Conformal metrics of constant curvature on planar domains

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J.A. Gálvez, P. Mira, The Liouville equation in a half-plane, *J. Diff. Equations* (2009).

The Liouville equation

Consider the following classical nonlinear problem:

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \mathbb{R}^2_+ = \{(s,t) \in \mathbb{R}^2 : t > 0\}, \\ \dfrac{\partial u}{\partial t} = g(u) & \text{on } \partial\mathbb{R}^2_+. \end{cases}$$

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Here, we will make the following choices:

$$\begin{cases} \Delta u + 2Ke^u = 0 & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial u}{\partial t} = -2\kappa e^{u/2} & \text{on } \partial\mathbb{R}^2_+ \equiv \mathbb{R}, \quad K, \kappa \in \mathbb{R}. \end{cases} \qquad \mathbf{(P)}$$

The equation $\Delta u + 2Ke^u = 0$ is called the *Liouville equation*.

Geometrical interpretation

A conformal metric $ds^2 = e^u(dx^2 + dy^2)$ on a planar domain $\Omega \subset \mathbb{R}^2$ satisfies

$$\Delta u + 2Ke^u = 0,$$

where K is the Gaussian curvature of ds^2.

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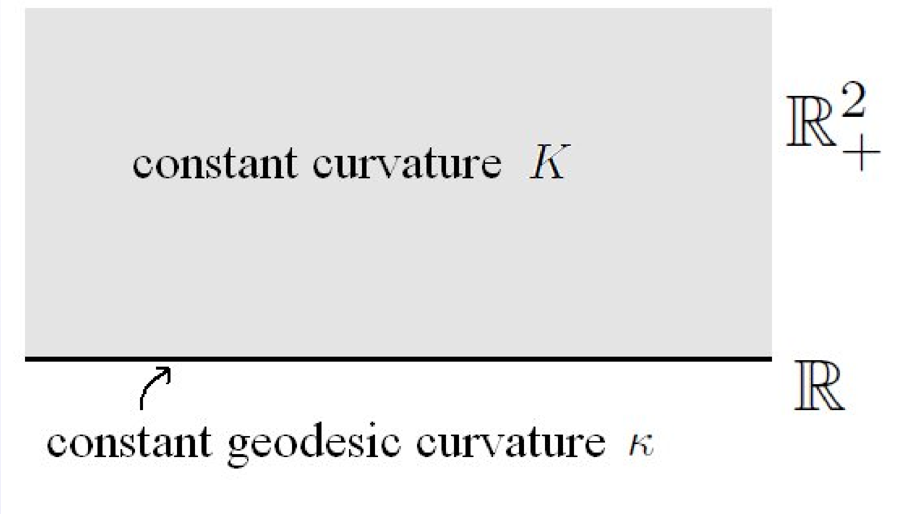
The boundary condition $u_t = -2\kappa e^{u/2}$ on $\mathbb{R} \equiv \partial\mathbb{R}^2_+$ means that ds^2 has constant geodesic curvature $\kappa \in \mathbb{R}$ on the boundary.

The half-plane problem



Previous results

(1) Y.Y. Li, M. Zhu (1995): Any solution to $(\mathbf{P})$ for $K = 1$ with

$$\int_{\mathbb{R}^2_+} e^u < +\infty, \qquad \int_{\mathbb{R}} e^{u/2} < +\infty$$

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(3) Y.Y. Li, M. Zhu (1995), and Chipot-Shafir-Fila (1996): Any solution to

$$\begin{cases} \Delta u + a u^{\frac{n+2}{n-2}} = 0, \ u > 0, & \text{in } \mathbb{R}^n_+, \\ \dfrac{\partial u}{\partial x_n} = c u^{\frac{n}{n-2}} & \text{on } \partial\mathbb{R}^n_+ \end{cases}$$

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Our objectives...

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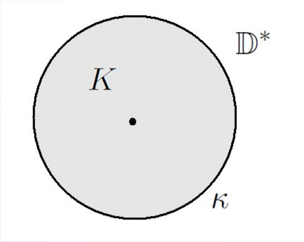
(II) To solve the analogous problem in $\mathbb{D}^*$:

$$\begin{cases} \Delta u + 2Ke^u = 0 & \text{in } \mathbb{D}^* = \{z \in \mathbb{R}^2 \equiv \mathbb{C} : 0 < |z| < 1\}, \\ \dfrac{\partial u}{\partial \nu} = -2\kappa e^{u/2} + 2 & \text{on } \mathbb{S}^1 = \{z : |z| = 1\}. \end{cases}$$



Other related theories

- Complex analysis.
- Minimal surfaces in $\mathbb{R}^3$ and maximal surfaces in $\mathbb{L}^3$.
- Constant mean curvature surfaces in $\mathbb{H}^3$ and $\mathbb{S}^3_1$.
- Flat surfaces in $\mathbb{H}^3$ and $\mathbb{S}^3_1$.
- Linear Weingarten surfaces.

The Neumann problem in $\mathbb{R}^2_+$

The Liouville equation and complex analysis

We fix $K \in \{-1, 1\}$ and identify $\mathbb{C} \equiv \mathbb{R}^2$ and $\mathbb{C}_+ \equiv \mathbb{R}^2_+$.

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Liouville's theorem:
Solutions to $\Delta u + 2Ke^u = 0$ on $\Omega \subset \mathbb{C}$ simply connected are:

$$u = \log \frac{4|g'|^2}{(1 + K|g|^2)^2}.$$

Here g is meromorphic (holomorphic with $|g| < 1$ if $K = -1$) with $g' \neq 0$ (and conversely).

Note: the developing map g gives a global isometric immersion of $(\Omega, e^u|dz|^2)$ into $\mathbb{Q}^2(K)$.

The extension lemma

Let u be a solution to

$$\begin{cases} \Delta u + 2Ke^u = 0 & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial u}{\partial t} = -2\kappa e^{u/2} & \text{on } \partial\mathbb{R}^2_+ \equiv \mathbb{R}, \quad K, \kappa \in \mathbb{R}. \end{cases} \qquad (\mathbf{P})$$

Then it holds

$$u_{zz} - \frac{1}{2}u_z^2 = \{g, z\} := \left(\frac{g''}{g'}\right)' - \frac{1}{2}\left(\frac{g''}{g'}\right)^2 \qquad (=: Q).$$

By the boundary condition, $\mathrm{Im} Q = 0$ on $\mathbb{R}$.

By Schwarz's reflection principle, Q (and g) can be meromorphically extended to $\mathbb{C}$.

The Cauchy problem for $\Delta u + 2Ke^u = 0$

The unique solution to the Cauchy problem

$$\begin{cases} \Delta u + 2Ke^u & = & 0, \\ u(s,0) & = & a(s), \\ u_t(s,0) & = & d(s) \end{cases}$$

can be constructed as follows.

The Cauchy problem for $\Delta u + 2Ke^u = 0$

The unique solution to the Cauchy problem

$$\begin{cases} \Delta u + 2Ke^u &= 0, \\ u(s,0) &= a(s), \\ u_t(s,0) &= d(s) \end{cases}$$

can be constructed as follows.Let $\alpha(s)$ be the unique curve in $\mathbb{Q}^2(K)$ with

$$v(s) = \int e^{a(r)/2} dr, \qquad \kappa_g(s) = \frac{-d(s)}{2e^{a(s)/2}}.$$

Let $g(s) := \pi(\alpha(s))$ denote its stereographic projection on $\bar{\mathbb{C}}$, and extend it holomorphically to $g(z)$. Then,

$$u = \log \frac{4|g'|^2}{(1 + K|g|^2)^2}.$$

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If $u_t(s,0) = -2\kappa e^{u(s,0)/2}$, then $\kappa_g(s) = \kappa \equiv$ constant !!!

The solution for $K = 1$

Let $u : \mathbb{C}_+ \to \mathbb{R}$ be a solution to $(\mathbf{P})$ for $K = 1$. Then, its developing map g is, over $\mathbb{R}$, of the form

$$g(s) = \alpha \exp\left(i \int^s \mu(r) dr \right),$$

where

$$\alpha := \frac{\operatorname{sg}(\kappa)}{\sqrt{\kappa^2 + 1} - |\kappa|}, \qquad \mu(s) := \operatorname{sg}(\kappa) \sqrt{\kappa^2 + 1}\, e^{a(s)/2}.$$

By the properties of g we see that $h(s) := 1/\mu(s)$ satisfies:

- $h(s) \neq 0$ and it can be extended to an entire function $h(z)$.
- $h(z)$ only has simple zeros with $h'(z_0) = \pm i$ at them.

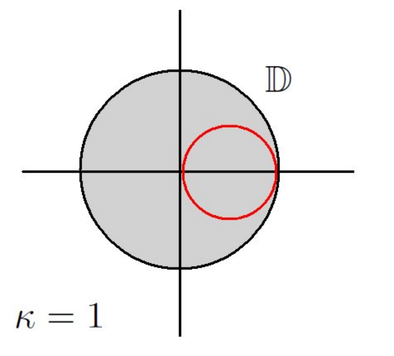
(And conversely...)

The solution for $K = -1$ and $\kappa \geq 1$

If $\kappa > 1 \Rightarrow$ similar to $K = 1$.

If $\kappa = 1$, then $g(\mathbb{C}_+) \subset \mathbb{D}$ and

$$g(s) = \frac{h(s)}{h(s) + 2i}, \qquad h(s) := \int_{s_0}^{s} e^{a(r)/2} dr, \qquad a(s) := u(s, 0)$$



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$h(s)$ extends to $\mathbb{C}$ with $h(\mathbb{C}_+) \subset \mathbb{C}_+$ and $h(\mathbb{C}_-) \subset \mathbb{C}_-$.

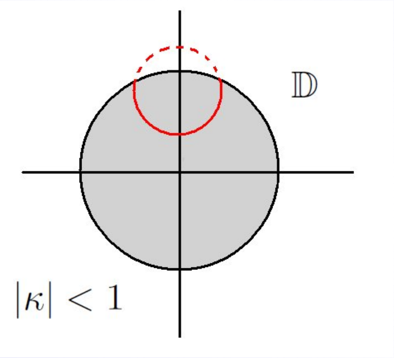
By the little Picard theorem, $h(z) = h_0\, z + h_1$ and so

$$u(s, t) = \log\left(\frac{h_0^2}{(1 + h_0 t)^2}\right).$$

$K = -1$ and $|\kappa| < 1$ is impossible

- $g(\mathbb{C}_+) \subset \mathbb{D}$.
- $g(\mathbb{R}) \subset C_\kappa$.
- $g(\bar{z}) = J(g(z))$.

So, $g(\mathbb{C})$ omits infinitely many points (Contradiction!).



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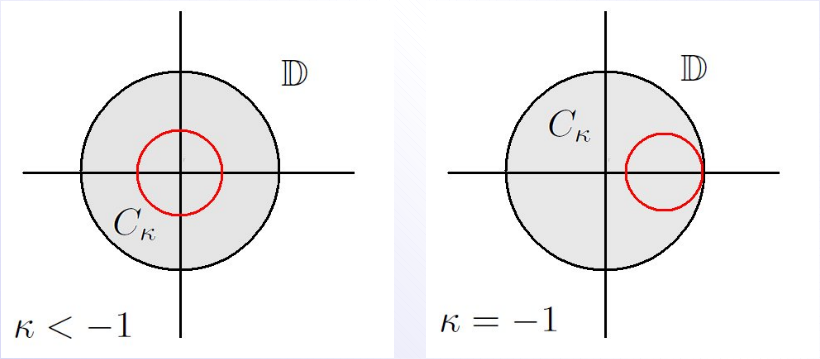

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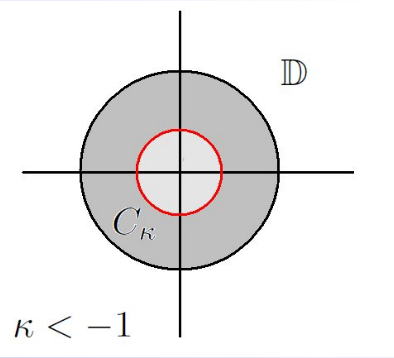
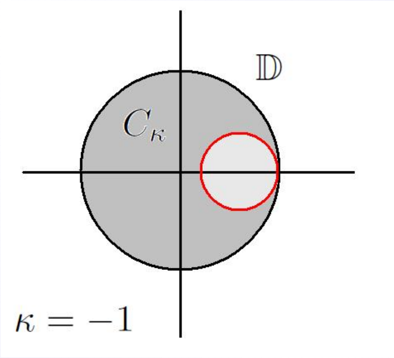




To sum up

We have obtained for the problem

$$\begin{cases} \Delta u + 2Ke^u = 0 & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial u}{\partial t} = -2\kappa e^{u/2} & \text{on } \partial\mathbb{R}^2_+ \equiv \mathbb{R}. \end{cases} \qquad (\mathbf{P})$$

- If $K = -1$ and $\kappa < 1$ $\Rightarrow$ the problem does not have a solution.
- If $K = -1$ and $\kappa = 1$ $\Rightarrow$ the unique solution is

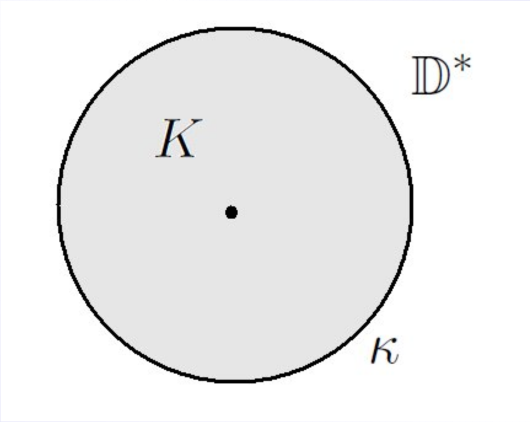
$$u(s,t) = \log\left(\frac{h_0^2}{(1+h_0 t)^2}\right).$$

- In the remaining cases $\Rightarrow$ there is an enormous family of solutions, all of which can be described by entire functions.

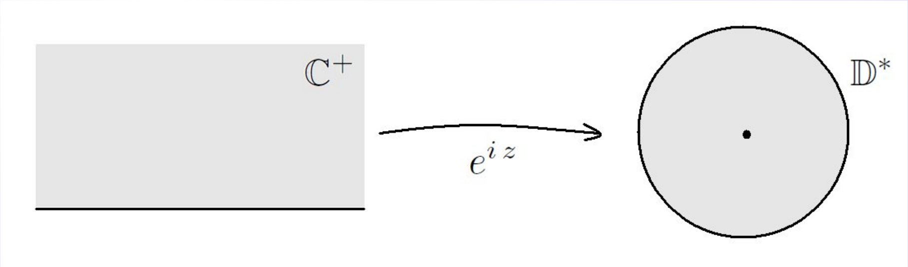
The punctured disc problem

Formulation of the problem

$$\begin{cases} \Delta u + 2Ke^u = 0 & \text{in } \mathbb{D}^*, \\ \dfrac{\partial u}{\partial \nu} = -2\kappa e^{u/2} + 2 & \text{on } \mathbb{S}^1, \quad K, \kappa \in \mathbb{R}. \end{cases} \qquad \textbf{(PDP)}$$



Reduction to the half-plane case



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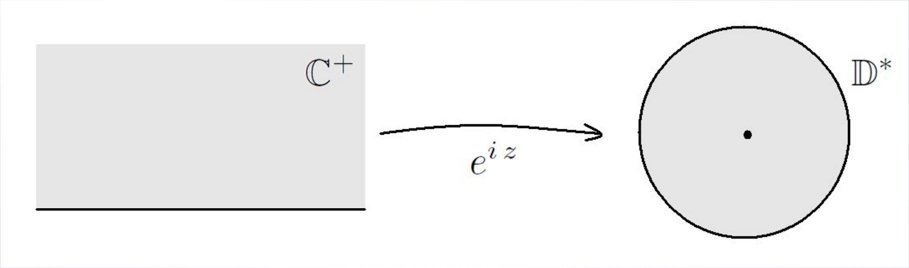

- Solutions to (**PDP**) come from 2π-periodic solutions to (**P**).
- In particular, (**PDP**) does not have a solution for $K = -1$ and $\kappa < 1$.

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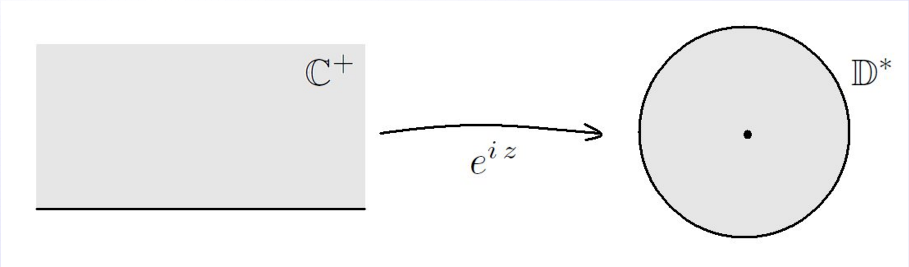

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- In particular, $(\mathbf{PDP})$ does not have a solution for $K = -1$ and $\kappa < 1$.

What are the finite area solutions to this problem?

The finite area case

Theorem: Any solution to $(\mathbf{PDP})$ such that

$$\int_{\mathbb{D}^*} e^u < \infty$$

is radially symmetric, i.e. $u = u(r)$ where $r = |z|$.

The finite area case

Theorem: Any solution to (**PDP**) such that

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is radially symmetric, i.e. $u = u(r)$ where $r = |z|$.

Moreover, all solutions can be explicitly given by simple expressions.

For instance, if $K = 1$ and $\kappa \geq 0$, then

$$u(r) = 2 \log \frac{2R\beta r^{\beta-1}}{1 + R^2 r^{2\beta}},$$

where

$$R := \frac{1}{\sqrt{\kappa^2 + 1} - |\kappa|}.$$

Sketch of proof: the solution u on $\mathbb{D}^*$ is

$$u = \log \frac{4|G'(\zeta)|^2}{(1+|G(\zeta)|^2)^2}, \qquad G(\zeta) \text{ multivalued on } \mathbb{C}.$$

- $Q^* := \{G, \zeta\}$ is holomorphic on $\mathbb{D}^*$, and

$$\operatorname{Im}(\zeta^2 Q^*) = 0 \qquad \text{on } \mathbb{S}^1.$$

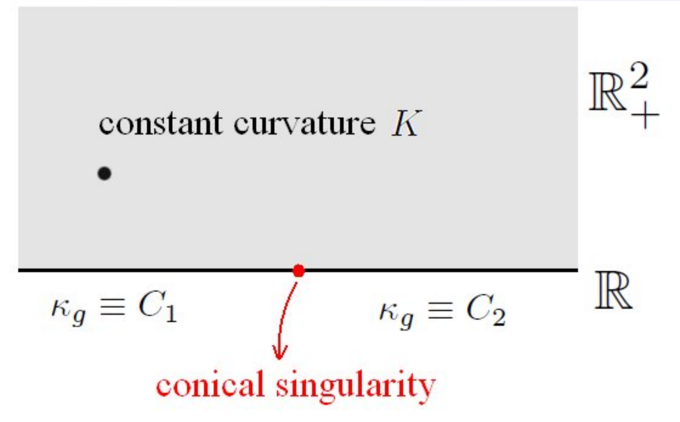
- $G(\zeta) = \zeta^\alpha F(\zeta)$, F single valued on $\mathbb{C}^*$. By the finite area assumption, F is meromorphic at 0 (Chou-Wan, 1994).

- $Q^* = r_0/\zeta^2$ and so $G(\zeta) = \mathcal{M}(\zeta^\beta)$.

- By $|G(\zeta)| = R$ on $\mathbb{S}^1$, then $\mathcal{M}(\zeta) = Re^{i\theta}\zeta$ and the result follows.

An open problem

The half-plane problem with corners



Jost, Wang, Zhou $\Rightarrow$ Classification for $K = 1$ when

$$\int_{\mathbb{R}^2_+} e^u < \infty, \qquad \int_{\mathbb{R}} e^{u/2} < \infty.$$