M$,

$$v_z \mapsto \Omega(v_z, \cdot),$$

  is a Dirac structure.

- Similarly, the graph of the Poisson structure $B : T^*M \times T^*M \to \mathbb{R}$, viewed as a map T^*M to $T^{**}M \cong TM$,

$$\alpha_z \mapsto B(\alpha_z, \cdot),$$

  is a Dirac structure.

- Furthermore, if the symplectic form and the Poisson structure are related, they induce the same Dirac structure on $TM \oplus T^*M$.

Motivating Example: Electrical Circuits

■ **Configuration space and constraints**

- The **configuration** $q \in E$ of the electrical circuit is given by specifying the current in each branch of the electrical circuit.

- Not all configurations are admissible, due to **Kirchhoff's Current Laws**:

$$\text{the sum of currents at a junction is zero.}$$

  This induce a **constraint KCL space** $\Delta \subset TE$.

- Its annihilator space $\Delta^{\circ} \subset T^*E$ is defined by
$$\Delta_q^{\circ} = \{e \in T_q^* E \,|\, \langle e, f \rangle = 0 \text{ for all } f \in \Delta_q\},$$
  which can be identified with the set of **branch voltages**, and encodes the **Kirchhoff's Voltage Laws**:

$$\text{the sum of voltages about a closed loop is zero.}$$

Motivating Example: Electrical Circuits

■ **Dirac structures and Tellegen's theorem**

- Given $\Delta \subset TE$ and $\Delta^\circ \subset T^*E$ which encode the Kirchhoff's current and voltage laws,

$$D_E = \Delta \oplus \Delta^\circ \subset TE \oplus T^*E$$

is a Dirac structure on E.

- Since $D = D^\perp$, we have that for each $(f, e) \in D_E$,

$$\langle e, f \rangle = 0.$$

This is a statement of **Tellegen's theorem**, which is an important result in the network theory of circuits.

Motivating Example: Electrical Circuits

■ Lagrangian for LC-circuits

- **Dirac's theory of constraints** was concerned with degenerate Lagrangians where the set of **primary constraints**, the image $P \subset T^*Q$ of the Legendre transformation, is not the whole space.

- The **magnetic energy** is given by
$$T(f) = \sum \frac{1}{2} L_i f_{L_i}^2.$$

- The **electric potential energy** is
$$V(q) = \sum \frac{1}{2} \frac{q_{C_i}^2}{C_i}.$$

- The **Lagrangian** of the LC circuit is given by
$$L(q, f) = T(f) - V(q).$$

Variational Principles

Continuous Hamilton–Pontryagin principle

■ **Pontryagin bundle and Hamilton–Pontryagin principle**

- Consider the **Pontryagin bundle** $TQ \oplus T^*Q$, which has local coordinates (q, v, p).

- The **Hamilton–Pontryagin principle** is given by

$$\delta \int [L(q, v) - p(v - \dot{q})] = 0,$$

where we impose the second-order curve condition, $v = \dot{q}$ using Lagrange multipliers p.

Continuous Hamilton–Pontryagin principle

■ Implicit Lagrangian systems

- Taking variations in q, v, and p yield

$$\delta \int [L(q,v) - p(v - \dot{q})]dt$$

$$= \int \left[\frac{\partial L}{\partial q}\delta q + \left(\frac{\partial L}{\partial v} - p\right)\delta v - (v - \dot{q})\delta p + p\delta\dot{q}\right] dt$$

$$= \int \left[\left(\frac{\partial L}{\partial q} - \dot{p}\right)\delta q + \left(\frac{\partial L}{\partial v} - p\right)\delta v - (v - \dot{q})\delta p\right] dt$$

where we used integration by parts, and the fact that the variation δq vanishes at the endpoints.

- This recovers the **implicit Euler–Lagrange equations**,

$$\dot{p} = \frac{\partial L}{\partial q}, \qquad p = \frac{\partial L}{\partial v}, \qquad v = \dot{q}.$$

Continuous Hamilton–Pontryagin principle

■ Hamilton's phase space principle

- By taking variations with respect to v, we obtain the **Legendre transform**,

$$\frac{\partial L}{\partial v}(q, v) - p = 0.$$

- The **Hamiltonian**, $H : T^*Q \to \mathbb{R}$, is defined to be,

$$H(q, p) = \underset{v}{\text{ext}} \left(pv - L(q, v) \right) = pv - L(q, v)|_{p=\partial L/\partial v(q,v)}.$$

- The Hamilton–Pontryagin principle reduces to,

$$\delta \int [p\dot{q} - H(q, p)] = 0,$$

which is the **Hamilton's principle in phase space**.

Continuous Hamilton–Pontryagin principle

■ Lagrange–d'Alembert–Pontryagin principle

- Consider a constraint distribution $\Delta_Q \subset TQ$.

- The **Lagrange–d'Alembert–Pontryagin principle** is given by

$$\delta \int L(q, v) - p(v - \dot{q})dt = 0,$$

for fixed endpoints, and variations $(\delta q, \delta v, \delta p)$ of $(q, v, p) \in TQ \oplus T^*Q$, such that $(\delta q, \delta v) \in (T\tau_Q)^{-1}(\Delta_Q)$, where $\tau_Q : TQ \to Q$.

Discrete Variational Principles

Geometry and Numerical Methods

■ **Dynamical equations preserve structure**

- Many continuous systems of interest have properties that are conserved by the flow:

  ○ Energy

  ○ Symmetries, Reversibility, Monotonicity

  ○ Momentum - Angular, Linear, Kelvin Circulation Theorem.

  ○ Symplectic Form

  ○ Integrability

- At other times, the equations themselves are defined on a manifold, such as a Lie group, or more general configuration manifold of a mechanical system, and the discrete trajectory we compute should remain on this manifold, since the equations may not be well-defined off the surface.

Motivation: Geometric Integration

■ **Main Goal of Geometric Integration:**

Structure preservation in order to reproduce long time behavior.

■ **Role of Discrete Structure-Preservation:**

Discrete conservation laws impart long time numerical stability to computations, since the structure-preserving algorithm exactly conserves a discrete quantity that is always close to the continuous quantity we are interested in.

Geometric Integration: Energy Stability

■ **Energy stability for symplectic integrators**

Control on global error

Continuous energy
Isosurface

Discrete energy
Isosurface
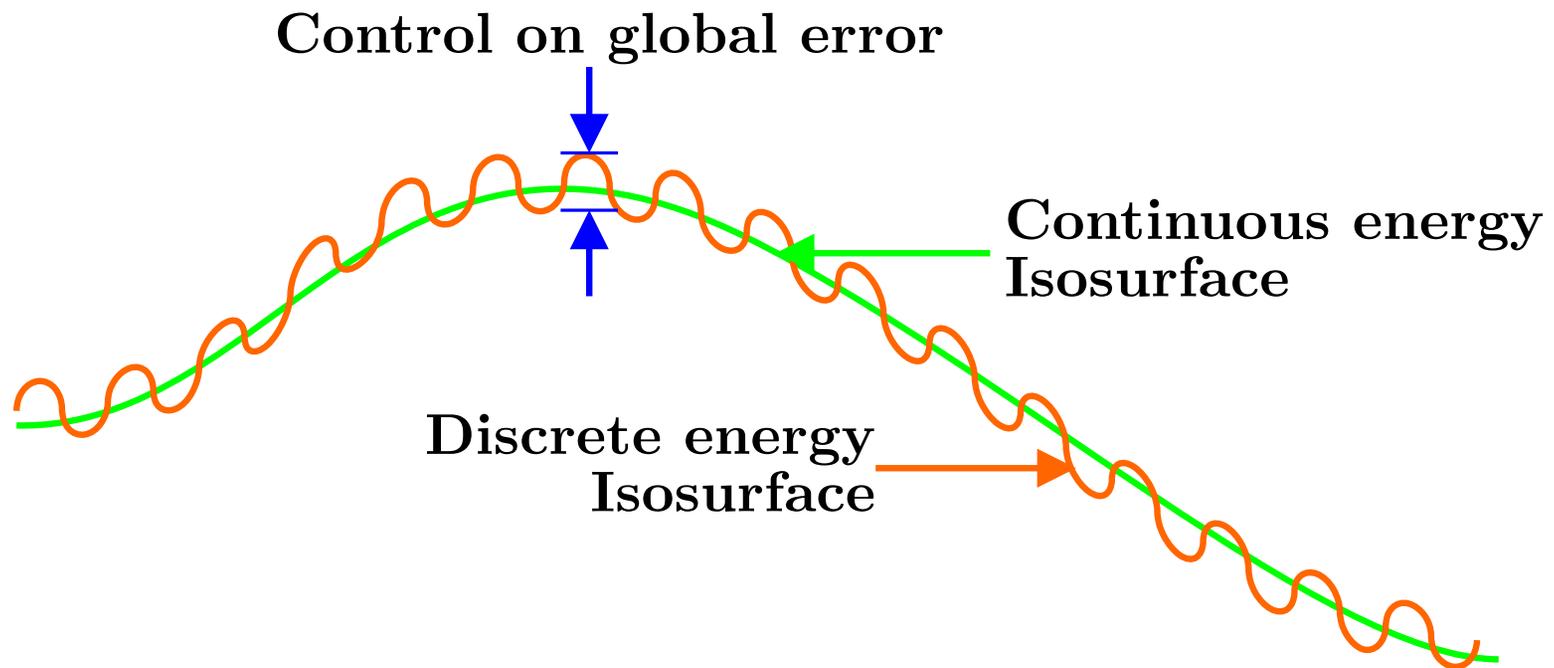
Geometric Integration: Energy Stability

■ **Energy behavior for conservative and dissipative systems**
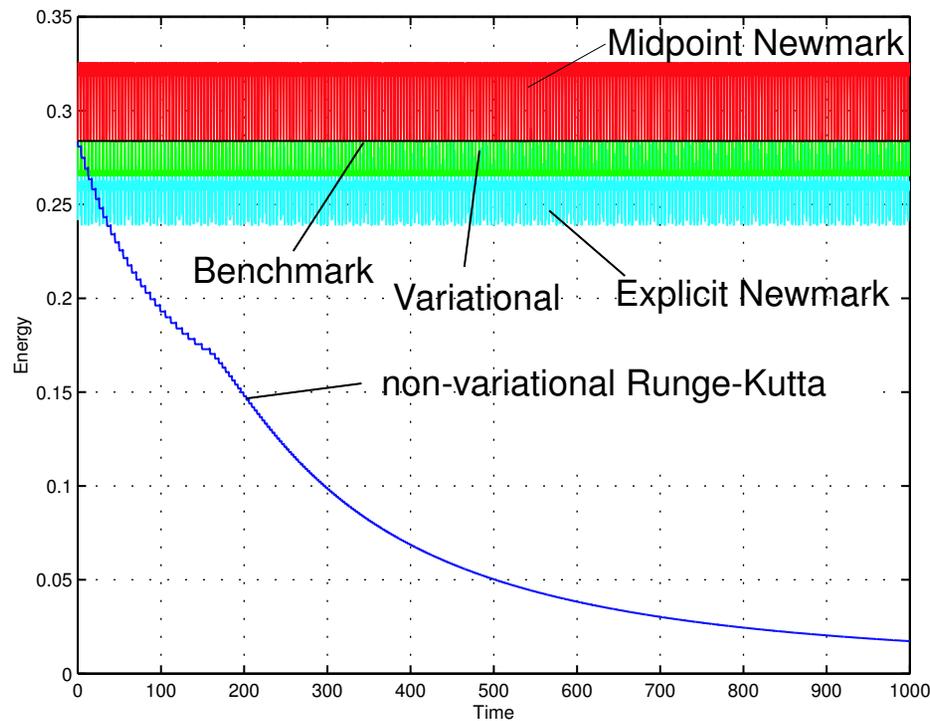
(a) Conservative mechanics
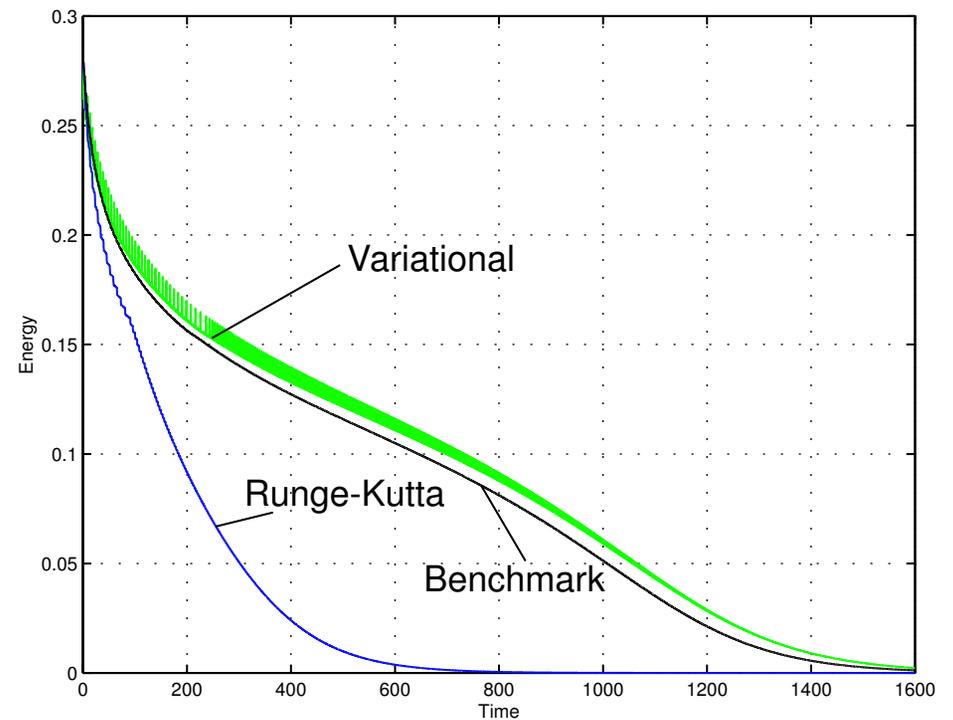
(b) Dissipative mechanics

Discrete Lagrangian Mechanics

■ Discrete Variational Principle
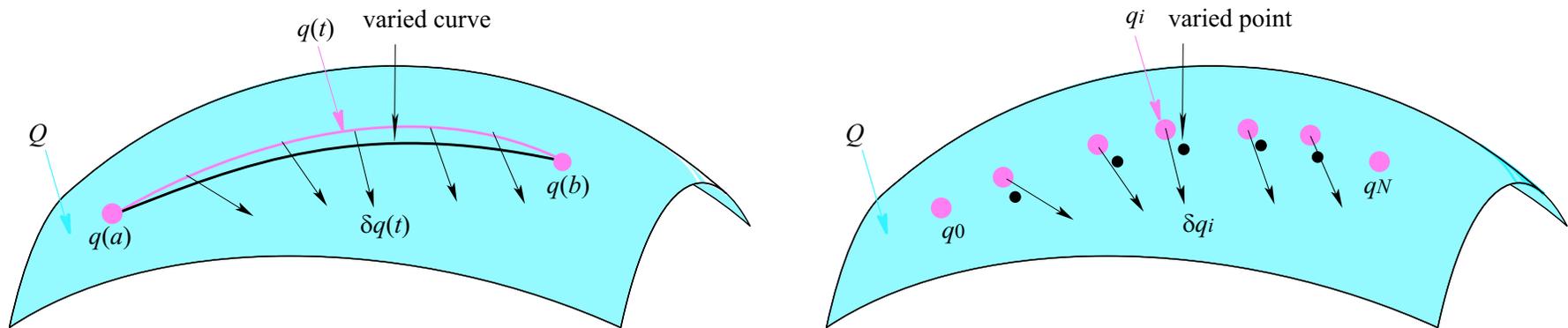
- **Discrete Lagrangian**

$$L_d \approx \int_0^h L\left(q(t), \dot{q}(t)\right) dt.$$

- **Discrete Euler-Lagrange equation**

$$D_2 L_d(q_0, q_1) + D_1 L_d(q_1, q_2) = 0.$$

Automatically symplectic and momentum preserving.

Discrete Lagrangian Mechanics

■ A Gallery of Simulations
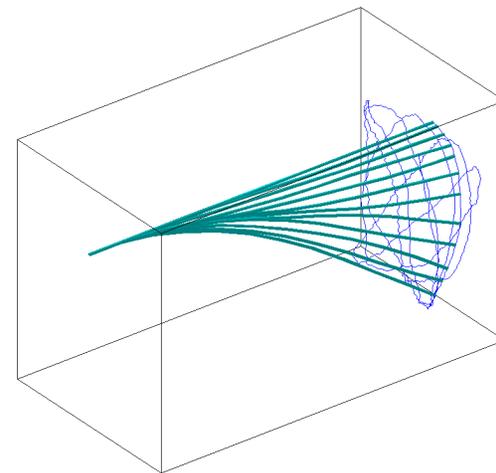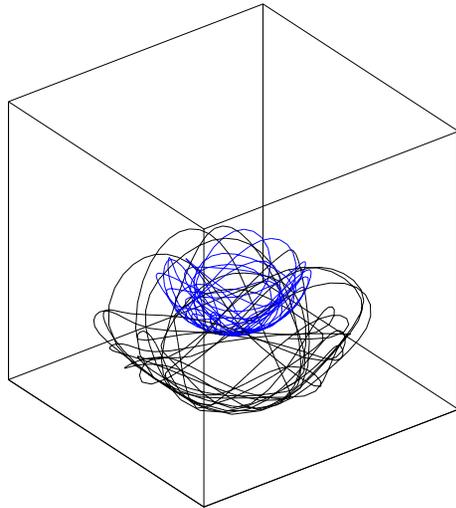
Joint work with Taeyoung Lee (FIT) and N. Harris McClamroch (Michigan).

Discrete Hamilton–Pontryagin principle

■ Discrete Pontryagin bundle and Hamilton–Pontryagin principle

- Consider the **discrete Pontryagin bundle** $(Q \times Q) \oplus T^*Q$, which has local coordinates (q_k^0, q_k^1, p_k).

- The **discrete Hamilton–Pontryagin principle** is given by

$$\delta \sum \left[L_d(q_k^0, q_k^1) - p_{k+1}(q_k^1 - q_{k+1}^0) \right] = 0,$$

  where we impose the second-order curve condition, $q_k^1 = q_{k+1}^0$ using Lagrange multipliers p_{k+1}

- The **discrete Lagrangian** L_d is a Type I generating function, and is chosen to be an approximation of **Jacobi's solution** of the Hamilton–Jacobi equation.

Discrete Hamilton–Pontryagin principle

■ **Implicit discrete Lagrangian systems**

- Taking variations in q_k^0, q_k^1, and p_k yield

$$\delta \sum \left[L_d(q_k^0, q_k^1) - p_{k+1}(q_k^1 - q_{k+1}^0) \right]$$
$$= \sum \left\{ [D_1 L_d(q_k^0, q_k^1) + p_k] \delta q_k^0 \right.$$
$$\left. -[q_k^1 - q_{k+1}^0] \delta p_{k+1} + [D_2 L_d(q_k^0, q_k^1) - p_{k+1}] \delta q_k^1 \right\}.$$

- This recovers the **implicit discrete Euler–Lagrange equations**,

$$p_k = -D_1 L_d(q_k^0, q_k^1), \qquad p_{k+1} = D_2 L_d(q_k^0, q_k^1), \qquad q_k^1 = q_{k+1}^0.$$

Discrete Hamilton–Pontryagin principle

■ Discrete Hamilton's phase space principle

- By taking variations with respect to q_k^1, we obtain the **discrete Legendre transform**,

$$D_2 L_d(q_k^0, q_k^1) - p_{k+1} = 0$$

- The **discrete Hamiltonian**, $H_{d+} : \mathcal{H}_+ \to \mathbb{R}$, is defined to be,

$$H_{d+}(q_k^0, p_{k+1}) = \operatorname*{ext}_{q_k^1} \ p_{k+1} q_k^1 - L_d(q_k^0, q_k^1)$$

$$= p_{k+1} q_k^1 - L_d(q_k^0, q_k^1) \Big|_{p_{k+1} = D_2 L_d(q_k^0, q_k^1)}.$$

- The discrete Hamilton–Pontryagin principle reduces to,

$$\delta \sum [p_{k+1} q_{k+1} - H_{d+}(q_k, p_{k+1})] = 0,$$

which is the **discrete Hamilton's principle in phase space**.

Discrete Hamilton's Equations and Discrete Hamiltonians

■ Discrete Hamilton's Equations

- The discrete Hamilton's principle in phase space yields the following **discrete Hamilton's equations**,

$$p_k = D_1 H_{d+}(q_k, p_{k+1}), \qquad q_{k+1} = D_2 H_{d+}(q_k, p_{k+1})$$

- From this, it is clear that the discrete Hamiltonian H_{d+} is a Type II generating function of a symplectic transformation.

Discrete Hamilton's Equations and Discrete Hamiltonians[4]

■ Exact Discrete Hamiltonian

- The **exact discrete Hamiltonian** generates the time-h flow map of the continuous Hamilton's equations, and is given by,

$$H_{d+}^{\text{exact}}(q_0, p_1) = \underset{\substack{(q,p)\in C^2([0,h],T^*Q) \\ q(0)=q_0, p(h)=p_1}}{\text{ext}} p(h)q(h) - \int_0^h (p(t)\dot{q}(t) - H(q(t), p(t)))dt$$

- Compare this to the **exact discrete Lagrangian** given by Jacobi's solution of the Hamilton–Jacobi equation,

$$L_d^{\text{exact}}(q_0, q_1) = \underset{\substack{q\in C^2([0,h],Q) \\ q(0)=q_0, q(h)=q_1}}{\text{ext}} \int_0^h L(q(t), \dot{q}(t))dt$$

- Approximations for both of these can be obtained by appropriately choosing a **finite-dimensional function space**, and a **numerical quadrature method**.

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[4]Joint work with Jingjing Zhang (Chinese Academy of Sciences).

Discrete Hamilton–Jacobi Theory[5]

■ Discrete Jacobi's Solution

- Consider the action sum, evaluated along a solution of the discrete Hamilton equations, and viewed as a function of the end point q_k,

$$S_{\mathrm{d}}^k(q_k) := \sum_{l=0}^{k-1} \left[p_{l+1} \cdot q_{l+1} - H_{\mathrm{d}}^+(q_l, p_{l+1}) \right]$$

- This satisfies the **discrete Hamilton–Jacobi equation**,

$$S_{\mathrm{d}}^{k+1}(q_{k+1}) - S_{\mathrm{d}}^k(q_k) - D_2 S_{\mathrm{d}}^{k+1}(q_{k+1}) \cdot q_{k+1}$$
$$+ H_{\mathrm{d}}^+ \left(q_k, D_2 S_{\mathrm{d}}^{k+1}(\hat{q}_0, q_{k+1}) \right) = 0.$$

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[5]Joint work with Anthony Bloch (Michigan) and Tomoki Ohsawa (Michigan).

Discrete Hamilton–Jacobi Theory

■ Derivation of Hamilton–Jacobi Equation

- Taking the finite difference of the discrete actions, we obtain,

$$S_{\mathrm{d}}^{k+1}(q_{k+1}) - S_{\mathrm{d}}^{k}(q_k) = p_{k+1} \cdot q_{k+1} - H_{\mathrm{d}}^{+}(q_k, p_{k+1}).$$

- Taking the derivative of both sides with respect to q_{k+1}, we have

$$p_{k+1} = DS_{\mathrm{d}}^{k+1}(q_{k+1}).$$

- Substituting this into the first equation yields,

$$S_{\mathrm{d}}^{k+1}(q_{k+1}) - S_{\mathrm{d}}^{k}(q_k) - D_2 S_{\mathrm{d}}^{k+1}(q_{k+1}) \cdot q_{k+1}$$
$$+ H_{\mathrm{d}}^{+}\left(q_k, D_2 S_{\mathrm{d}}^{k+1}(\hat{q}_0, q_{k+1})\right) = 0,$$

which is the **discrete Hamilton–Jacobi equation**[6].

[6]This equation first appeared in N. A. Elnatanov and Jeremy Schiff, The Hamilton–Jacobi difference equation, *Functional Differential Equations*, 3(279–286), 1996.

Discrete Hamilton–Jacobi Theory

■ Geometric Hamilton–Jacobi Theory

- Suppose that $\{S_{\mathrm{d}}^k\}_{k=1}^N$ satisfies the discrete Hamilton–Jacobi equation, and consider the set of points $\{c_k\}_{k=1}^N \subset Q$ such that

$$c_{k+1} = \pi_Q \circ \tilde{F}_{L_{\mathrm{d}}}\left(dS_{\mathrm{d}}^k(c_k)\right),$$

where $\tilde{F}_{L_{\mathrm{d}}} : T^*Q \to T^*Q$ is the discrete Hamiltonian map,

$$\tilde{F}_{L_{\mathrm{d}}} : (q_k, p_k) \mapsto (q_{k+1}, p_{k+1}).$$

- Then the set of points $\{(c_k, p_k)\}_{k=0}^N \subset T^*Q$ with

$$p_k := DS_{\mathrm{d}}^k(c_k), \quad k = 0, 1, \ldots, N$$

is a solution of the discrete Hamilton's equations.

Discrete Lagrange–d'Alembert–Pontryagin principle

■ **Continuous constraints from discrete constraints**

- Given $\Delta_Q^d \subset Q \times Q$, consider compatible curves on Q,

$$\mathcal{C}_{\Delta_Q^d} := \{\varphi \in C^\infty([-1,1], Q) |\, \exists \epsilon > 0,$$

$$\forall \tau \in (0, \epsilon), (\varphi(-\tau), \varphi(0)), (\varphi(0), \varphi(\tau)) \in \Delta_Q^d\}.$$

- Identify $v_q \in T_q Q$ with $[\varphi]$, the **equivalence class of curves** where $\varphi(0) = q$, and $D\varphi(0) = v$, and define $\Delta_Q \subset TQ$,

$$\varphi \in \mathcal{C}_{\Delta_Q^d} \implies [\varphi] \in \Delta_Q.$$
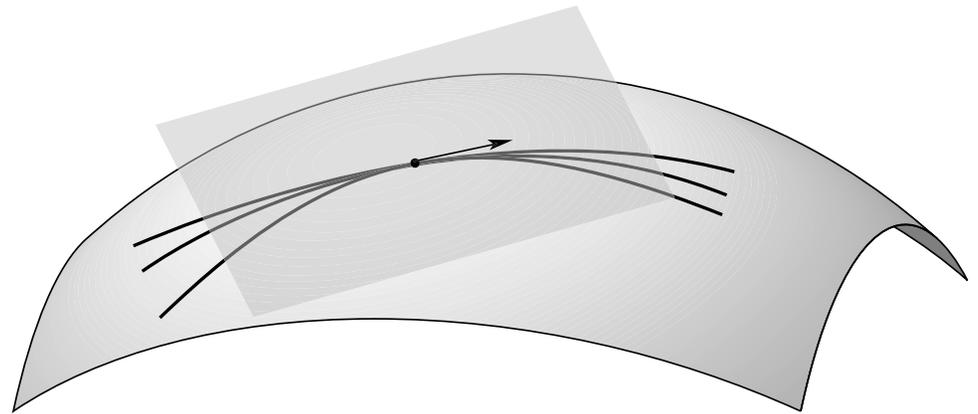
Discrete Lagrange–d'Alembert–Pontryagin principle

■ **Discrete Lagrange–d'Alembert–Pontryagin principle**

- The **Discrete Lagrange–d'Alembert–Pontryagin principle** is given by

$$\delta \sum \left[L_d(q_k^0, q_k^1) - p_{k+1}(q_k^1 - q_{k+1}^0) \right] = 0,$$

for fixed endpoints q_0^0 and q_N^0, and variations $(\delta q_k^0, \delta q_k^1, \delta p_k)$ of $(q_k^0, q_k^1, p_k) \in (Q \times Q) \oplus T^*Q$ such that $\delta q_k^0 \in \Delta_Q(q_k^0)$, $\delta q_k^1 \in \Delta_Q(q_k^1)$, and $(q_k^0, q_k^1) \in \Delta_Q^d$.

Dirac Structures

Continuous Dirac Mechanics

■ **The Big Diagram**

$$\gamma_Q$$

$$T^*T^*Q \xleftarrow{\quad \Omega^\flat \quad} TT^*Q \xrightarrow{\quad \kappa_Q \quad} T^*TQ$$

$$\pi^2 \qquad \rho_{TT^*Q} \qquad \pi^1$$

$$\pi_{T^*Q} \qquad \tau_{T^*Q} \qquad T\pi_Q \qquad \pi_{TQ}$$

$$T^*Q \xleftarrow{\quad pr_2 \quad} TQ \oplus T^*Q \xrightarrow{\quad pr_1 \quad} TQ$$

$$(q, p, -\delta p, \delta q) \longleftarrow (q, p, \delta q, \delta p) \longrightarrow (q, \delta q, \delta p, p)$$

$$(q, p) \xleftarrow{\quad} (q, \delta q) \oplus (q, p) \xrightarrow{\quad} (q, \delta q)$$
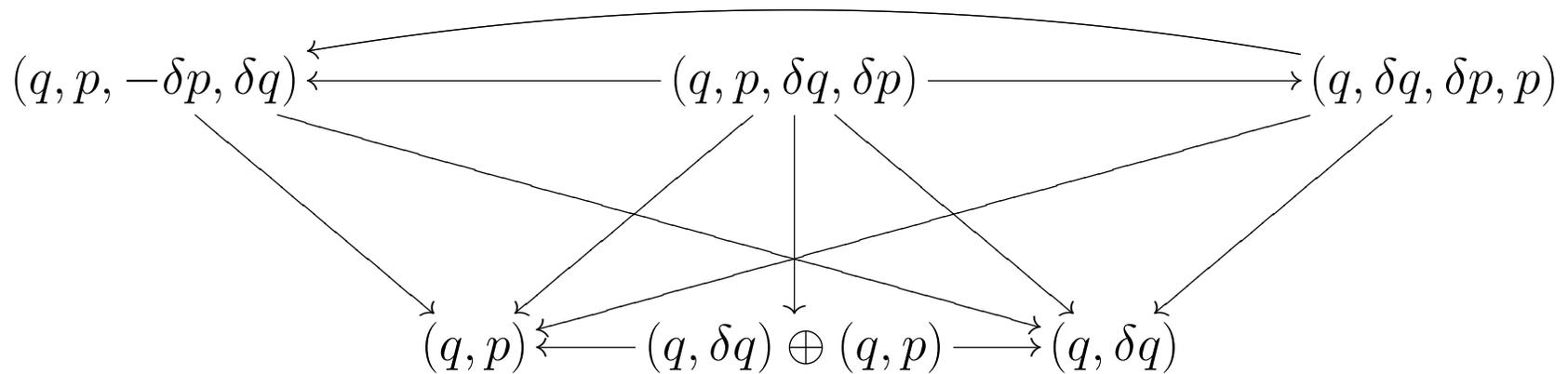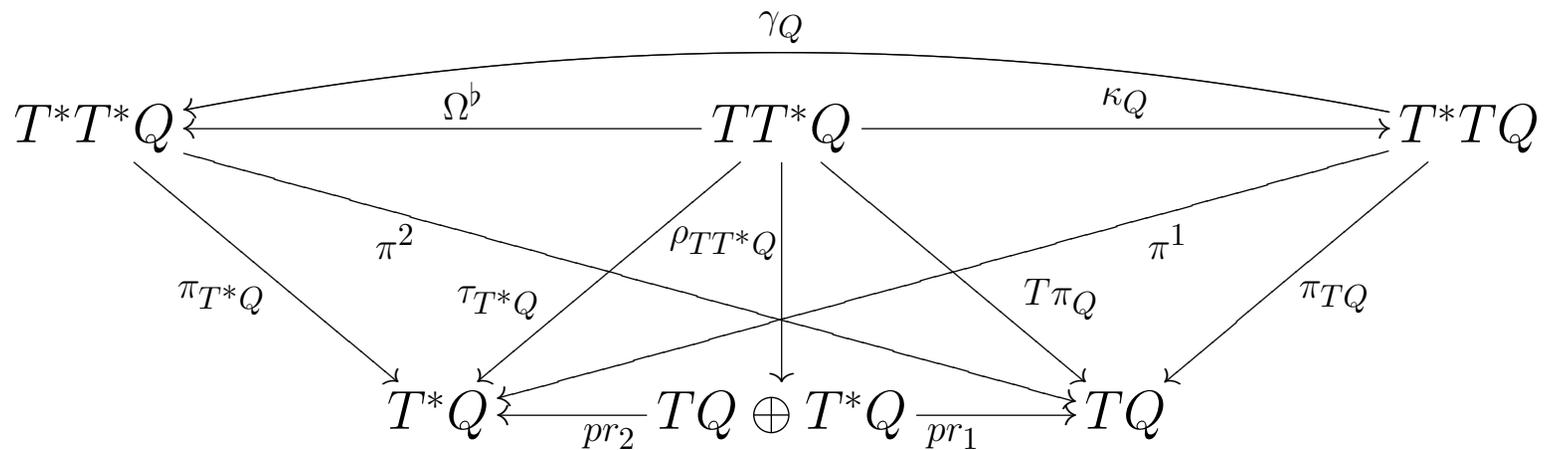
Continuous Dirac Mechanics

■ Dirac Structures and Constraints

- A constraint distribution $\Delta_Q \subset TQ$ induces a **Dirac structure** on T^*Q,

$$D_{\Delta_Q}(z) := \Big\{ (v_z, \alpha_z) \in T_z T^*Q \times T_z^* T^*Q \,\Big|$$
$$v_z \in \Delta_{T^*Q}(z), \; \alpha_z - \Omega^\flat(v_z) \in \Delta_{T^*Q}^\circ(z) \Big\}$$

  where $\Delta_{T^*Q} := (T\pi_Q)^{-1}(\Delta_Q) \subset TT^*Q$.

- Holonomic and nonholonomic constraints, as well as constraints arising from interconnections can be incorporated into the Dirac structure.

Continuous Dirac Mechanics

■ Implicit Lagrangian Systems

- Let $\gamma_Q := \Omega^\flat \circ (\kappa_Q)^{-1} : T^*TQ \to T^*T^*Q$.

- Given a Lagrangian $L : TQ \to \mathbb{R}$, define $\mathfrak{D}L := \gamma_Q \circ dL$.

- An **implicit Lagrangian system** (L, Δ_Q, X) is,

$$(X, \mathfrak{D}L) \in D_{\Delta_Q},$$

where $X \in \mathfrak{X}(T^*Q)$.

- This gives the **implicit Euler–Lagrange equations**,

$$\dot{q} = v \in \Delta_Q(q), \qquad p = \frac{\partial L}{\partial v}, \qquad \dot{p} - \frac{\partial L}{\partial q} \in \Delta_Q^\circ(q).$$

- In the special case $\Delta_Q = TQ$, we obtain,

$$\dot{q} = v, \quad \dot{p} = \frac{\partial L}{\partial q}, \quad p = \frac{\partial L}{\partial v}.$$

Continuous Dirac Mechanics

■ Implicit Hamiltonian Systems

- Given a Hamiltonian $H : T^*Q \to \mathbb{R}$, an **implicit Hamiltonian system** (H, Δ_Q, X) is,

$$(X, dH) \in D_{\Delta_Q},$$

  which gives the **implicit Hamilton's equations**,

$$\dot{q} = \frac{\partial H}{\partial p} \in \Delta_Q(q), \qquad \dot{p} + \frac{\partial H}{\partial q} \in \Delta_Q^\circ(q).$$

- In the special case $\Delta_Q = TQ$, we recover the standard Hamilton's equations,

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}.$$

The Geometry of Symplectic Flows

■ Hamiltonian Flows and the $\Omega^\flat$ map

- The flow F_X of a vector field $X \in \mathfrak{X}(T^*Q)$ is symplectic if locally,

$$i_X \Omega = dH, \qquad \text{for some function } H : T^*Q \to \mathbb{R}.$$

- We require that the following diagram commutes,

$$
\begin{array}{ccc}
TT^*Q & \xrightarrow{\ \ \Omega^\flat\ \ } & T^*T^*Q \\
\ \ \nwarrow{\scriptstyle X} & & \nearrow{\scriptstyle dH} \\
& T^*Q &
\end{array}
\qquad
\begin{array}{ccc}
(q,p,\dot{q},\dot{p}) & \longrightarrow & (q,p,\partial H/\partial q, \partial H/\partial p) \\
\ \ \nwarrow & & \nearrow \\
& (q,p) &
\end{array}
$$

- This gives rise to the map $\Omega^\flat : TT^*Q \to T^*T^*Q$,

$$\Omega^\flat : (q,p,\delta q, \delta p) \mapsto (q,p,-\delta p, \delta q).$$

The Geometry of Symplectic Flows

■ Lagrangian Flows and the κ_Q map

- The second-order vector field $X_L \in \mathfrak{X}(TQ)$ preserves the Lagrangian symplectic form if, $\pounds_{X_L}\Omega_L = 0$.

- Consider the Lagrange one-form, given by,

$$\Theta_L = (\mathbb{F}L)^*\Theta = \frac{\partial L}{\partial v}dq.$$

- Since $\pounds_{X_L}\Theta_L$ is closed, by the Poincaré lemma, we have a local function $L : TQ \to \mathbb{R}$ such that,

$$\pounds_{X_L}\Theta_L = dL,$$

  which is the intrinsic Euler–Lagrange equation.

- In terms of the $\mathbb{F}L$-related vector field $X \in \mathfrak{X}(T^*Q)$, we have,

$$p = \frac{\partial L}{\partial v}, \qquad \dot{q} = v, \qquad \dot{p} = \frac{\partial L}{\partial q}.$$

The Geometry of Symplectic Flows

■ **Lagrangian Flows and the κ_Q map**

- We require that the following diagram commutes,

$$
\begin{array}{ccccc}
TT^*Q & \xleftarrow{\ T\mathbb{F}L\ } & TTQ & \xrightarrow{\ \kappa_Q\ } & T^*TQ \\[2pt]
\uparrow{\scriptstyle X} & & \uparrow{\scriptstyle X_L} & {\scriptstyle dL} & \nearrow \\[2pt]
T^*Q & \xleftarrow{\ \mathbb{F}L\ } & TQ & &
\end{array}
$$

$$
\begin{array}{ccc}
(q,p,\dot q,\dot p) \longleftarrow (q,v,\dot q,\dot v) & \quad & (q,v,\partial L/\partial q,\partial L/\partial v) \\
\uparrow \qquad\qquad \uparrow & & \\
(q,p) \longleftarrow (q,v) & &
\end{array}
$$

- This gives rise to the map κ_Q,

$$
\kappa_Q : (q,p,\delta q,\delta p) \mapsto (q,\delta q,\delta p, p).
$$

Discrete Dirac Structures

The Geometry of Symplectic Maps

■ Generating Functions

- The Lagrangian and the Hamiltonian induce a Lagrangian and Hamiltonian vector field.

- In discrete time, the analogue would be the generating functions of a symplectic map.

- In particular, a discrete Lagrangian is a Type I generating function, and discrete Hamiltonians are Type II or III generating functions.

The Geometry of Generating Functions

■ Generating Functions of Type I and the κ_Q^d map

- The flow F on T^*Q is symplectic iff there exists $S_1 : Q \times Q \to \mathbb{R}$,

$$(i_F^{Q \times Q})^* \Theta_{T^*Q \times T^*Q} = dS_1.$$

  which gives

$$p_0 = -D_1 S_1, \qquad p_1 = D_2 S_1.$$

- We require that the following diagram commutes,

$$
\begin{array}{ccc}
T^*Q \times T^*Q & \xrightarrow{\;\kappa_Q^d\;} & T^*(Q \times Q) \\
& & \\
\nwarrow^{i_F^{Q \times Q}} & \nearrow_{dS_1} & \\
& Q \times Q &
\end{array}
\qquad
((q_0, p_0), (q_1, p_1)) \to (q_0, q_1, D_1 S_1, D_2 S_1)
$$

$$(q_0, q_1)$$

- This gives rise to a map $\kappa_Q^d : T^*Q \times T^*Q \to T^*(Q \times Q)$

$$\kappa_Q^d : ((q_0, p_0), (q_1, p_1)) \mapsto (q_0, q_1, -p_0, p_1).$$

The Geometry of Generating Functions

■ Generating Functions of Type II

- Consider $\mathcal{H}_+$, whose local coordinates are (q_0, p_1).

- Then the flow F on T^*Q is symplectic if and only if there exists $S_2 : \mathcal{H}_+ \to \mathbb{R}$ such that

$$(i_F^{\mathcal{H}_+})^* \Theta^{(2)}_{T^*Q \times T^*Q} = dS_2,$$

which gives

$$p_0 = D_1 S_2, \qquad q_1 = D_2 S_2.$$

The Geometry of Generating Functions

■ **Generating Functions of Type II and the $\Omega^\flat_{d+}$ map**

- We require that the following diagram commutes,

$$T^*Q \times T^*Q \xrightarrow{\quad \Omega^\flat_{d+} \quad} T^*\mathcal{H}_+ \qquad\qquad ((q_0, p_0), (q_1, p_1)) \to (q_0, p_1, D_1 S_2, D_2 S_2)$$

$$i_F^{\mathcal{H}_+} \qquad\qquad dS_2$$

$$\mathcal{H}_+ \qquad\qquad\qquad (q_0, p_1)$$

- This gives rise to a map $\Omega^\flat_{d+} : T^*Q \times T^*Q \to T^*\mathcal{H}_+$

$$\Omega^\flat_{d+} : ((q_0, p_0), (q_1, p_1)) \mapsto (q_0, p_1, p_0, q_1).$$

(+)-Discrete Dirac Mechanics

■ The Big Diagram

$$\gamma_Q^{d+}$$

$$T^*\mathcal{H}_+ \xleftarrow{\;\;\Omega^\flat_{d+}\;\;} T^*Q \times T^*Q \xrightarrow{\;\;\kappa^d_Q\;\;} T^*(Q \times Q)$$

$$\pi_{\mathcal{H}_+} \qquad \pi^2_{d+} \qquad \tau_{\mathcal{H}_+} \qquad \rho^{d+}_{(T^*Q)^2} \qquad \pi^1_{d+} \qquad \pi_Q \times \pi_Q \qquad \pi_{Q \times Q}$$

$$\mathcal{H}_+ \xleftarrow{\;pr^{d+}_2\;} (Q \times Q) \oplus \mathcal{H}_+ \xrightarrow{\;pr^{d+}_1\;} Q \times Q$$

$$(q_0, p_1, p_0, q_1) \xleftarrow{\quad} ((q_0, p_0), (q_1, p_1)) \xrightarrow{\quad} (q_0, q_1, -p_0, p_1)$$

$$(q_0, p_1) \xleftarrow{\quad} (q_0, q_1) \oplus (q_0, p_1) \xrightarrow{\quad} (q_0, q_1)$$
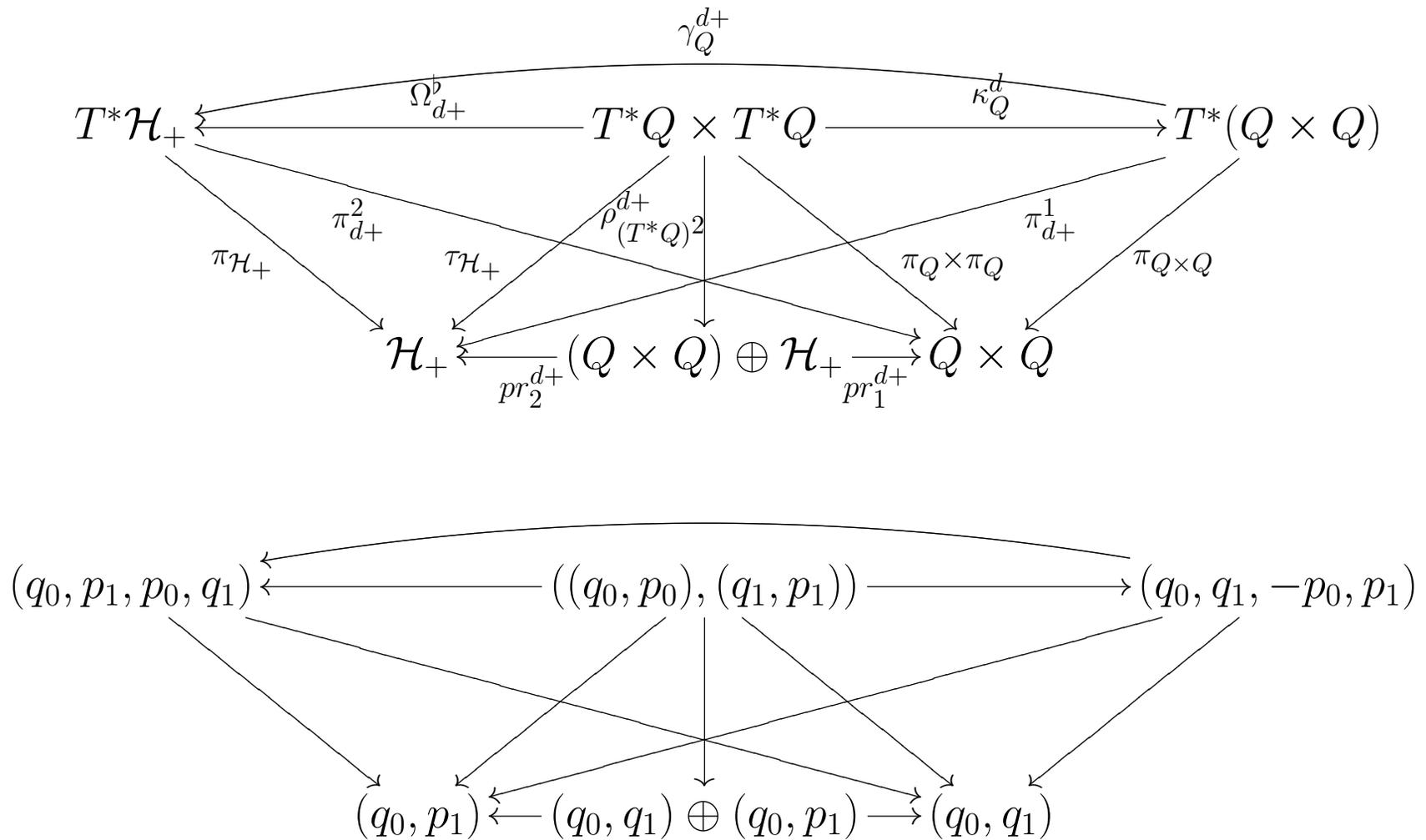
(+)-Discrete Dirac Mechanics

■ **Discrete Dirac Structures and Discrete Constraints**

- A discrete constraint distribution $\Delta_Q^d \subset Q \times Q$ induces a continuous constraint distribution $\Delta_Q \subset TQ$.

- These two distributions yield a **discrete Dirac structure**,

$$
D_{\Delta_Q}^{d+}(z) := \Big\{ ((z, z^1), \alpha_{z_+}) \in (\{z\} \times T^*Q) \times T_{z_+}^* \mathcal{H}_+ \Big|
$$
$$
\left(z, z^1 \right) \in \Delta_{T^*Q}^d, \ \alpha_{z_+} - \Omega_{d+}^\flat \left((z, z^1) \right) \in \Delta_{\mathcal{H}_+}^\circ \Big\},
$$

where

$$
\Delta_{T^*Q}^d := (\pi_Q \times \pi_Q)^{-1}(\Delta_Q^d) \subset T^*Q \times T^*Q,
$$
$$
\Delta_{\mathcal{H}_+}^\circ := \left(\Omega_{d+}^\flat \right) \left(\Delta_Q^\circ \times \Delta_Q^\circ \right) \subset T^*\mathcal{H}_+.
$$

(+)-Discrete Dirac Mechanics

■ Implicit Discrete Lagrangian Systems

- Let $\gamma_Q^{d+} := \Omega_{d+}^\flat \circ (\kappa_Q^d)^{-1} : T^*(Q \times Q) \to T^*\mathcal{H}_+$.

- Given a discrete Lagrangian $L_d : Q \times Q \to \mathbb{R}$, define $\mathfrak{D}^+ L_d := \gamma_Q^{d+} \circ dL_d$.

- An **implicit discrete Lagrangian system** is given by

$$\left(X_d^k, \mathfrak{D}^+ L_d(q_k^0, q_k^1) \right) \in D_{\Delta_Q}^{d+},$$

where $X_d^k = ((q_k^0, p_k^0), (q_{k+1}^0, p_{k+1}^0)) \in T^*Q \times T^*Q$.

- This gives the **implicit discrete Euler–Lagrange equations**,

$$p_{k+1}^0 - D_2 L_d(q_k^0, q_k^1) \in \Delta_Q^\circ(q_k^1), \qquad p_k^0 + D_1 L_d(q_k^0, q_k^1) \in \Delta_Q^\circ(q_k^0),$$

$$q_k^1 = q_{k+1}^0, \qquad (q_k^0, q_{k+1}^0) \in \Delta_Q^d.$$

(+)-Discrete Dirac Mechanics

■ Implicit Discrete Hamiltonian Systems

- Given a discrete Hamiltonian $H_{d+} : \mathcal{H}_+ \to \mathbb{R}$, an **implicit discrete Hamiltonian system** $(H_{d+}, \Delta_Q^d, X_d)$ is,

$$\left(X_d^k, dH_{d+}(q_k^0, p_k^1) \right) \in D_{\Delta_Q}^{d+},$$

which gives the **implicit discrete Hamilton's equations**,

$$p_k^0 - D_1 H_{d+}(q_k^0, p_k^1) \in \Delta_Q^\circ(q_k^0), \qquad q_{k+1}^0 = D_2 H_{d+}(q_k^0, p_k^1),$$

$$p_k^1 - p_{k+1}^0 \in \Delta_Q^\circ(q_k^1), \qquad (q_k^0, q_{k+1}^0) \in \Delta_Q^d,$$

Connections to Mechanics on Lie algebroids[7]

■ **Tulczyjew's triple on Lie algebroids**

$$(\mathcal{L}^{\tau^*}E)^* \xleftarrow{\quad \flat_{E^*} \quad} \mathcal{L}^{\tau^*}E \equiv \rho^*(TE^*) \xrightarrow{\quad A_E \quad} (\mathcal{L}^{\tau}E)^*$$

$$(\tau^{\tau^*})^* \qquad \tau^{\tau^*} \qquad pr_1 \qquad (\tau^{\tau})^*$$

$$E^* \qquad\qquad\qquad E$$

■ **Tulczyjew's triple on tangent bundles**

$$T^*T^*Q \xleftarrow{\quad \Omega^\flat \quad} TT^*Q \xrightarrow{\quad \kappa_Q \quad} T^*TQ$$

$$\pi_{T^*Q} \qquad \tau_{T^*Q} \qquad T\pi_Q \qquad \pi_{TQ}$$

$$T^*Q \qquad\qquad\qquad TQ$$

[7]Joint work with Diana Sosa Martín (La Laguna)

Connections to Mechanics on Lie algebroids

■ Dirac Mechanics on Lie algebroids

- Introduce the Lie algebroid analogue of the Pontryagin bundle,

$$E \oplus E^*.$$

- Construct the Lie algebroid analogue of the Dirac structure by using the two vector bundle isomorphisms,

$$A_E : \rho^*(TE^*) \to (\mathcal{L}^\tau E)^*$$
$$\flat_{E^*} : \mathcal{L}^{\tau^*} E \to (\mathcal{L}^{\tau^*} E)^*$$

- Generalizes Dirac mechanics to Lie algebroids, thereby unifying Lagrangian and Hamiltonian mechanics on Lie algebroids.

- Interesting to consider the Lie groupoid analogue of the Tulczyjew's triple, viewed as a generalization of discrete Dirac mechanics.

Connections to Multisymplectic Classical Field Theories

■ Tulczyjew's triple in classical field theories

- Bundle $\pi_{XY} : Y \to X$.

- Lagrangian density $\mathbb{L} : Z \to \Lambda^{n+1}X$, for first-order field theories $Z = J^1Y$.

- We have the following Tulczyjew's triple,

$$
\begin{array}{ccccc}
\Lambda_2^{n+1}Z^* & \xleftarrow{\quad\tilde{\beta}\quad} & \widetilde{J^1Z^*} & \xrightarrow{\quad\tilde{\alpha}\quad} & \Lambda_2^{n+1}Z \\
& \searrow{\scriptstyle\pi_{Z^*\Lambda_2^{n+1}Z^*}} \quad \swarrow{\scriptstyle\tilde{\rho}} & & \searrow{\scriptstyle\widetilde{j^1\pi_{YZ^*}}} \quad \swarrow{\scriptstyle\pi_{Z\Lambda_2^{n+1}Z}} & \\
& Z^* & & Z &
\end{array}
$$

- Provides a means of developing multisymplectic Dirac mechanics for classical field theories.

Conclusion

■ Discrete Dirac Structures

- We have constructed a discrete analogue of a Dirac structure by considering the geometry of generating functions of symplectic maps.

- Unifies geometric integrators for degenerate, interconnected, and nonholonomic Lagrangian and Hamiltonian systems.

- Provides a characterization of the discrete geometric structure associated with Hamilton–Pontryagin integrators.

■ Discrete Hamilton–Pontryagin principle

- Provides a unified discrete variational principle that recovers both the discrete Hamilton's principle, and the discrete Hamilton's phase space principle.

- Is sufficiently general to characterize all near to identity Dirac maps.

Conclusion

■ Current Work and Future Directions

- Discrete Dirac structures are intimately related to the geometry of Lagrangian submanifolds and the Hamilton–Jacobi equation.

- Derive the Dirac analogue of the Hamilton–Jacobi equation, with nonholonomic Hamilton–Jacobi theory as a special case.

- Discrete Augmented Variational Principles, with the Hamilton–Pontryagin principle, Clebsch variational principle, optimal control and the symmetric representation of rigid bodies as special cases.

- Continuous and discrete Dirac mechanics on Lie algebroids and Lie groupoids.

- Continuous and discrete multisymplectic Dirac mechanics.

Questions?

M. Leok, T. Ohsawa, Discrete Dirac Structures and Variational Discrete Dirac Mechanics, arXiv:0810.0740