Vakonomic Constraints in Higher-Order Classical Field Theory

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Order of the day

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The Skinner-Rusk formalism in HOFT

Joint work with:

- Manuel de León, *ICMAT*
- David Martín de Diego, *ICMAT*
- Joris Vankerschaver, *CalTech*

C. M. Campos, M. de León, D. Martín de Diego, J. Vankerschaver.
Unambiguous formalism for higher-order Lagrangian field theories
To appear in J. Phys. A, arXiv:0906.0389v2.

C. M. Campos, M. de León, D. Martín de Diego.
Vakonomic constraints in higher-order field theories
Work in progress.

Introduction

Classically we have...

- The Euler-Lagrange equations: $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0.$
- The Hamilton equations: $\frac{\partial H}{\partial q_i} = -\dot{p}^i$, $\frac{\partial H}{\partial p^i} = \dot{q}_i$.
- The Cartan form: $\Omega_L := -\mathrm{d}\Theta_L = \mathrm{d}q^i \wedge \mathrm{d}\hat{p}_i$.
- The Legendre transform: $FL(q^i, \dot{q}^i) = (q^i, \hat{p}_i = \frac{\partial L}{\partial \dot{q}^i})$.
- There is an equivalence between the Lagrangian y Hamiltonian formalisms.

Introducción

Question: ¿what can we do in degenerate cases?

Mark J. Gotay, James M. Nester, and George Hinds.
Presymplectic manifolds and the Dirac-Bergmann theory of constraints.
J. Math. Phys. **19** (1978), no. 11, 2388–2399.

Alternative: to combine the phase space and space of velecities.

Ray Skinner and Raymond Rusk.
Generalized Hamiltonian dynamics. I. Formulation on $T^*Q \oplus TQ$.
J. Math. Phys. **24** (1983), no. 11, 2589–2594.

Adaptation: Classical Field Theory.

M. de León, J. C. Marrero, and D. Martín de Diego.
A new geometric setting for classical field theories.
Banach Center Publ., vol. 59, Polish Acad. Sci., Warsaw, 2003.

Others: Cariñena, Gràcia, Muñoz, Pons, Román-Roy, Ibort, *et al.*

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Introduction

- General framework that recovers the well known tools of mechanics ($m = 1$).
- Nice description of the Euler-Lagrange equations.
- Cartan forms may be obtained, but not canonically.
- There is no well defined Legendre transform.

D. J. Saunders and M. Crampin.
On the Legendre map in higher-order field theories
J. Phys. A **23** (1990), no. 14, 3169–3182.

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Different approaches

1. Time-independent mechanics: symplectic framework.
2. Time-dependent mechanics: cosymplectic framework.
3. Classical field theory (and higher-order): multisymplectic framewrok.

Huge literature on regard to these subjects.

- M. de León, J. C. Marrero, and D. Martín de Diego.
  A new geometric setting for classical field theories.
  Banach Center Publ., vol. 59, Polish Acad. Sci., Warsaw, 2003.
- A. Echeverría-Enríquez, C. López, J. Marín-Solano, M.C. Muñoz-Lecanda, N. Román-Roy.
  Lagrangian-Hamiltonian unified formalism for field theory.
  J. Math. Phys. **45** (2004), no. 1, 360-380.

1. Let $\pi : E \longrightarrow M$ be a fiber bundle ($\dim M = m$ and $\dim E = m + n$).
2. $J^1\pi$ denotes its first prolongation, the first jet bundle.
3. $J^1\pi^\dagger$ denotes its affine dual.
4. Coordinates:
   - (x^i) for M, $\mathrm{d}^m x = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m$, $\mathrm{d}^{m-1}x_i = i_{\partial_i}\mathrm{d}^m x$,
   - (x^i, u^α) for E,
   - $(x^i, u^\alpha, u_i^\alpha)$ for $J^1\pi$,
   - $(x^i, u^\alpha, p, p_\alpha^i)$ for $J^1\pi^\dagger$.
5. The Lagrangian function $L : J^1\pi \longrightarrow \mathbb{R}$.
6. The mixed space of velocities and momenta: $W = J^1\pi \times_E J^1\pi^\dagger$.
7. The pairing $\Phi : w \in W \mapsto \langle pr_2(w), pr_1(w)\rangle = p + p_\alpha^i u_i^\alpha \in \mathbb{R}$.
8. The dynamical function $H = \Phi - L \circ pr_1$.
9. The premultisymplectic $(m+1)$-form $\Omega = pr_2^*\Omega_{J^1\pi^\dagger}$.
10. The dynamical equation $i_X\Omega = \mathrm{d}H$.

Note: we have not mentioned neither the Legendre transform, nor the Cartan form.

The multi-index notation

Definition

- A multi-index is an m-tuple $I \in \mathbb{N}^m$ whose i-th component is $I(i)$.
- The sum and "substraction" is defined componentwise $(I \pm J)(i) = I(i) \pm J(i)$.
- The length of I is the sum $|I| = \sum_i I(i)$, and its factorial $I! = \Pi_i I(i)!$.
- $1_i = (\delta_j^i) = (0, \ldots, 1, \ldots, 0)$.

The partial derivatives of a function $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ are

$$f_I = \frac{\partial^{|I|} f}{\partial x^I} := \frac{\partial^{I(1)+I(2)+\cdots+I(m)} f}{\partial x_1^{I(1)} \partial x_2^{I(2)} \ldots \partial x_m^{I(m)}}.$$

For instance, given $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$,

$$f_{(2,1,0)} = \frac{\partial^3 f}{\partial x_1^2 \partial x_2} = f_{(1,1,0)+1_1} = f_{(2,0,0)+1_2}.$$

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El formalismo de Skinner y Rusk de orden superior

1. Let $\pi : E \longrightarrow M$ be a fiber bundle ($\dim M = m$ and $\dim E = m + n$).
2. $J^k\pi$ is the kth-jet bundle and $J^k\pi^\dagger$ is its affine dual.
3. Coordinates:
   - (x^i, u^α_J) for $J^1\pi$, where $|J| \leq k$,
   - $(x^i, u^\alpha_I, p, p^{I,i}_\alpha)$ for $J^1\pi^\dagger$, where $|I| \leq k - 1$.
4. The Lagrangian function $L : J^k\pi \longrightarrow \mathbb{R}$.
5. The mixed space: $W := J^k\pi \times_{J^{k-1}\pi} J^k\pi^\dagger$.
6. The pairing $\Phi(x^i, u^\alpha_I, u^\alpha_K, p^{I,i}_\alpha, p) = p^{I,i}_\alpha u^\alpha_{I+1_i} + p$.
7. The dynamical function $H := \Phi - L \circ pr_1$.
8. The canonical multisymplectic $(m+1)$-form
   $\Omega = -\mathrm{d}p \wedge \mathrm{d}^m x - \mathrm{d}p^{I,i}_\alpha \wedge \mathrm{d}u^\alpha_I \wedge \mathrm{d}^{m-1}x_i$.
9. The premultisymplectic $(m+1)$-form $\Omega_H := \Omega + \mathrm{d}H \wedge \eta$.

We look for Ehresmann conexions in the fiber bundle $\pi_{W,M} : W \longrightarrow M$ whose horizontal projector $\mathbf{h}$ satisfies

$$i_{\mathbf{h}}\Omega_H = (m-1)\Omega_H.$$

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Solving the dynamical equation

In first place, we restrict to the space where such solutions exist:

$$W_1 := \{ w \in W \ / \ \exists \mathbf{h} : T_w W \longrightarrow T_w W \text{ lineal t.q. } \mathbf{h}^2 = \mathbf{h}, \ \ker \mathbf{h} = (V\pi_{W,M})_w, i_{\mathbf{h}}\Omega_H(w) = (m-1)\Omega_H(w) \}.$$

The projectors must be of the form:

$$\mathbf{h} = \left(\frac{\partial}{\partial x^i} + A^\alpha_{Ji} \frac{\partial}{\partial u^\alpha_J} + B^{Ij}_{\alpha i} \frac{\partial}{\partial p^{I,j}_\alpha} + C_j \frac{\partial}{\partial p} \right) \otimes \mathrm{d}x^i.$$

After some computation...

... we finally obtain:

$$B_{\alpha j}^{\ j} = \frac{\partial L}{\partial u^\alpha}; \tag{1}$$

$$\sum_{I+1_j=J} p_\alpha^{I,i} = \frac{\partial L}{\partial u_J^\alpha} - B_{\alpha j}^{Jj}, \text{ where } |J| = 1, \ldots, k-1; \tag{2}$$

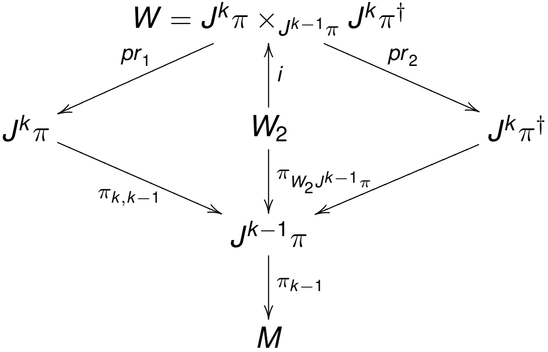
$$\sum_{I+1_j=K} p_\alpha^{I,i} = \frac{\partial L}{\partial u_K^\alpha}, \text{ where } |K| = k; \tag{3}$$

$$A_{Ii}^\alpha = u_{I+1_i}^\alpha, \text{ where } |I| = 0, \ldots, k-1.$$

Adding the contition $H(w) = 0$ ($p = L - p^{I,i} u_{I+1_i}$), we define

$$W_2 := \{w \in W_1 \ : \ H(w) = 0\} = \{w \in W \ : \ (3) \text{ and } H(w) = 0\}.$$

Framework



$$W_2 = \left\{ w \in W \;:\; \sum_{I+1_i = K} p_\alpha^{I,i} = \frac{\partial L}{\partial u_K^\alpha}, \quad p = L - p^{I,i} u_{I+1_i} \right\}$$

Tangency contitions

$$W_2 := \{w \in W_1 \ : \ H(w) = 0\} = \{w \in W \ : \ (3) \text{ and } H(w) = 0\}$$

Now, we have to guarantee that the solutions are tangent to W_2, that is that $\mathbf{h}(T_w W) \subset i_*(TW_2)\ \forall w \in W_2$, which is equivalent to the following equations

$$\begin{aligned} \sum_{I+1_i=K} B^{li}_{\alpha j} &= \frac{\partial^2 L}{\partial x^j \partial u^\alpha_K} + u^\beta_{I+1_j} \frac{\partial^2 L}{\partial u^\beta_I \partial u^\alpha_K} + A^{\alpha'}_{K'j} \frac{\partial^2 L}{\partial u^{\alpha'}_{K'} \partial u^\alpha_K}, \qquad (4) \\ C_j &= \frac{\partial L}{\partial x^j} + A^\alpha_{Jj} \frac{\partial L}{\partial u^\alpha_J} + A^\alpha_{I+1_i j} p^{I,i}_\alpha + B^{li}_{\alpha j} u^\alpha_{I+1_i}. \end{aligned}$$

where $|K| = k$.

The higher-order Euler-Lagrange equations

Let consider an Ehresmann conexion in $\pi_{WM} : W \longrightarrow M$ along W_2 whose horizontal projector $\mathbf{h}$ is a solution of the dynamical equation

$$i_{\mathbf{h}}\Omega_H = (m-1)\Omega_H.$$

Theorem

Let σ be a section of $\pi_{W_2M} : W_2 \longrightarrow M$ and denote $\bar{\sigma} = i \circ \sigma$ and $\phi = \pi_{W_2E} \circ \sigma$. If σ is an integral section of $\mathbf{h}$, then σ is holonomic,

$$pr_1 \circ \bar{\sigma} = j^k\phi,$$

and satisfies the higher-order Euler-Lagrange equations:

$$j^{2k}\phi^* \left(\sum_{|J|=0}^{k} (-1)^{|J|} \frac{\mathrm{d}^{|J|}}{\mathrm{d}x^J} \frac{\partial L}{\partial u_J^\alpha} \right) = 0.$$

Some results

Theorem

Let Γ be a conexion in $\pi_{WM} : W \longrightarrow M$ along W_2 whose horizontal projector $\mathbf{h}$ satisfies

$$i_{\mathbf{h}}\Omega_H = (m-1)\Omega_H.$$

The integral sections of Γ satisfy the DeDonder equations.

Theorem

(W_2, Ω_2) is multisymplectic iff L is regular ($\det\left(\frac{\partial^2 L}{\partial u_K^\alpha \partial u_{K'}^{\alpha'}}\right) \neq 0$).

Some results

Consider the equations in which appear B^{Ii}_j's with $|I| = k - 1$ (equations (2) y (4)),

$$B^{Jj}_{\alpha j} = \frac{\partial L}{\partial u^\alpha_J} - \sum_{I+1_i=J} p^{I,i}_\alpha, \text{ where } |J| = k-1;$$

$$\sum_{I+1_i=K} B^{Ii}_{\alpha j} = \frac{\partial^2 L}{\partial x^j \partial u^\alpha_K} + u^\beta_{I+1_j} \frac{\partial^2 L}{\partial u^\beta_I \partial u^\alpha_K} + A^{\alpha'}_{K'j} \frac{\partial^2 L}{\partial u^{\alpha'}_{K'} \partial u^\alpha_K}, \text{ where } |K| = k.$$

This is a linear system of equations in the B's which is

- overdetermined when $k = 1$ or $m = 1$,
- completly determined when $k = m = 2$,
- not determined otherwise.

Theorem

If $k, m \geq 2$, the above system has maximal rank.

Some results

Theorem

If $k, m \geq 2$, the above system has maximal rank.

Case $k = 2$ and $m = 3$: 11 equations with 12 unknowns.

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

Case $k = 5$ y $m = 6$: 1638 equations with 4536 unknowns.

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Introducing constraints

- The Lagrangian function $L : J^k\pi \longrightarrow \mathbb{R}$.
- The constraint submanifold $\mathcal{C} = \{\Psi^\mu = 0\} \hookrightarrow J^k\pi$, $1 \leq \mu \leq l$.
- The mixed space $W = J^k\pi \times_{J^{k-1}\pi} J^k\pi^\dagger$.
- The restricted mixed space $W_0^{\mathcal{C}} = \left\{ w \in pr_1^{-1}(\mathcal{C}) : H(w) = 0 \right\}$.

Theorem

Given $w \in W_0^{\mathcal{C}}$ and $X \in \Lambda_d^m(T_w W_0^{\mathcal{C}})$, let $\bar{X} = i_* X \in \Lambda_d^m(T_w W)$. We have

$$i_X \Omega_0^{\mathcal{C}} = 0 \quad \Leftrightarrow \quad i_{\bar{X}} \Omega \in T_w^0 W_0^{\mathcal{C}},$$

where $T_w^0 W_0^{\mathcal{C}}$ is the annihilator of $i_*(T_w W_0^{\mathcal{C}})$ in $T_w W$.

We look for solutions of the equation

$$(-1)^m i_{\bar{X}} \Omega = \lambda_\mu \mathrm{d}\Psi^\mu + \lambda \mathrm{d}H, \text{ with } i_{\bar{X}}\eta = 1.$$

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Solving the vakonomic dynamical equation

Let $\bar{X} = \bigwedge_j \left(\frac{\partial}{\partial x^j} + A^\alpha_{Jj} \frac{\partial}{\partial u^\alpha_J} + B^{Ii}_{\alpha j} \frac{\partial}{\partial p^{I,i}_\alpha} + C_j \frac{\partial}{\partial p} \right)$. We obtain that $\lambda = -1$, besides the relations of holonomy, dynamics and tangency

$$
\begin{aligned}
A^\alpha_{Ii} &= u^\alpha_{I+1_i} \\
0 &= \lambda_\mu \frac{\partial \Psi^\mu}{\partial u^\alpha} + \frac{\partial L}{\partial u^\alpha} - B^{\;j}_{\alpha j}, \;\; |J| = 0 \\
\sum_{I+1_i = J} p^{Ii}_\alpha &= \lambda_\mu \frac{\partial \Psi^\mu}{\partial u^\alpha_J} + \frac{\partial L}{\partial u^\alpha_J} - B^{Jj}_{\alpha j}, \;\; |J| = 1, \ldots, k-1 \\
\sum_{I+1_i = K} p^{Ii}_\alpha &= \lambda_\mu \frac{\partial \Psi^\mu}{\partial u^\alpha_K} + \frac{\partial L}{\partial u^\alpha_K}, \;\; |K| = k \\
C_i &= \lambda_\mu \left(\frac{\partial \Psi^\mu}{\partial x^i} + A^\alpha_{Ii} \frac{\partial \Psi^\mu}{\partial u^\alpha_I} \right) + \frac{\partial L}{\partial x^i} + A^\alpha_{Ii} \frac{\partial L}{\partial u^\alpha_I} + \ldots
\end{aligned}
$$

Suppose that the constraints are of maximal order, $\Psi^\mu = u^\alpha_{\hat{K}} - \Phi^\alpha_{\hat{K}} = 0$. With a right manipulation on the relations of dynamics, we obtain

$$
\begin{aligned}
0 + \sum_{I+1_i=\hat{K}} p^{Ii}_\beta \frac{\partial \Phi^\beta_{\hat{K}}}{\partial u^\alpha} &= \frac{\partial \tilde{L}}{\partial u^\alpha} - B^{\,j}_{\alpha j}, \quad |J| = 0 \\
\sum_{I+1_i=J} p^{Ii}_\alpha + \sum_{I+1_i=\hat{K}} p^{Ii}_\beta \frac{\partial \Phi^\beta_{\hat{K}}}{\partial u^\alpha_J} &= \frac{\partial \tilde{L}}{\partial u^\alpha_J} - B^{Jj}_{\alpha j}, \quad |J| = 1, \ldots, k-1 \\
\sum_{I+1_i=\bar{K}} p^{Ii}_\alpha + \sum_{I+1_i=\hat{K}} p^{Ii}_\beta \frac{\partial \Phi^\beta_{\hat{K}}}{\partial u^\alpha_{\bar{K}}} &= \frac{\partial \tilde{L}}{\partial u^\alpha_{\bar{K}}}, \quad |\bar{K}| = k
\end{aligned}
$$

where $\tilde{L}$ is the restricted Lagrangian.

Getting rid of the constraints

More generally, if $\Psi^\mu = u^\mu - \Phi^\mu = 0$, with a right manipulation on the relations of dynamics, we obtain

$$\sum_{\nu+1_i=\bar{\mu}} p_\nu + \sum_{\nu+1_i=\mu} p_\nu \frac{\partial \Phi^\mu}{\partial u^{\bar{\mu}}} = \underbrace{\frac{\partial L}{\partial u^{\bar{\mu}}} + \frac{\partial L}{\partial u^\mu}\frac{\partial \Phi^\mu}{\partial u^{\bar{\mu}}}}_{\frac{\partial \tilde{L}}{\partial u^{\bar{\mu}}}} - B_{\bar{\mu}j}^{\ j} - B_{\mu j}^{\ j}\frac{\partial \Phi^\mu}{\partial u^{\bar{\mu}}},$$

where $\tilde{L}$ is the restricted Lagrangian.

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What we left behind and what is ahead

Conclusions:

- Global framework for field theory.
- There is no well defined Legendre transform or Cartan form.
- The reduction algorithm stops inevitably.
- The cases $k = 1$ (first order), $m = 1$ (mechanics) and $k = m = 2$ are special.

Future work:

- Control.
- Geometric discretization and integration.
- Space plus time decomposition.
- Jets of infinite dimension, $J^\infty\pi$.

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In the end

- Yes, we can.
(Barack Obama)

- ... do so may things. What are we waiting for?

Thank you for your attention and for your votes!