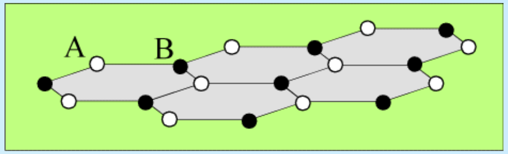
Optics and magneto-optics of graphene

Vladimir Falko



Introduction: symmetries and notations.

Optics and magneto-optics of graphene: absorption.

Abergel, VF - PRB 75, 155430 (2007)
Abergel, Russell, VF - APL 91, 063125 (2007)

Magneto-phonon resonance and filling factor dependent fine structure of the G-line in the Raman spectrum of phonons.

Goerbig, Fuchs, Kechedzhi, VF - PRL 99, 087402 (2007)
Kashuba, VF – unpublished (2009)

Electronic excitations in the Raman spectrum of graphene.

Kashuba, VF – arxiv:09065251 (2009)

Magneto-optics of bilayer graphene.

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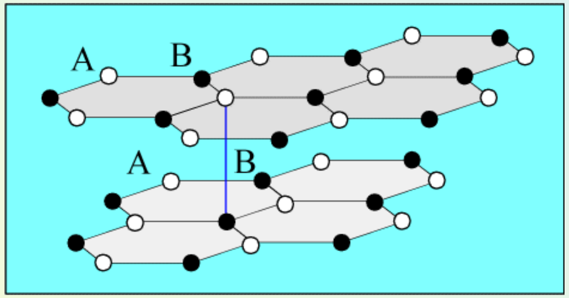
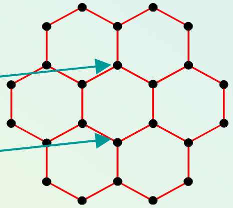
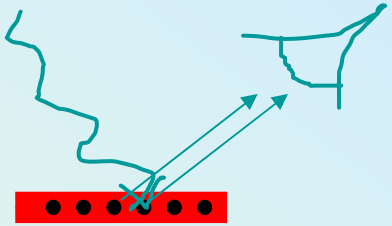
Mucha-Kruczynski, McCann, VF - SSC 149, 1111 (2009)

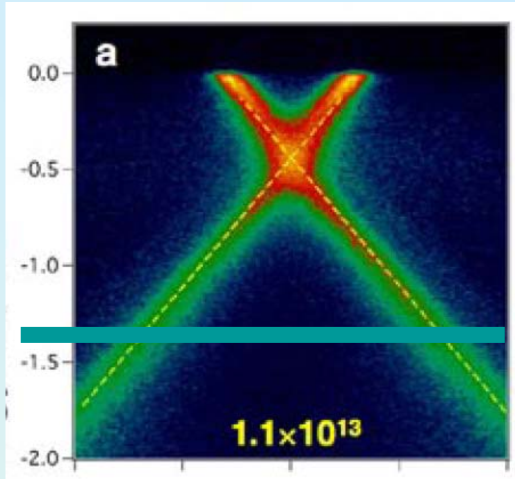


Electrons in graphene as observed in ARPES

$$\psi = \begin{pmatrix} \varphi_A \\ \varphi_B \end{pmatrix}$$





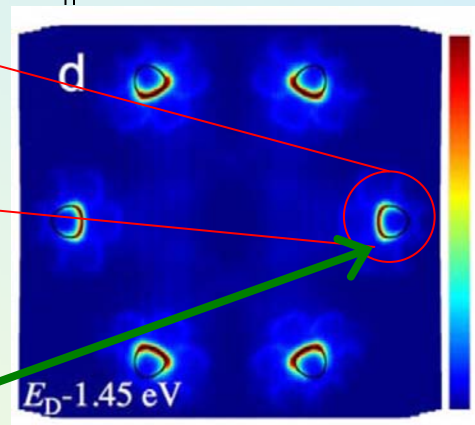


$$\varepsilon = -vp$$

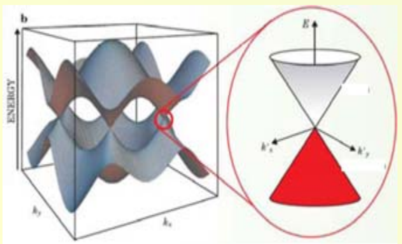
$$I_{ARPES} \sim |\varphi_A + \varphi_B|^2$$

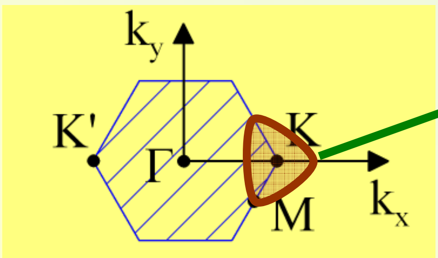
$$\sim \sin^2\left(\frac{\vec{k}\cdot\vec{R}_{BA}}{2} + \frac{\vartheta}{2}\right)$$

$$\vec{k}_{\parallel} = \vec{G} \pm \vec{K} + \vec{p}$$



Mucha-Kruczynski, Tsyplyatyev, Grishin, McCann, VF, Boswick, Rotenberg - PRB 77, 195403 (2008)





ARPES of heavily doped graphene synthesized on silicon carbide

Bostwick *et al* - Nature Physics, 3, 36 (2007)

Electrons in graphene observed using ARPES

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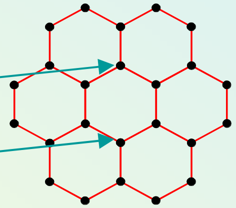
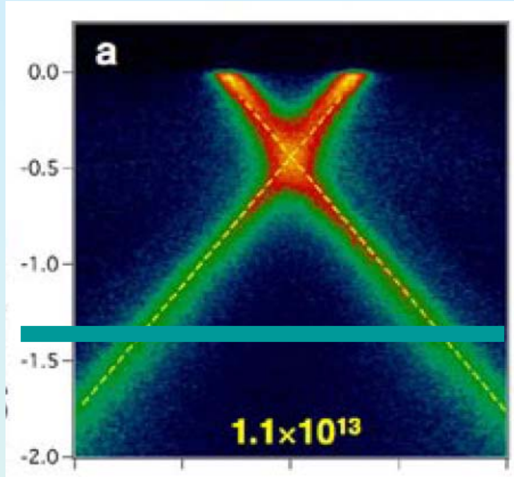


$$\begin{pmatrix} A,K \\ B,K \\ B,K' \\ A,K' \end{pmatrix} \quad \begin{matrix} \varsigma = +1 \\ \varsigma = -1 \end{matrix}$$



$$\varepsilon = -vp$$

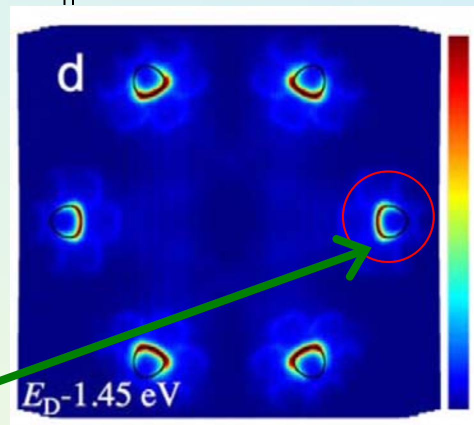
$$\pi = p_x + ip_y \qquad \pi^+ = p_x - ip_y$$

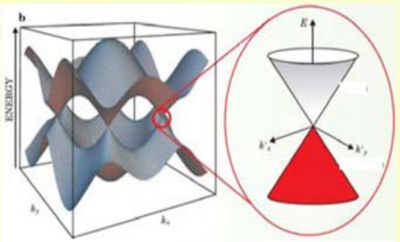
$$\vec{k}_{\parallel} = \vec{G} \pm \vec{K} + \vec{p}$$

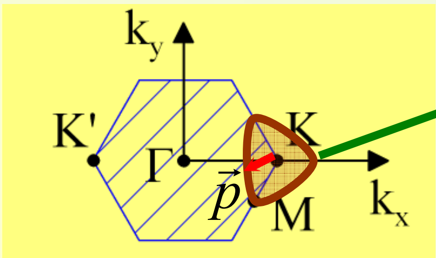
valley

'trigonal warping' terms

$$H_1 \approx \varsigma v \begin{pmatrix} 0 & \pi^+ \\ \pi & 0 \end{pmatrix} + \mu \begin{pmatrix} 0 & \pi^2 \\ (\pi^+)^2 & 0 \end{pmatrix}$$







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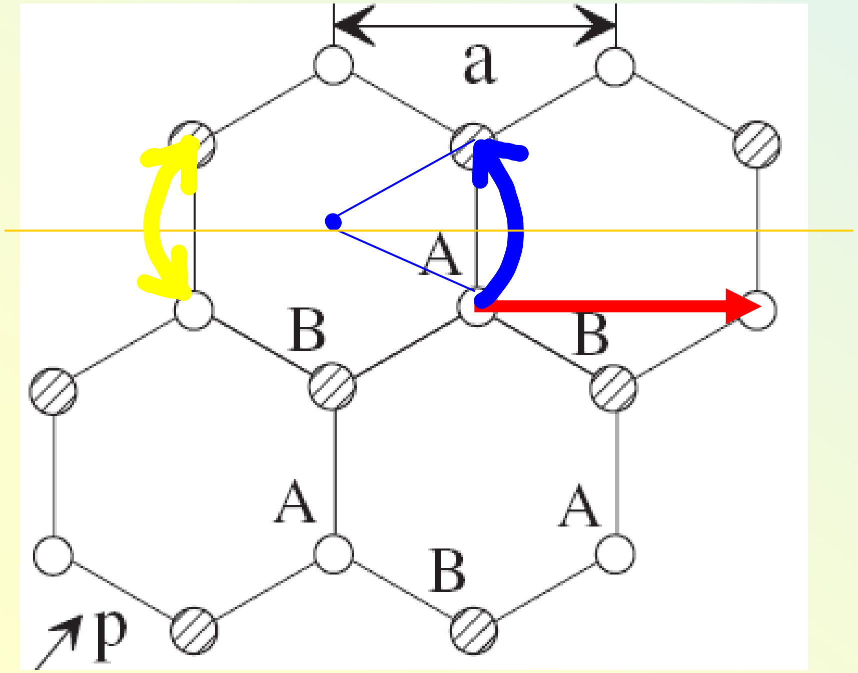
4-dimensional representation of the symmetry group of the honeycomb lattice

$$G\{C_{6v}\otimes T\}$$

$$\psi = \begin{pmatrix} A,K \\ B,K \\ B,K' \\ A,K' \end{pmatrix} \qquad \begin{pmatrix}1\\0\\0\\0\end{pmatrix};\begin{pmatrix}0\\1\\0\\0\end{pmatrix};\begin{pmatrix}0\\0\\1\\0\end{pmatrix};\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Generating elements: $T_{A\to A}, C_{\frac{\pi}{3}}, S_x$



Translation $T_{A\to A}$

$$\begin{pmatrix} e^{i\frac{4\pi}{3}} & & & \\ & e^{i\frac{4\pi}{3}} & & \\ & & e^{-i\frac{4\pi}{3}} & \\ & & & e^{-i\frac{4\pi}{3}} \end{pmatrix}$$

Rotation $C_{\frac{\pi}{3}}$

$A\longleftrightarrow B$

$K\longleftrightarrow K'$

$$\begin{pmatrix} & & e^{i\frac{2\pi}{3}} & \\ & & & e^{-i\frac{2\pi}{3}} \\ e^{i\frac{2\pi}{3}} & & & \\ & e^{-i\frac{2\pi}{3}} & & \end{pmatrix}$$

Mirror reflection S_x

$A\longleftrightarrow B$

$$\begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}$$

Basis of 4x4 matrices: 16 generators of $\mathbf{U_4}$

$$\begin{pmatrix} A,K \\ B,K \\ B,K' \\ A,K' \end{pmatrix} \begin{matrix} \varsigma = +1 \\ \varsigma = -1 \end{matrix}$$

sublattice matrices

$\mathbf{SU_2}$ Lie algebra with

$$\Sigma_x = \begin{bmatrix} \sigma_x & 0 \\ 0 & -\sigma_x \end{bmatrix} \quad \Sigma_y = \begin{bmatrix} \sigma_y & 0 \\ 0 & -\sigma_y \end{bmatrix} \quad \Sigma_z = \begin{bmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{bmatrix}$$

$$[\Sigma_{s_1}, \Sigma_{s_2}] = 2i\varepsilon^{s_1 s_2 s_3} \Sigma_{s_3}$$

valley matrices

$\mathbf{SU_2}$ Lie algebra with

$$\Lambda_x = \begin{bmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{bmatrix} \quad \Lambda_y = \begin{bmatrix} 0 & -i\sigma_z \\ i\sigma_z & 0 \end{bmatrix} \quad \Lambda_z = \begin{bmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{bmatrix}$$

$$[\Lambda_{l_1}, \Lambda_{l_2}] = 2i\varepsilon^{l_1 l_2 l_3} \Lambda_{l_3}$$

$$[\Sigma_s, \Lambda_l] = 0$$

$t \rightarrow -t$

$\vec{\Sigma},\ \vec{\Lambda}$ invert sign

$I,\ \vec{\Sigma} \otimes \vec{\Lambda}$ symmetric

Irreducible matrix representation of $G\{T, C_{6v}\}$

$$\hat{X} \to U[\hat{X}] = \hat{U}^{+} \hat{X} \hat{U}$$

four 1D-representations

four 2D-representations

one 4D-representation

$\Sigma_{(x,y)}\ \Lambda_{(x,y)}$

	$C_{\pi/3}$	s_x	T	
I	1	1	1	A_1
Σ_z	1	-1	1	A_2
$\Lambda_z\Sigma_z$	-1	-1	1	B_1
Λ_z	-1	1	1	B_2

	$C_{\pi/3}$	s_x	T	
$\begin{bmatrix} \Sigma_x \\ \Sigma_y \end{bmatrix}$	$\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	E_1
$\begin{bmatrix} \Lambda_z\Sigma_x \\ \Lambda_z\Sigma_y \end{bmatrix}$	$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	E_2
$\begin{bmatrix} \Lambda_x \\ \Lambda_y \end{bmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	
$\begin{bmatrix} \Lambda_x\Sigma_z \\ \Lambda_y\Sigma_z \end{bmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	

Kechedzhi, McCann, VF, Suzuura, Ando, Altshuler - Eur. Phys. J. Special Topics 148, 39–54 (2007)

Basis of 4x4 matrices: 16 generators of $\mathbf{U_4}$

$$\begin{pmatrix} A,K \\ B,K \\ B,K' \\ A,K' \end{pmatrix} \begin{matrix} \varsigma = +1 \\ \\ \varsigma = -1 \end{matrix}$$

$$\Sigma_{x/y} = \begin{bmatrix} \sigma_{x/y} & 0 \\ 0 & -\sigma_{x/y} \end{bmatrix} \qquad \Sigma_z = \begin{bmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{bmatrix}$$

sublattice matrices

$\mathbf{SU_2}$ Lie algebra with

$$[\Sigma_{s_1}, \Sigma_{s_2}] = 2i\varepsilon^{s_1 s_2 s_3}\Sigma_{s_3}$$

hopping between sublattices

asymmetry between sublattices

$$\Lambda_x = \begin{bmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{bmatrix} \quad \Lambda_y = \begin{bmatrix} 0 & -i\sigma_z \\ i\sigma_z & 0 \end{bmatrix} \qquad \Lambda_z = \begin{bmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{bmatrix} \rightarrow \zeta$$

valley matrices

$\mathbf{SU_2}$ Lie algebra with:

$$[\Lambda_{l_1}, \Lambda_{l_2}] = 2i\varepsilon^{l_1 l_2 l_3}\Lambda_{l_3}$$

intervalley scattering

asymmetry between valleys

$$[\Sigma_s, \Lambda_l] = 0$$

$t \rightarrow -t$ — $\vec{\Sigma},\ \vec{\Lambda}$ invert sign; $I,\ \vec{\Sigma} \otimes \vec{\Lambda}$ symmetric

Dirac term

warping term

$$\hat{H} \approx v\vec{\Sigma}\cdot\vec{p} - \frac{v^2}{6\gamma_0}\zeta\, \Sigma_x\left(\vec{\Sigma}\cdot\vec{p}\right)\Sigma_x\left(\vec{\Sigma}\cdot\vec{p}\right)\Sigma_x$$

$t \to -t$

$\vec{\Sigma},\ \vec{\Lambda}$

invert sign

$\vec{\Sigma}\otimes\vec{\Lambda}$

symmetric

the same
in both valleys

asymmetric
between
valleys

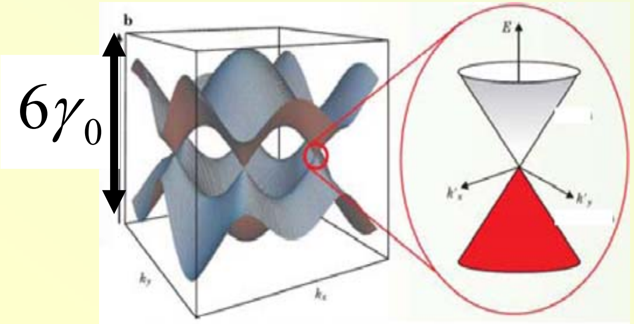
$$\Sigma_{x/y} = \begin{bmatrix} \sigma_{x/y} & 0 \\ 0 & -\sigma_{x/y} \end{bmatrix}$$

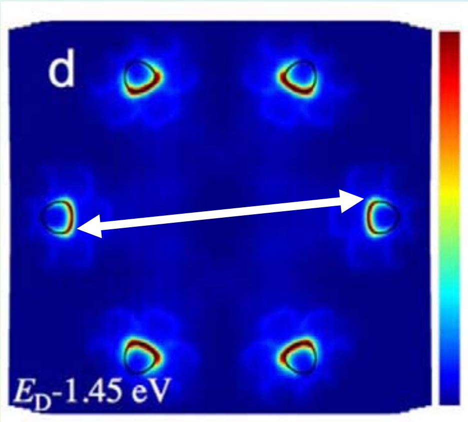
$$\Lambda_z = \begin{bmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{bmatrix} \to \xi$$

captures

$K, \vec{p} \Leftrightarrow K', -\vec{p}$

symmetry





McCann, Kechedzhi, VF, Suzuura, Ando, Altshuler - PRL 97, 146805 (2006)

Dirac electron interaction with photons

$$\hat{H} = v\vec{\Sigma}\cdot(\vec{p} - \tfrac{e}{c}\vec{A})$$

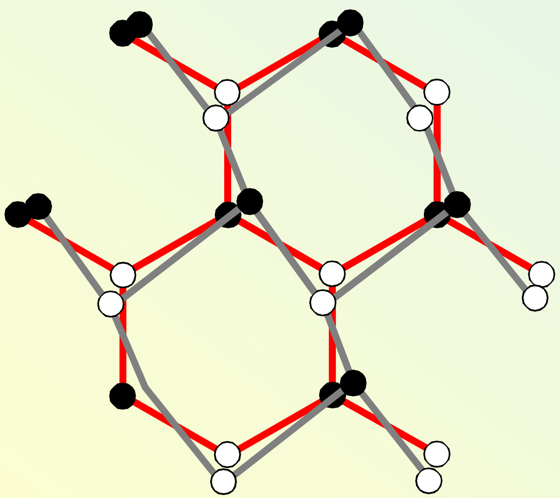
positive parity (valley-symmetric):
same in both valleys

$$\Lambda_z = \begin{bmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{bmatrix} \to \zeta$$

nagative ‘parity’ (opposite signs in different valleys):
valley-antisymmetric

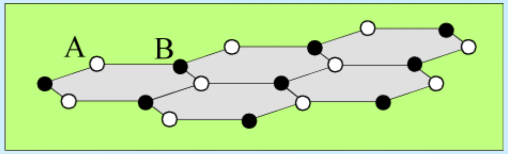
$$\hat{H} = v\vec{\Sigma}\cdot\vec{p} - \zeta\,\vec{\Sigma}\cdot(\vec{\ell}_z \times \vec{u})g\sqrt{2M\omega_0}$$

Γ-point optical phonons (‘G-line’)



$$\vec{u} = \vec{u}_A - \vec{u}_B$$

Ferrari, *et al*, PRL 97, 187401 (2006)



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Abergel, VF - PRB 75, 155430 (2007)
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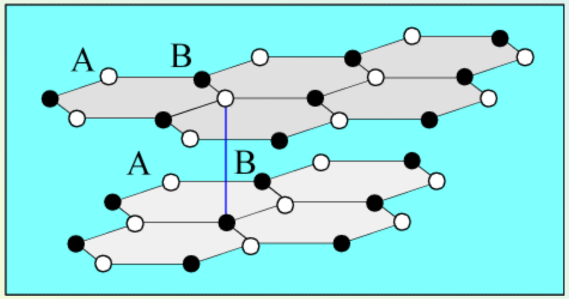
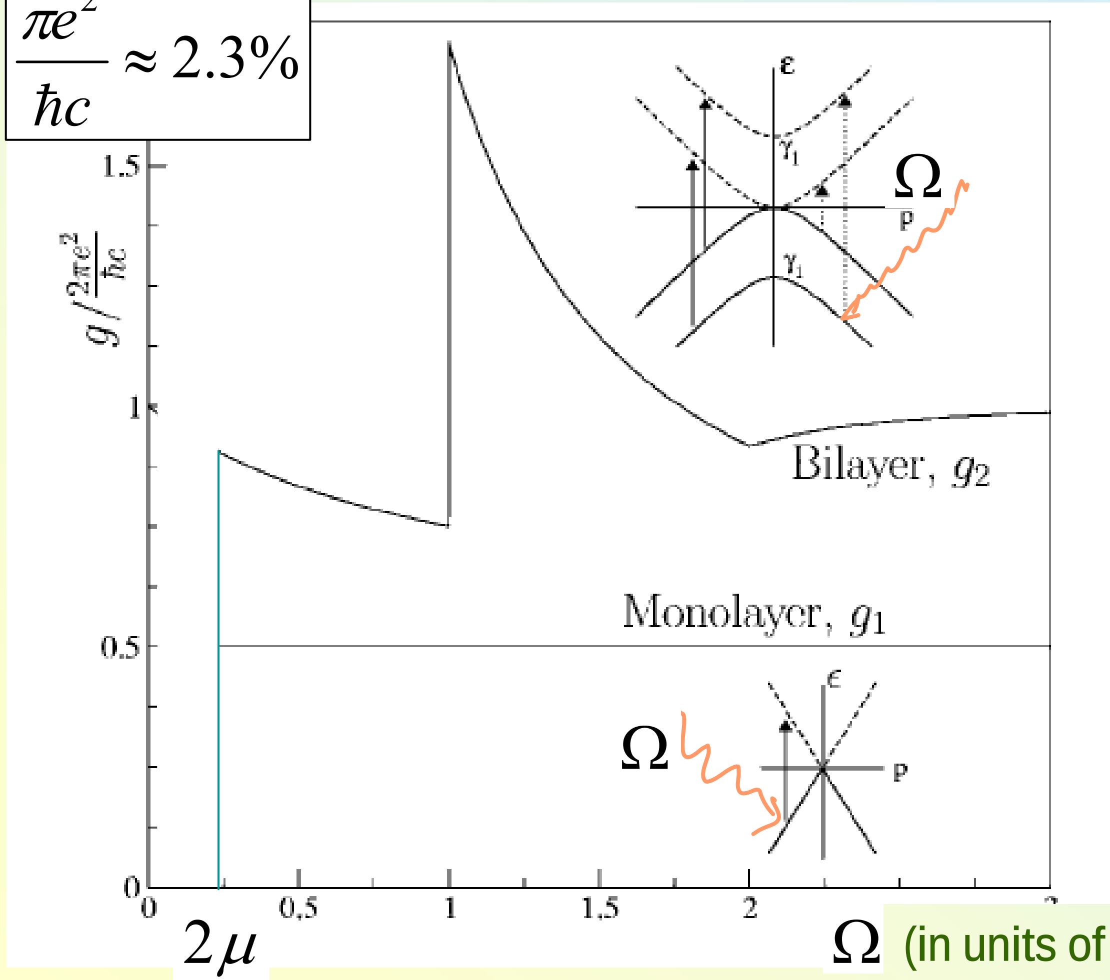
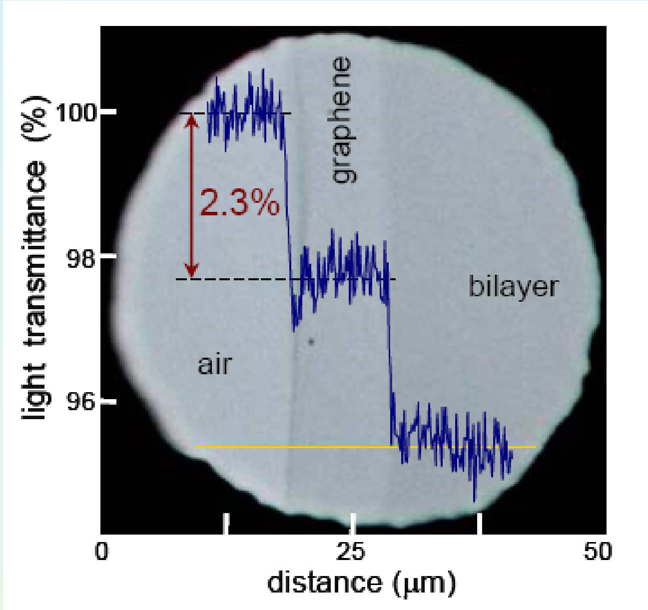


Absorption coefficient

$$\frac{\pi e^2}{\hbar c} \approx 2.3\%$$



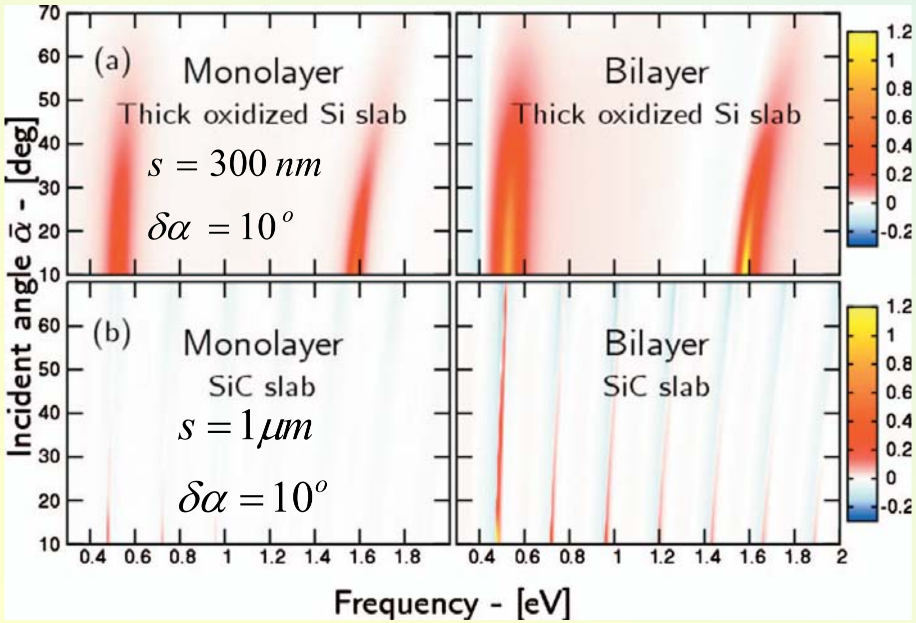
2μ Ω (in units of γ_1)



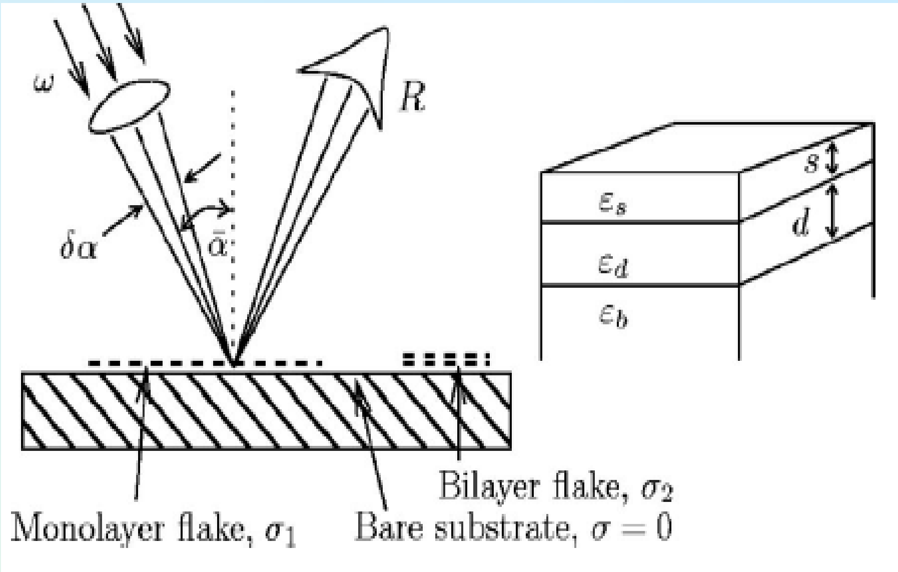
Nair, Blake, Grigorenko, Novoselov, Booth, Stauber, Peres, Geim - Science (2008)

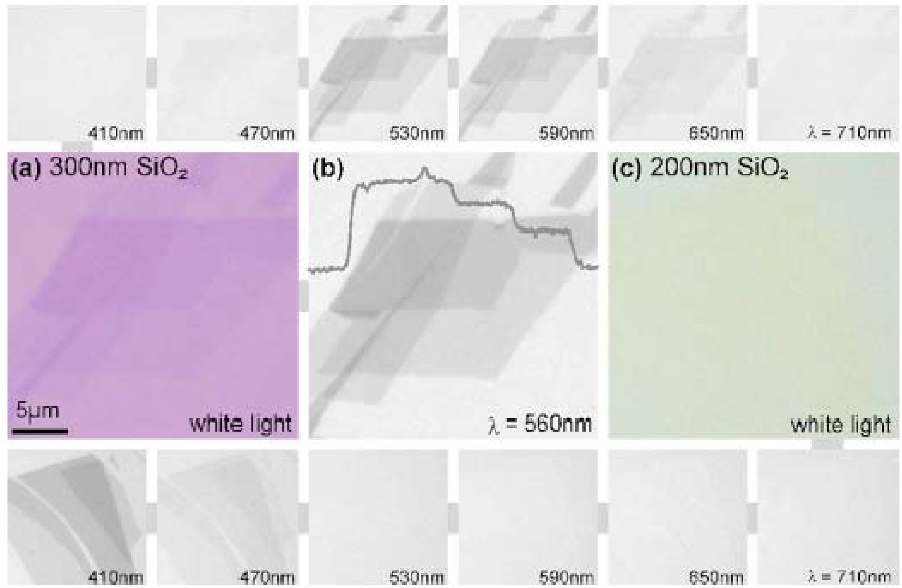
Graphene flakes are better visible in reflection when the oxide layer in SiO_2/Si wafer acts as a 'clearing' optical film if

$$\frac{\lambda}{2} = \frac{\sqrt{\varepsilon_s - \sin^2\alpha}}{N + \frac{1}{2}} s$$



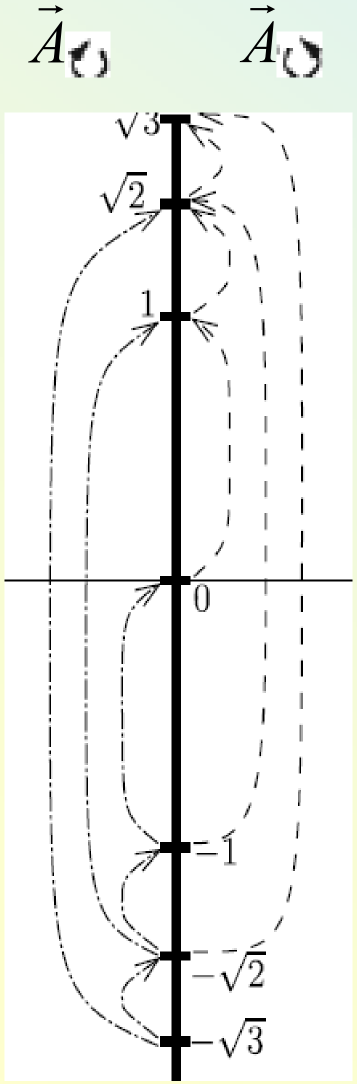
Abergel, Russell, VF - Appl. Phys. Lett. 91, 063125 (2007)





Blake, Hill, Castro Neto, Novoselov, Jiang, Yang, Booth, Geim - Appl. Phys. Lett. 91, 063124 (2007)

Infrared absorptions due to inter-LL transitions



McClure, Phys. Rev. 104, 666 (1956)

Landau level $n^{\pm}$ $\qquad \varepsilon^{\pm} = \pm\sqrt{2n}\hbar v / \lambda_B$

$$\omega_{n^- \to (n+1)^+} = \omega_{(n+1)^- \to n^+} = \sqrt{2}\,\frac{\hbar v}{\lambda_B}\left(\sqrt{n} + \sqrt{n+1}\right)$$

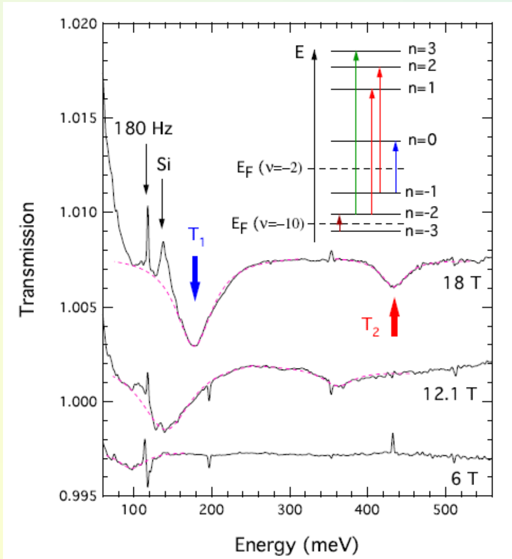
Sadowski et al - PRL 97, 266405 (2006)

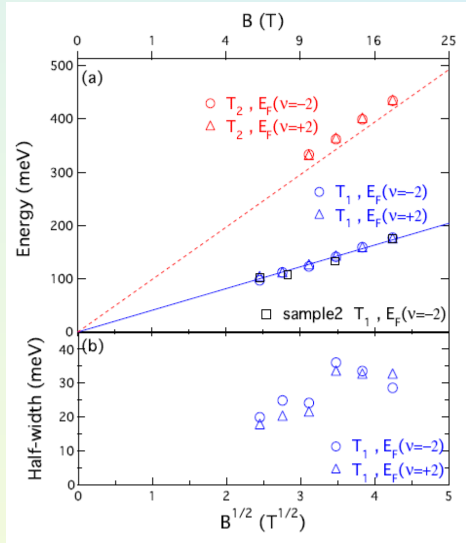
$(n+1)^- \to n^+$

$M_z = -1$ $\qquad \vec{A}_{\circlearrowleft}$

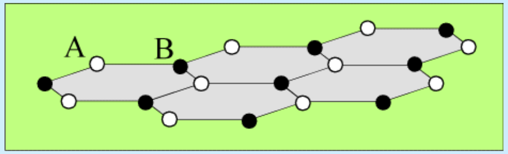
$n^- \to (n+1)^+$

$M_z = +1$ $\qquad \vec{A}_{\circlearrowright}$





Jiang, Henriksen, Tung, Wang, Schwartz, Han, Kim, Stormer - PRL (2007)



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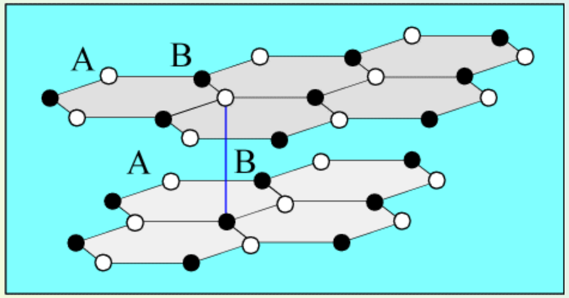
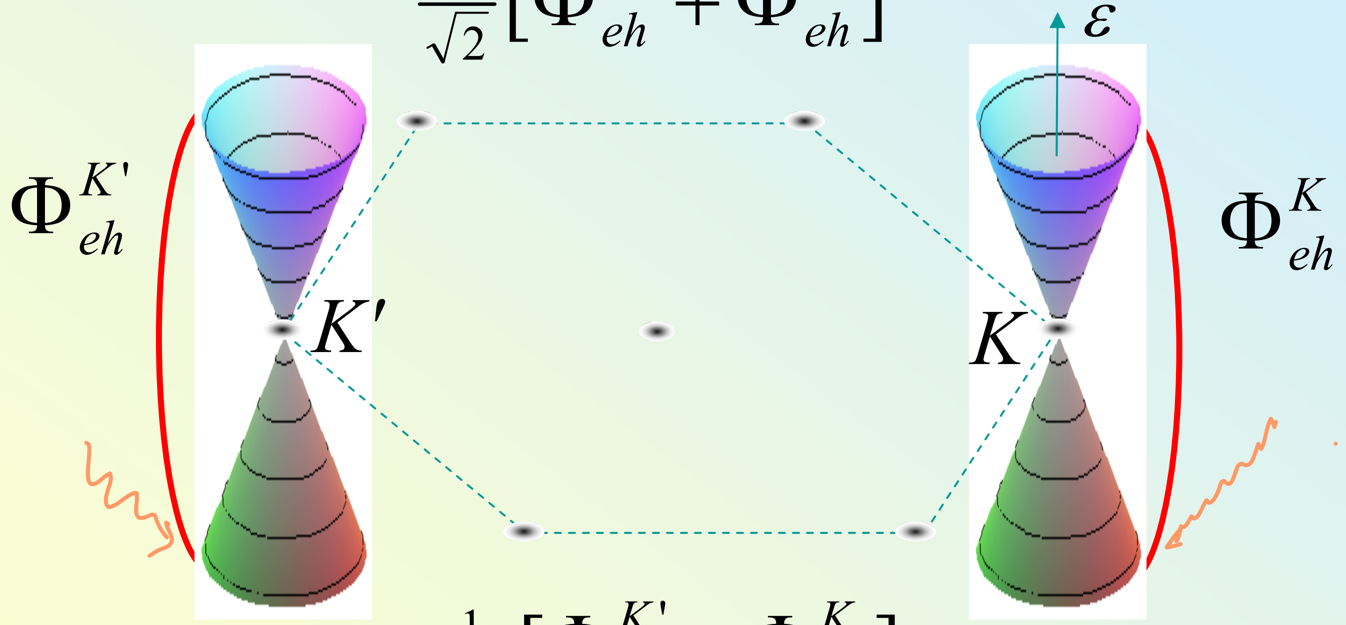


$$\hat{H} = v\vec{\Sigma}\cdot(\vec{p} - \tfrac{e}{c}\vec{A}_{\Omega})$$

Absorption of a photon generates ‘valley-symmetric’ excitations (positive parity)

$$\tfrac{1}{\sqrt{2}}[\Phi_{eh}^{K'} + \Phi_{eh}^{K}]$$

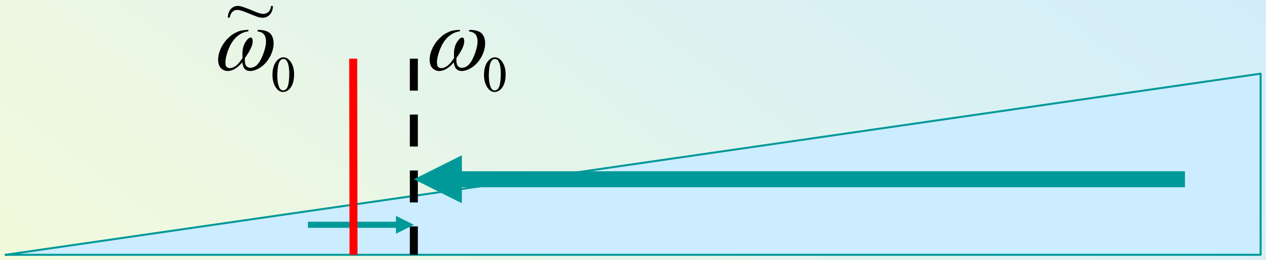


$$\tfrac{1}{\sqrt{2}}[\Phi_{eh}^{K'} - \Phi_{eh}^{K}]$$

Absorption of a phonon generates ‘valley-antisymmetric’ excitations (negative parity)

$$\hat{H} = v\vec{\Sigma}\cdot\vec{p} - \zeta\vec{\Sigma}\cdot(\vec{\ell}_z \times \vec{u})g\sqrt{2M\omega_0}$$

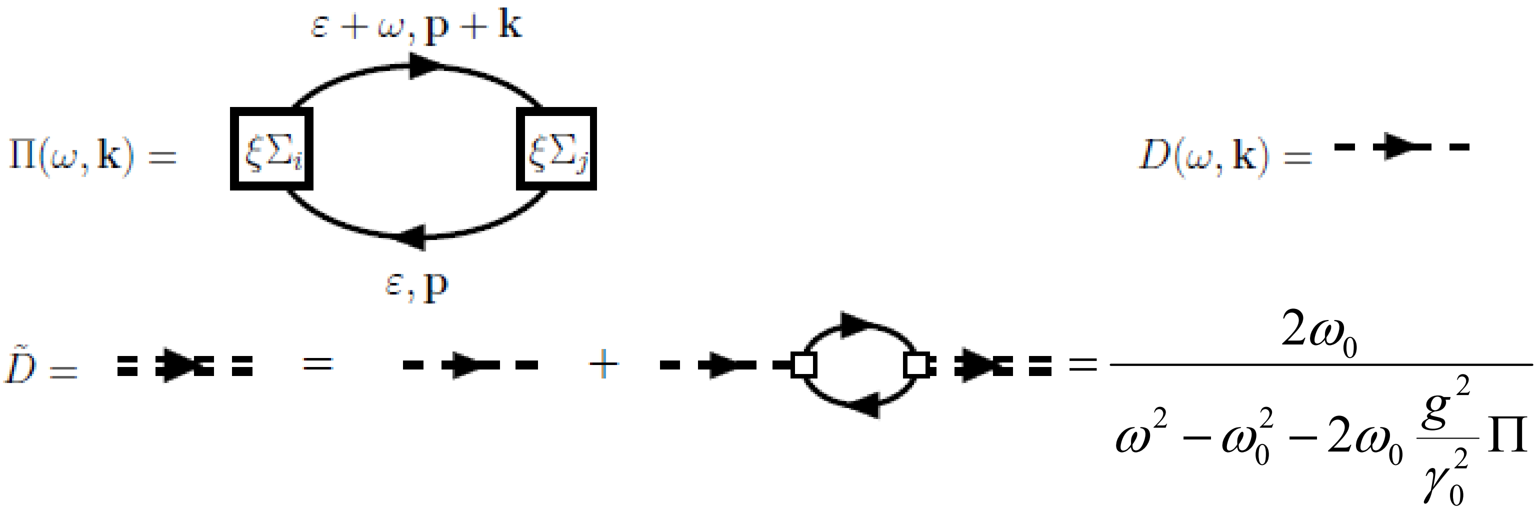
Coupling of optical phonon with valley anti-symmetric electronic excitations shifts the phonon energy.



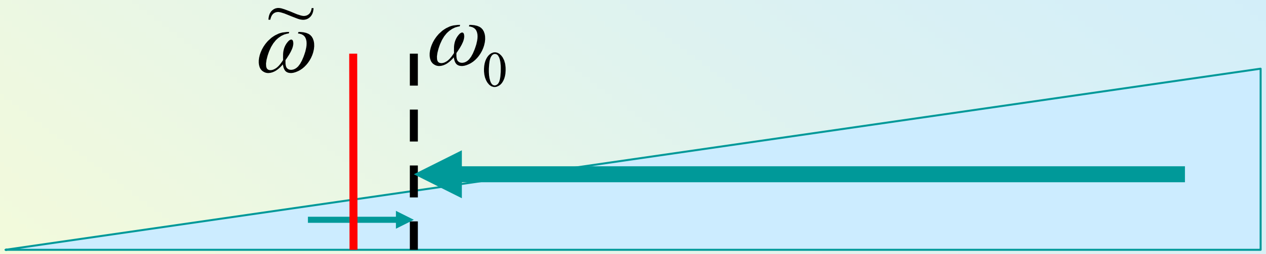
Continuous spectrum of $\frac{1}{\sqrt{2}}[\Phi_{eh}^{K'} - \Phi_{eh}^{K}]$

$$\tilde{\omega}_0 \approx \omega_0 + \frac{g^2}{\gamma_0^2}\Pi$$

Castro Neto, Guinea, PRB 75, 045404 (2007)
Ando, J. Phys. Soc. Jpn. 76, 024712 (2007)



Coupling of optical phonon with valley anti-symmetric electronic excitations

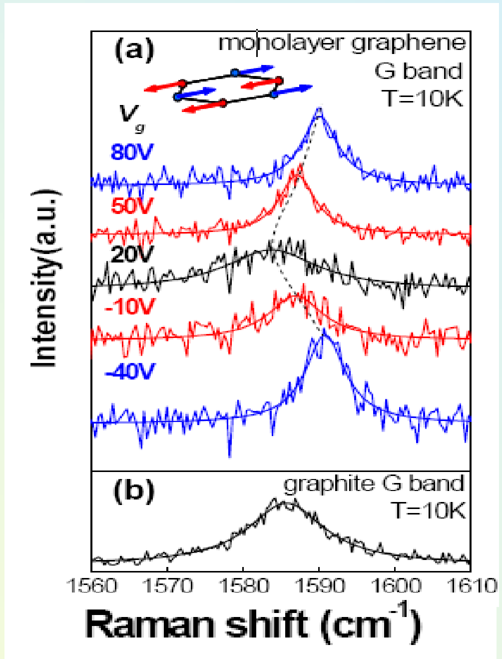


Continuous spectrum of

$$\frac{1}{\sqrt{2}}[\Phi_{eh}^{K'} - \Phi_{eh}^{K}]$$

$$\tilde{\omega} \approx \omega_0 + \frac{g^2}{\gamma_0^2}\Pi(\mu)$$

Castro Neto, Guinea, PRB 75, 045404 (2007)
Ando, J. Phys. Soc. Jpn. 76, 024712 (2007)



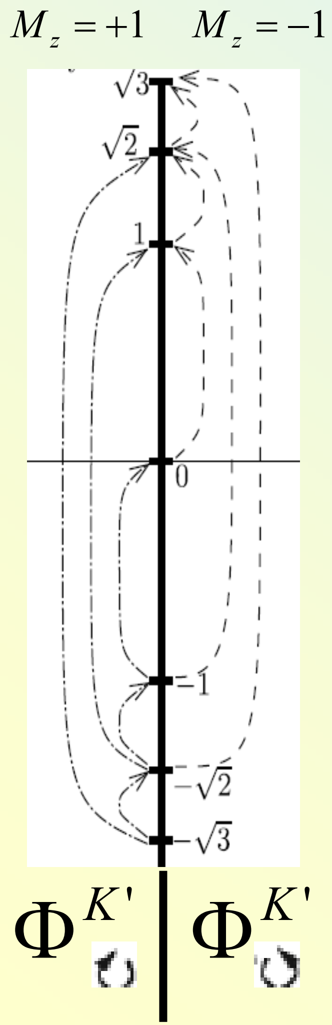
Yan, Zhang, Kim, Pinczuk - PRL 98, 166802 (2007)
S. Pisana, et al, Nature Mat. 6, 198 (2007)

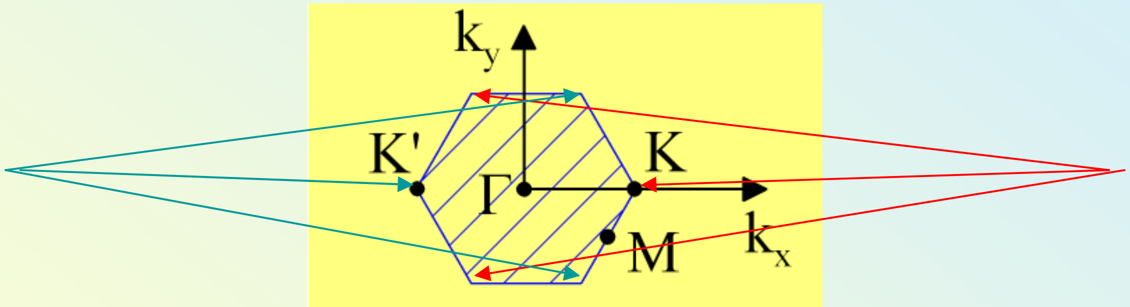
Coupling of modes in a magnetic field

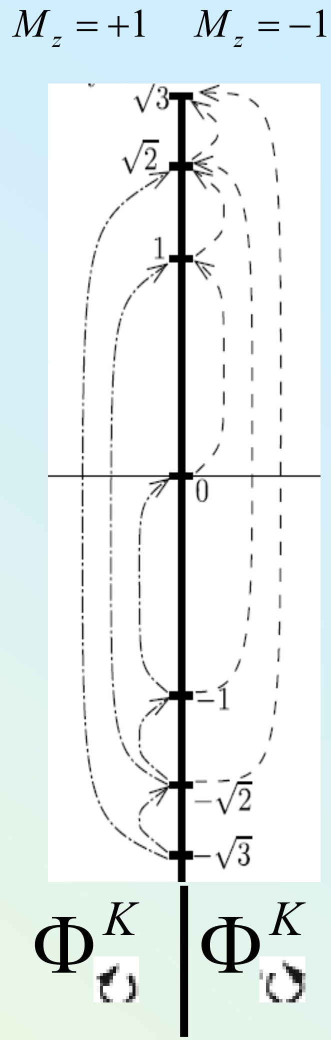
$$\vec{A}_{\circlearrowleft} \sim \vec{\ell}_x + i\vec{\ell}_y \qquad \vec{A}_{\circlearrowright} \sim \vec{\ell}_x - i\vec{\ell}_y$$

Optically active (coupled to IR light)

$$\frac{1}{\sqrt{2}}[\Phi^{K'} + \Phi^{K}]$$





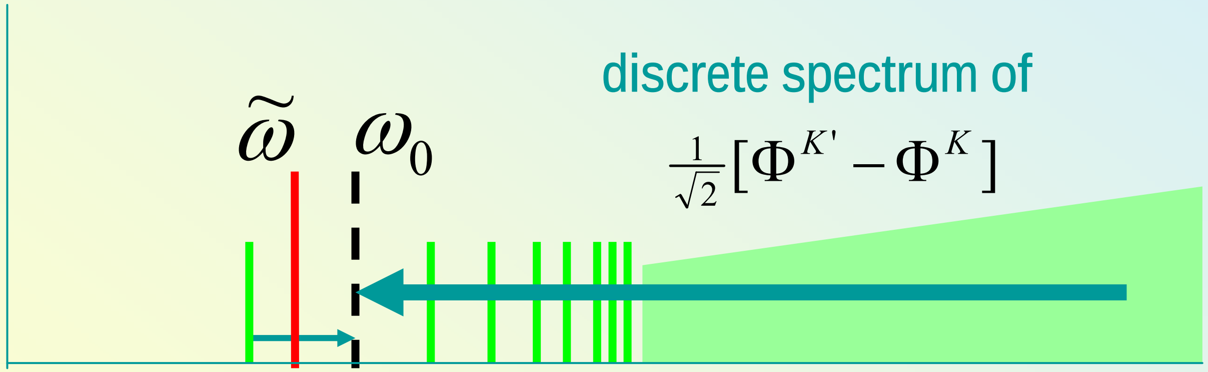


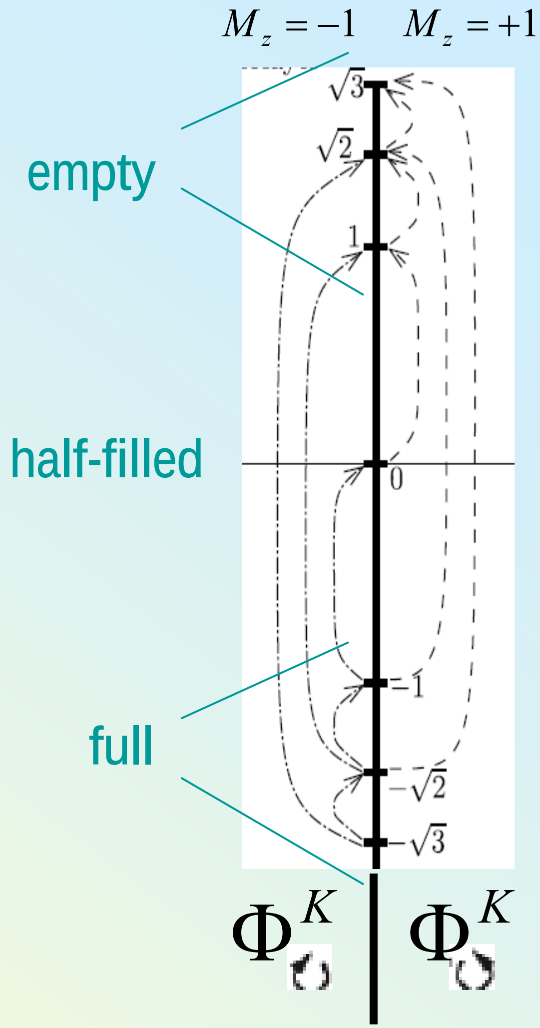
$$\frac{1}{\sqrt{2}}[\Phi^{K'} - \Phi^{K}]$$

Optically inactive and coupled to the phonon

$$\vec{u}_{\circlearrowleft} \sim \vec{\ell}_x + i\vec{\ell}_y \qquad \vec{u}_{\circlearrowright} \sim \vec{\ell}_x - i\vec{\ell}_y$$

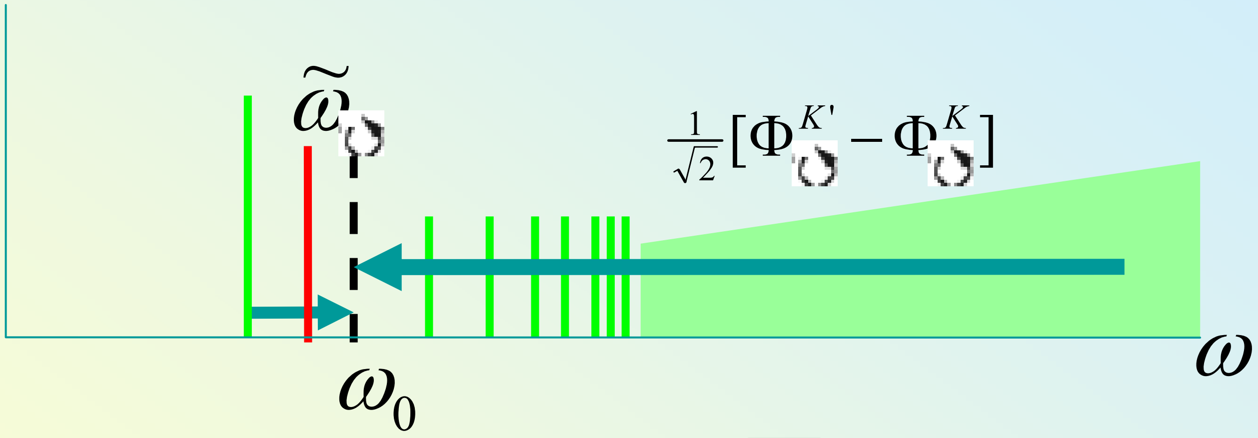
Coupling of optical phonon with valley antisymmetric magneto-excitons in undoped graphene





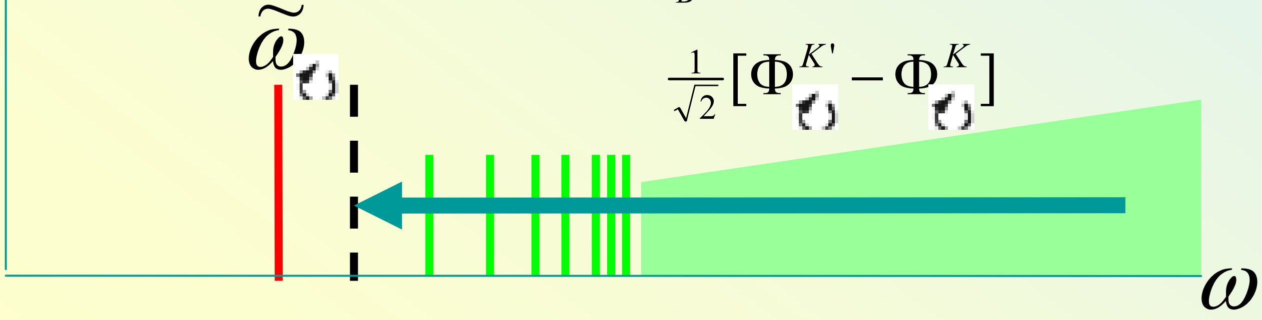
Ando - J. Phys. Soc. Jpn. 76, 024712 (2007)
Goerbig, Fuchs, Kechedzhi, VF - PRL 99, 087402 (2007)

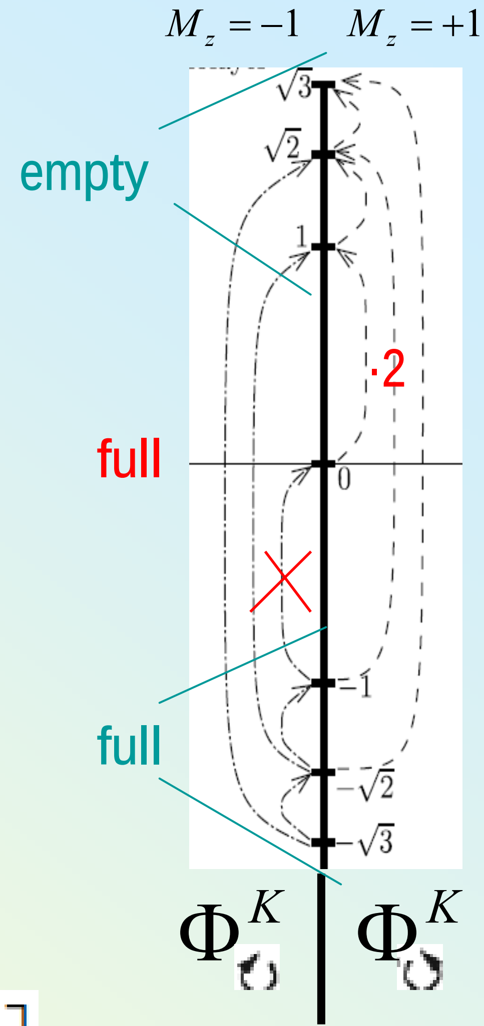
Coupling of optical phonon with magneto-excitons in doped graphene



$$g_{\circlearrowright}(n) = \frac{ga}{2\lambda_B}\sqrt{\frac{3\sqrt{3}}{2\pi}}\sqrt{(1+\delta_{n,0})[\nu_{n^-} - \nu_{(n+1)^+}]}$$

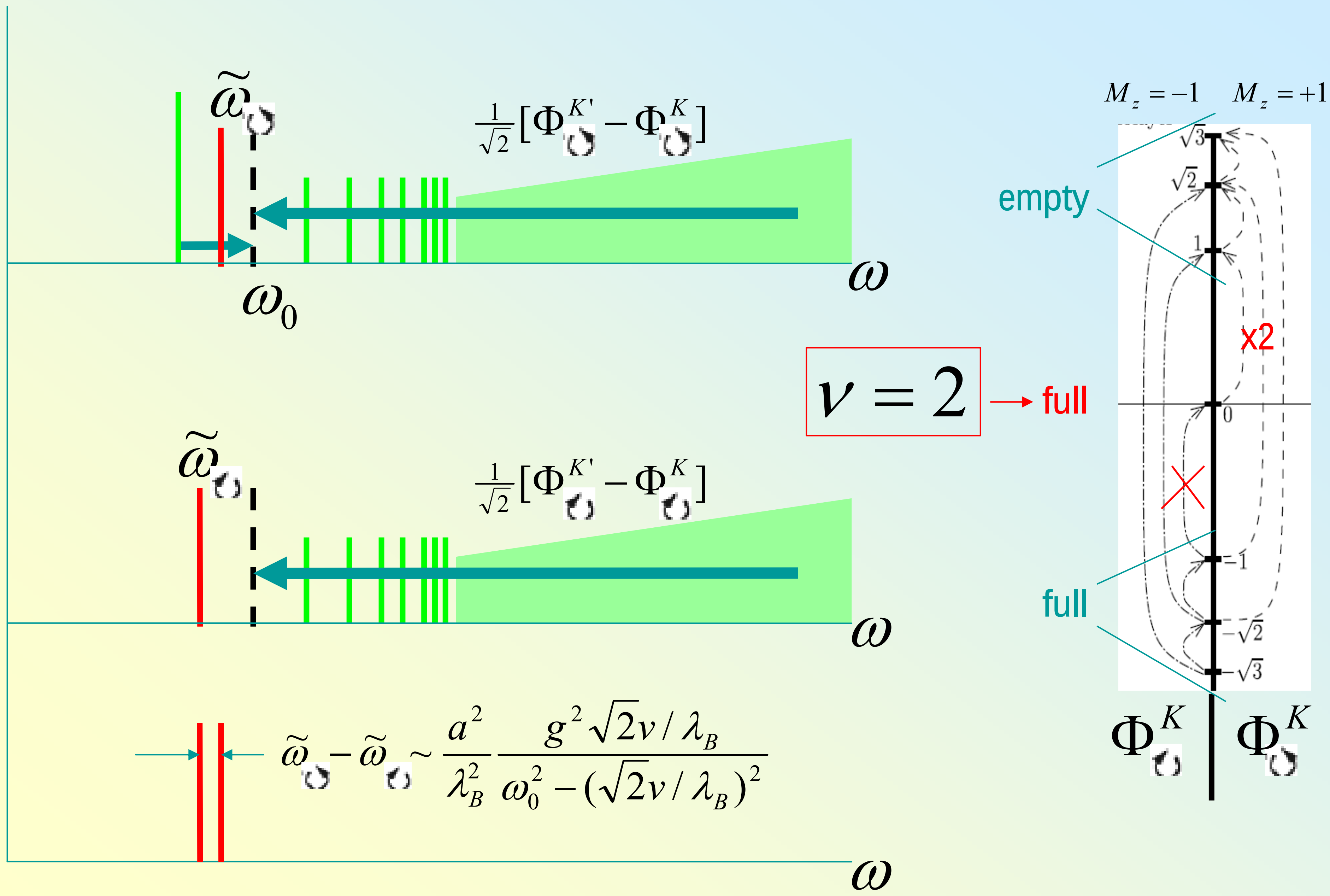
$$g_{\circlearrowleft}(n) = \frac{ga}{2\lambda_B}\sqrt{\frac{3\sqrt{3}}{2\pi}}\sqrt{(1+\delta_{n,0})[\nu_{(n+1)^-} - \nu_{n^+}]}$$





$$\tilde{\omega}^2_{\mathcal{A}} - \omega^2 = 4\omega\left[\sum_{n=n_F+1}^{N} \frac{\Omega_n g^2_{\mathcal{A}}(n)}{\tilde{\omega}^2_{\mathcal{A}} - \Omega_n^2} + \frac{\Delta_{n_F} g^2_{\mathcal{A}}(n_F)}{\tilde{\omega}^2_{\mathcal{A}} - \Delta^2_{n_F}}\right],$$

Coupling of optical phonon with magneto-excitons in doped graphene

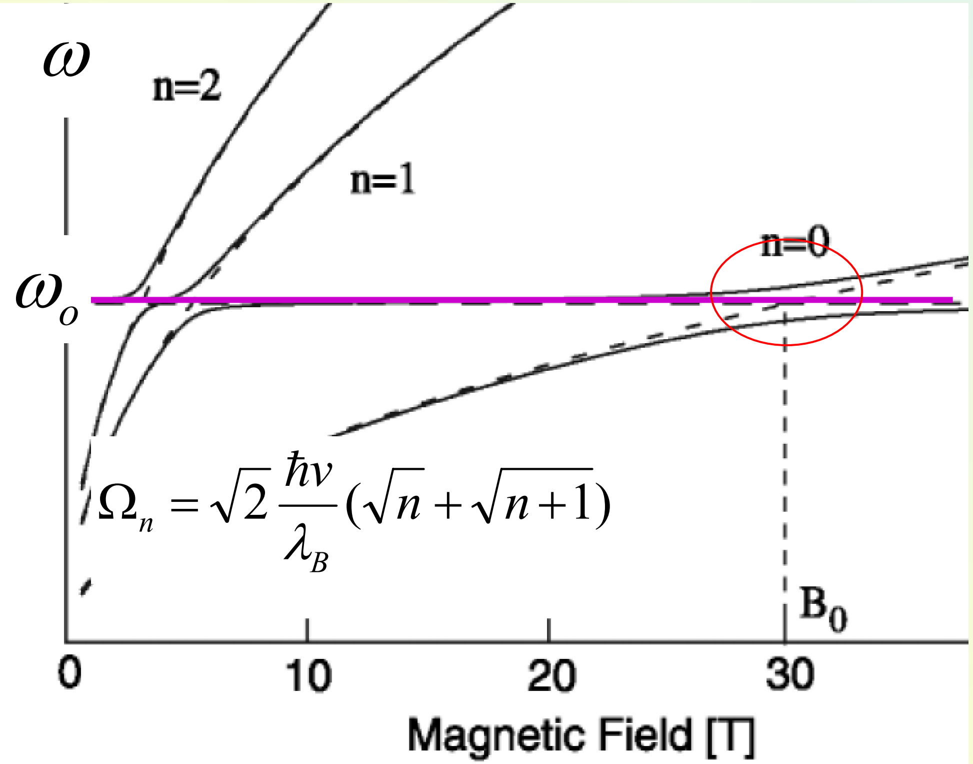


Magneto-phonon resonance in the Raman spectrum

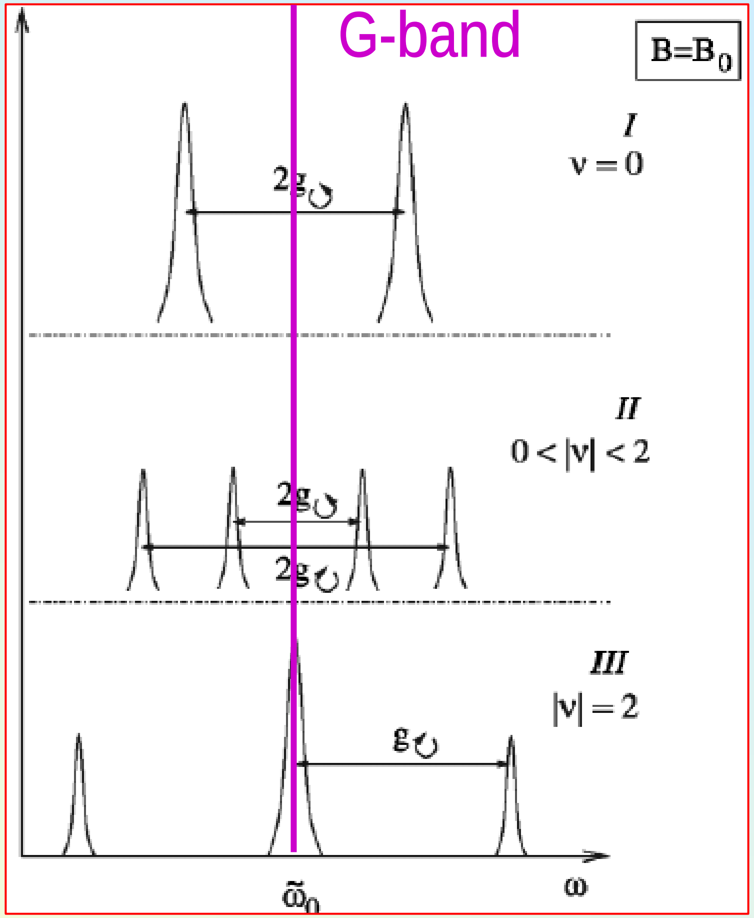
$$\tilde{\omega}^{\pm}_{\mathcal{A}}(n) = \tfrac{1}{2}(\Omega_n + \tilde{\omega}_0) \mp \sqrt{\tfrac{1}{4}(\Omega_n - \tilde{\omega}_0)^2 + g^2_{\mathcal{A}}(n)}$$

$$g_{\circlearrowright}(n) = \frac{ga}{2\lambda_B}\sqrt{\frac{3\sqrt{3}}{2\pi}}\sqrt{(1+\delta_{n,0})[\nu_{n^-} - \nu_{(n+1)^+}]}$$

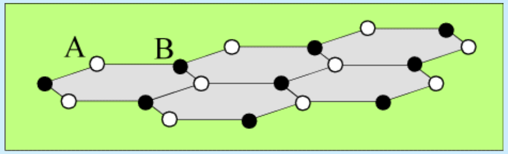
$$g_{\circlearrowleft}(n) = \frac{ga}{2\lambda_B}\sqrt{\frac{3\sqrt{3}}{2\pi}}\sqrt{(1+\delta_{n,0})[\nu_{(n+1)^-} - \nu_{n^+}]}$$



$$\Omega_n = \sqrt{2}\,\frac{\hbar v}{\lambda_B}(\sqrt{n} + \sqrt{n+1})$$



Goerbig, Fuchs, Kechedzhi, VF - PRL 99, 087402 (2007)



Introduction: symmetries and notations.

Optics and magneto-optics of graphene: absorption.

Abergel, VF - PRB 75, 155430 (2007)
Abergel, Russell, VF - APL 91, 063125 (2007)

Magneto-phonon resonance and a filling factor dependent fine structure of the G-line in the Raman spectrum of phonons.

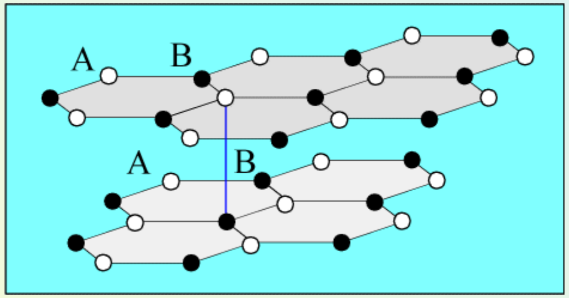
Goerbig, Fuchs, Kechedzhi, VF - PRL 99, 087402 (2007)
Kashuba, VF – unpublished (2009)

Electronic excitations in the Raman spectrum of graphene.

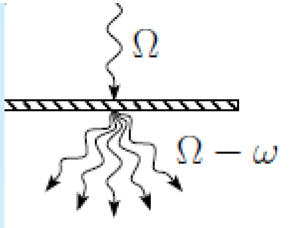
Kashuba, VF – arxiv:09065251 (2009)

Magneto-optics of bilayer graphene.

Abergel, VF - PRB 75, 155430 (2007)
Mucha-Kruczynski, McCann, VF - SSC 149, 1111 (2009)



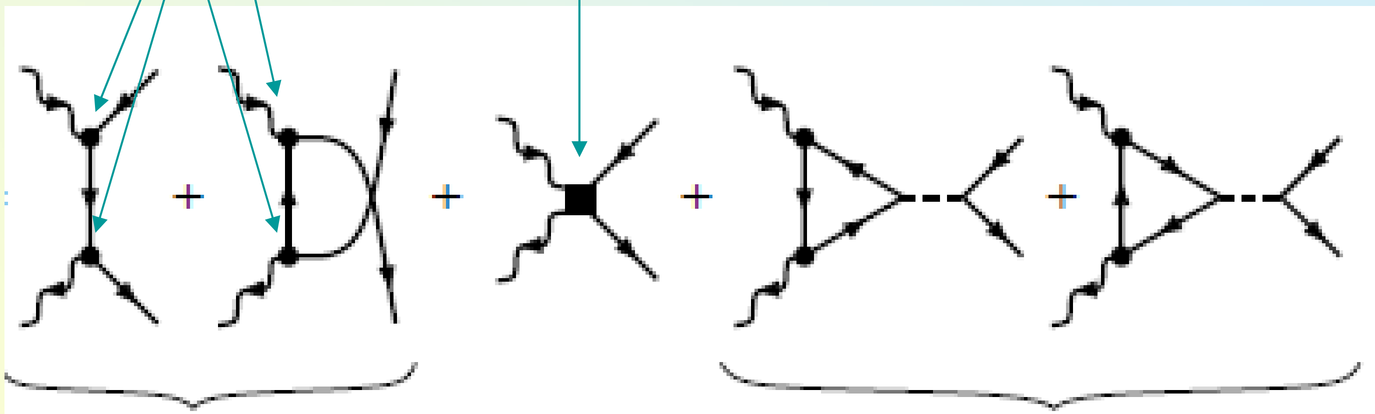
Electronic excitations in the Raman spectrum: electron-hole pair left after scattering a photon



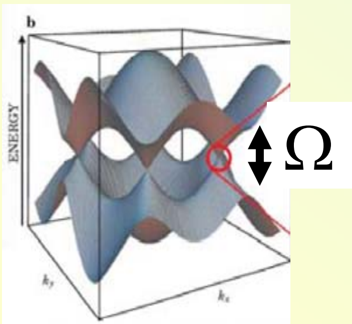
$$\omega = \varepsilon^{+}(\vec{p}) - \varepsilon^{-}(\vec{p})$$

$$\frac{\partial H}{\partial \vec{p}}$$

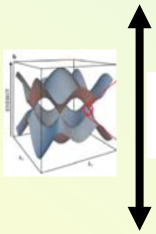
$$\frac{\partial^2 H}{\partial p_i \partial p_j}$$



dominant if



dominant if

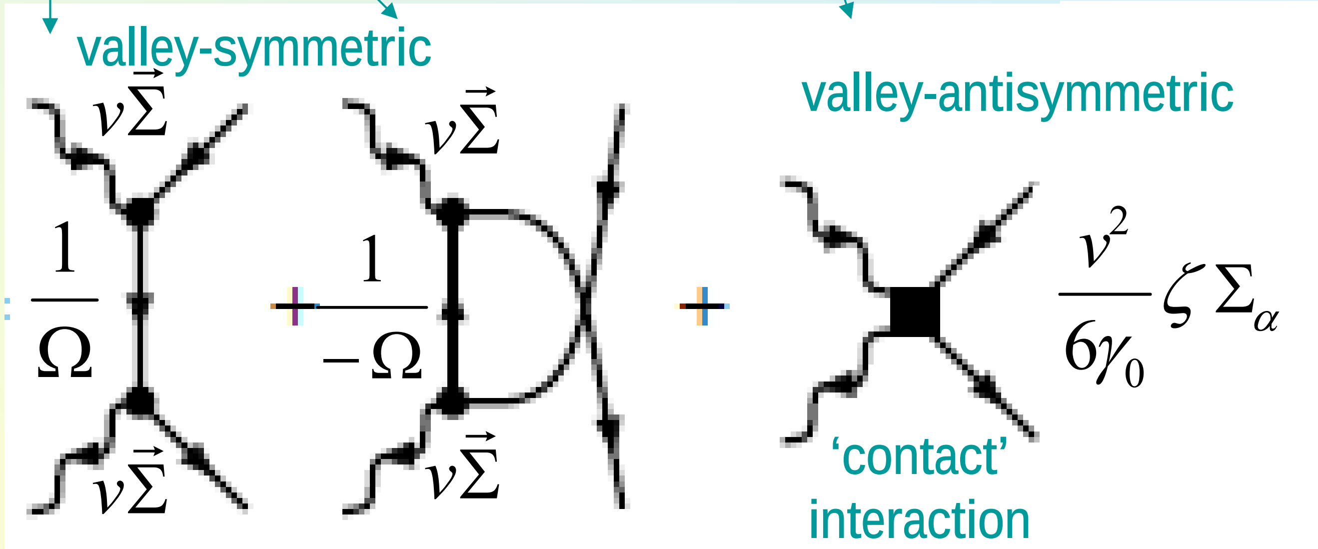


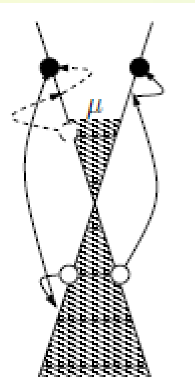
Ω

small due to the e-h symmetry

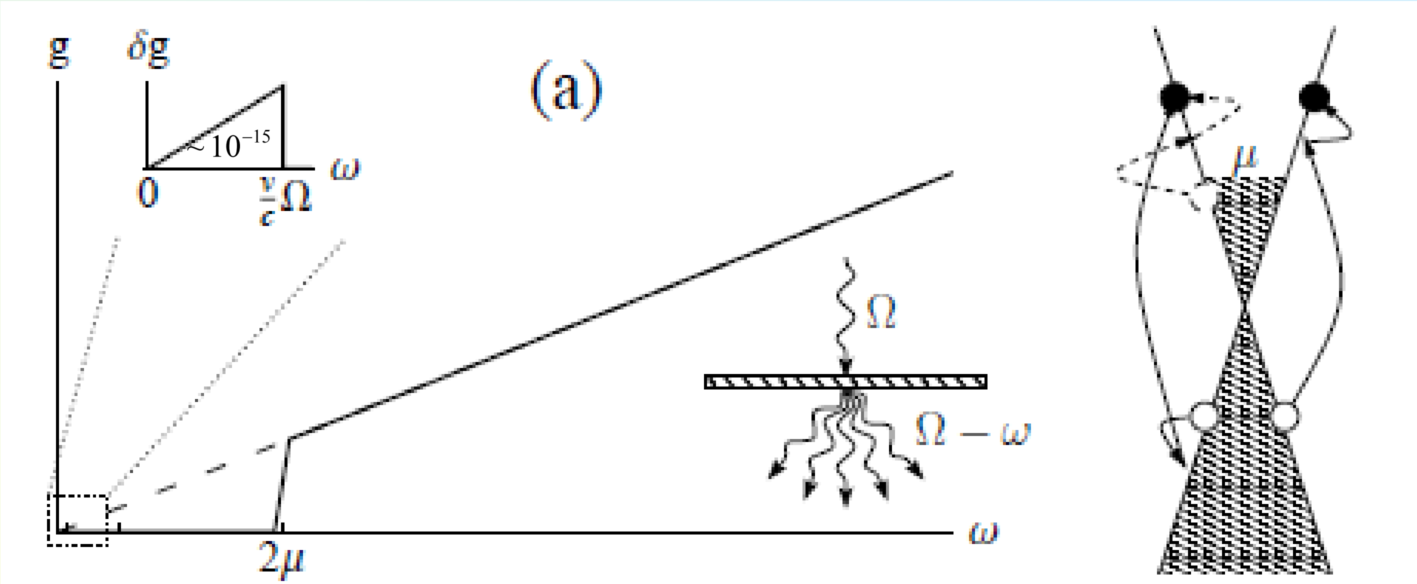
Platzmann and Wolff, ***Waves and interactions in solid state plasmas***

$$\hat{H} \approx v\vec{\Sigma}\cdot\vec{p} - \frac{v^2}{6\gamma_0}\zeta\,\Sigma_x\left(\vec{\Sigma}\cdot\vec{p}\right)\Sigma_x\left(\vec{\Sigma}\cdot\vec{p}\right)\Sigma_x$$





valley-symmetric e-h excitations generated by scattering of light

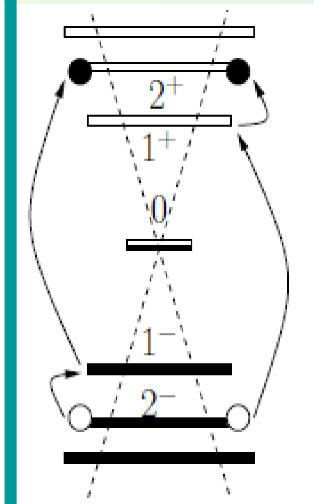


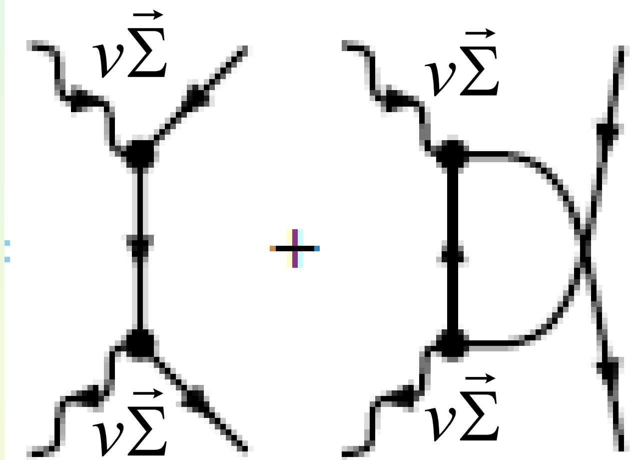
$$I(\omega) \approx \frac{|\vec{\ell}_{in} \times \vec{\ell}^{*}_{out}|^2}{4} \left(\frac{e^2}{\pi\hbar v} \frac{v^2}{c^2} \right)^2 \frac{\omega}{\Omega^2} \theta(\omega - 2\mu)$$

Kashuba, VF – arxiv:09065251 (2009)

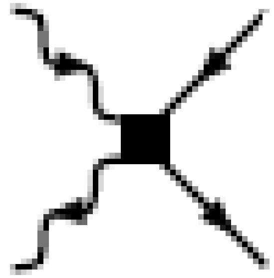
valley-symmetric (positive parity), decoupled from the phonon

valley-antisymmetric (negative parity), coupled to the phonon

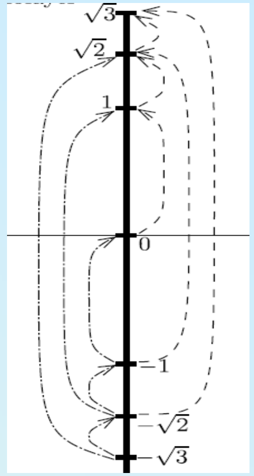




+



$$\frac{v^2}{6\gamma_0}\zeta\Sigma_\alpha$$



$$\sigma_{in}^{\pm} \to \sigma_{out}^{\pm}$$
$$\Delta n = 0$$
$$M_z = 0$$

$$\omega_{n^- \to n^+} = 2\sqrt{2}\,\frac{\hbar v}{\lambda_B}$$

$$\Delta n = 2$$ excitations are weak in Raman, due to the cancellation between diagrams

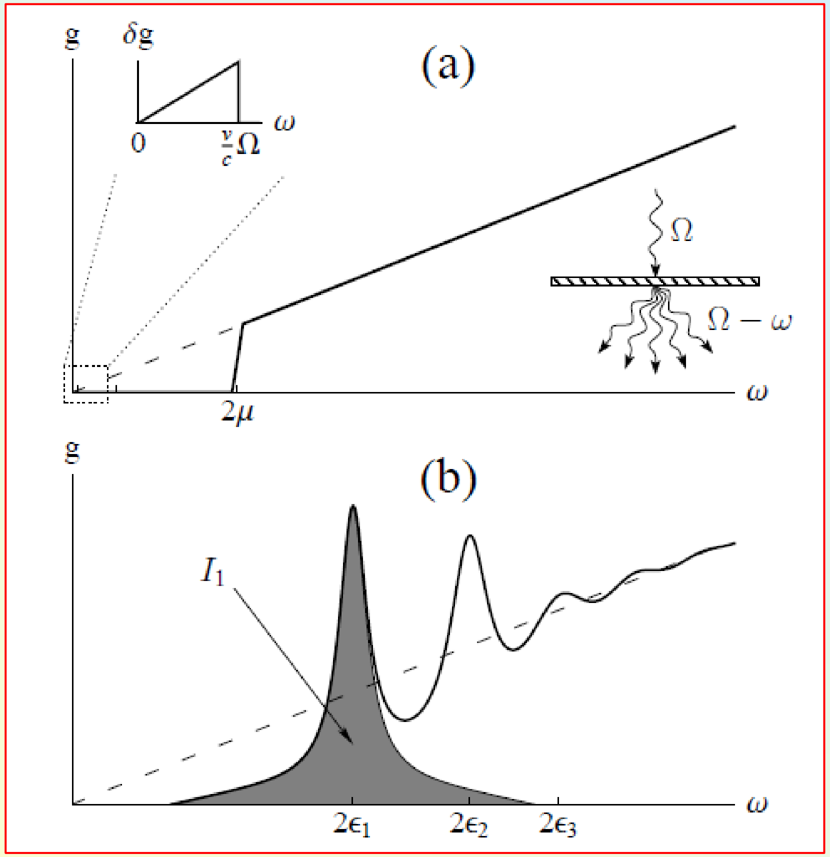
$$\omega = \sqrt{2}\,\frac{\hbar v}{\lambda_B}\left(\sqrt{n} + \sqrt{n+1}\right)$$

$$\sigma_{in}^{\pm} \to \sigma_{out}^{\mp}$$
$$\Delta n = \mp 1$$
$$\Delta M_z = \pm 3$$

transferred to the lattice

Kashuba, VF – arxiv:09065251 (2009)

Signature of the electronic excitations in the Raman spectrum



$$I \propto | \vec{\ell}_{in} \times \vec{\ell}^{*}_{out} |^2$$

$$n^- \rightarrow n^+$$

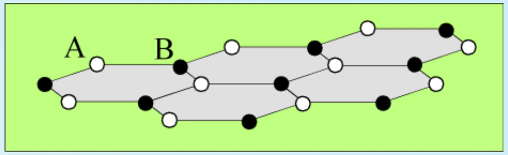
$$\sigma^{\pm}_{in} \rightarrow \sigma^{\pm}_{out}$$

$$I_1 \sim \left(\frac{v^2}{c^2} \frac{e^2 / \lambda_B}{\pi\Omega} \right)^2 \sim 10^{-12}$$

for $B \sim 20T, \Omega \sim 1eV$

Kashuba, VF – arxiv:09065251 (2009)

Conclusions



Magneto-phonon resonance and a filling factor dependent fine structure of the 'G-line' in the Raman spectrum of phonons.

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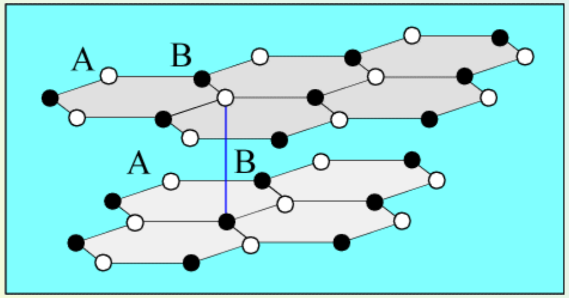
Kashuba, VF – unpublished (2009)

Electronic excitations in the Raman spectrum of graphene.

Kashuba, VF – arxiv:09065251 (2009)

Magneto-optics of bilayer graphene.

Abergel, VF - PRB 75, 155430 (2007)
Mucha-Kruczynski, McCann, VF - SSC 149, 1111 (2009)



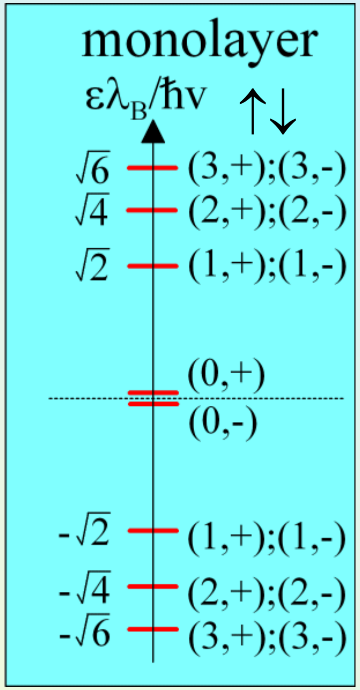
monolayer:

energy scale $\hbar v/\lambda_B$

where $\lambda_B = \sqrt{\frac{\hbar}{eB}}$

state at zero energy:

$$\pi\phi_0 = 0$$



Monolayer, *J=1*

McClure, Phys. Rev. 104, 666 (1956)

$$\varepsilon^{\pm} = \pm\sqrt{2n}\,\frac{\hbar v}{\lambda_B}$$

$$g\begin{pmatrix} 0 & (\pi^{+})^{J} \\ \pi^{J} & 0 \end{pmatrix}\psi = \varepsilon\psi$$

bilayer:

energy scale $\hbar\omega_c$

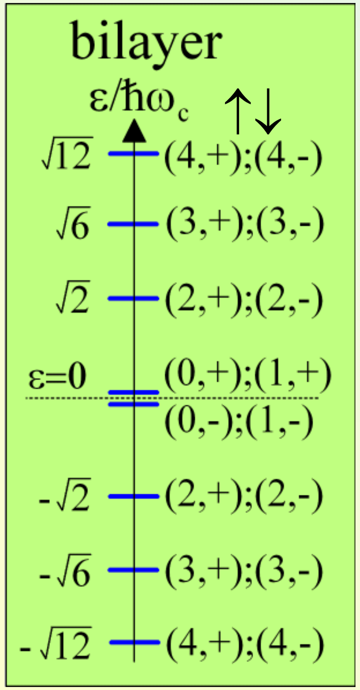
where $\omega_c = \frac{eB}{m}$

$m \sim 0.035 m_e$

states at zero energy:

$$\pi^2\phi_0 = 0$$

$$\pi^2\phi_1 = 0$$



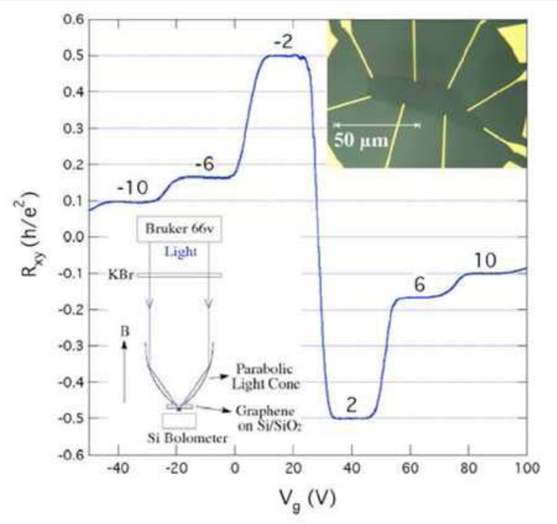
Bilayer, *J=2*

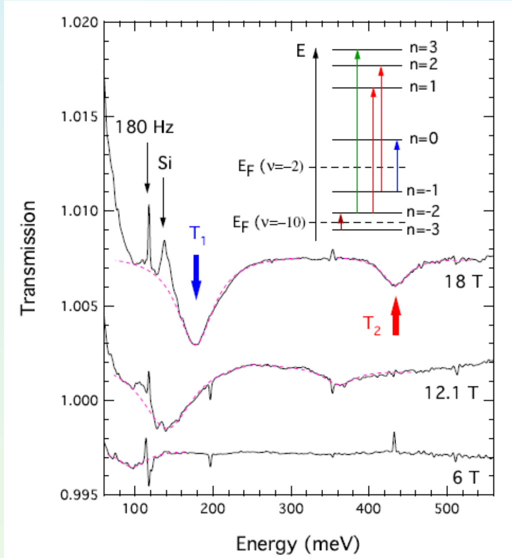
$$\varepsilon^{\pm} = \pm\hbar\omega_c\sqrt{n(n-1)}$$

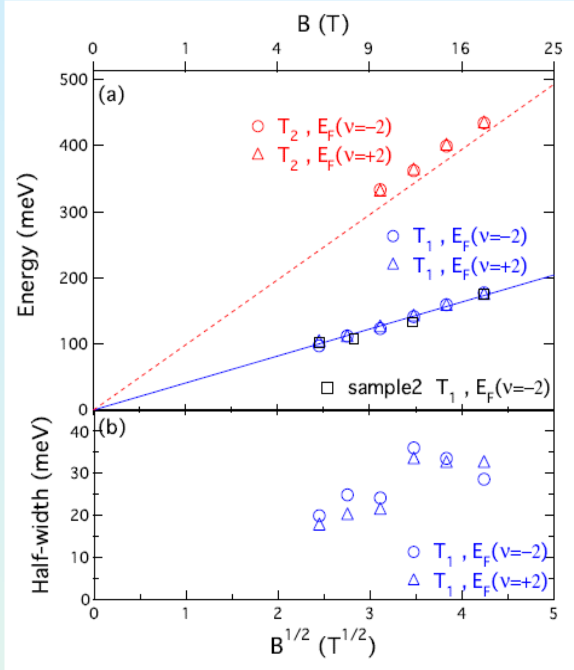
with *8*-fold degenerate $\varepsilon=0$ Landau level

McCann, VF - Phys. Rev. Lett. 96, 086805 (2006)

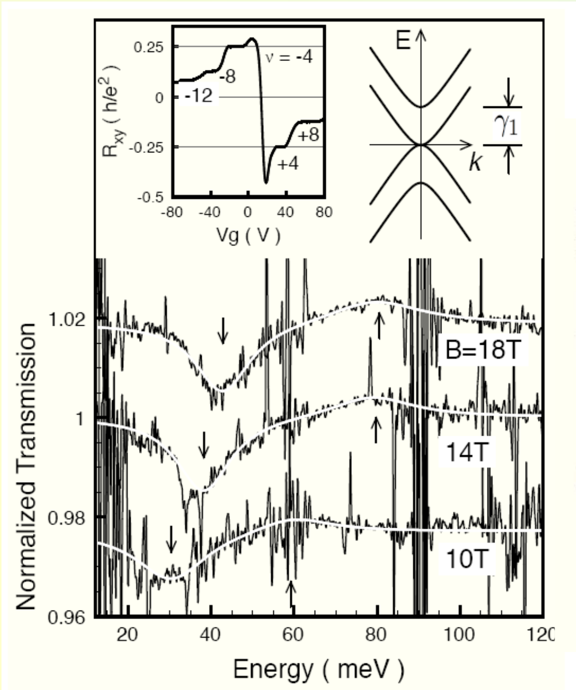
Infrared absorptions in mono/bi-layer graphene - experiment

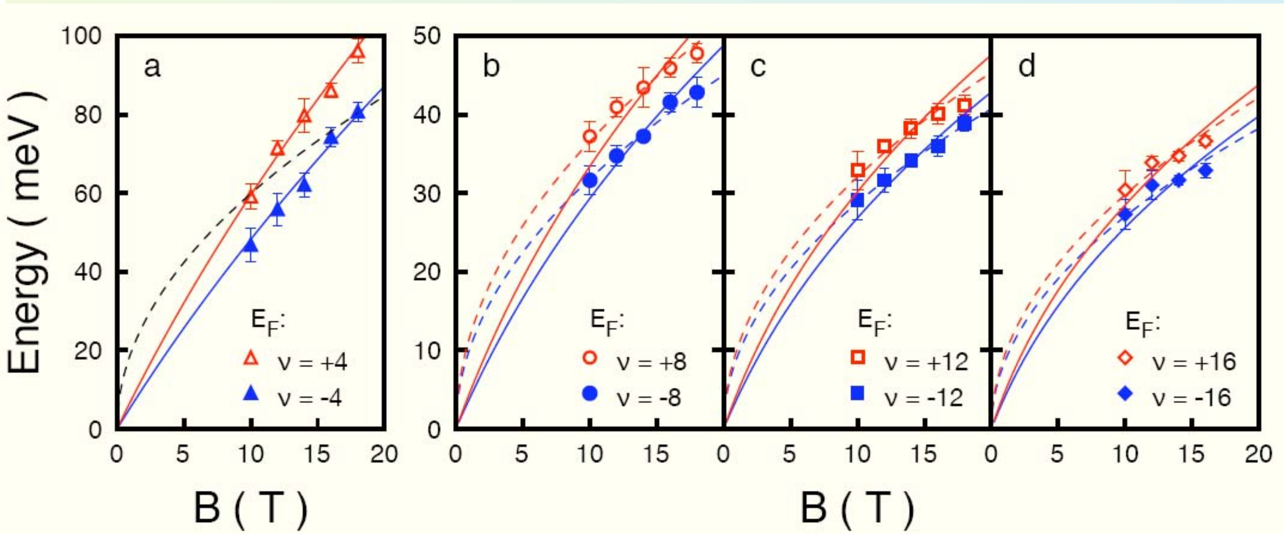






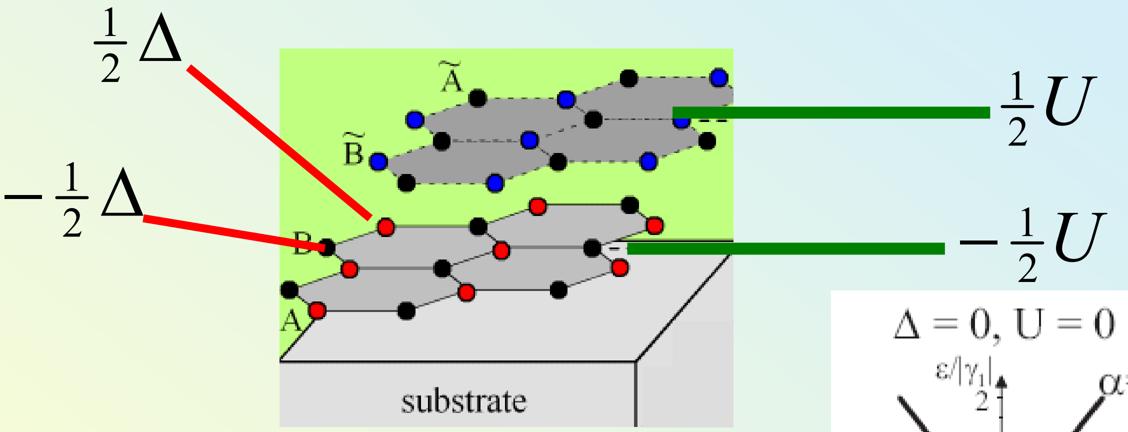
Jiang, Henriksen, Tung, Wang, Schwartz, Han, Kim, Stormer (2007)





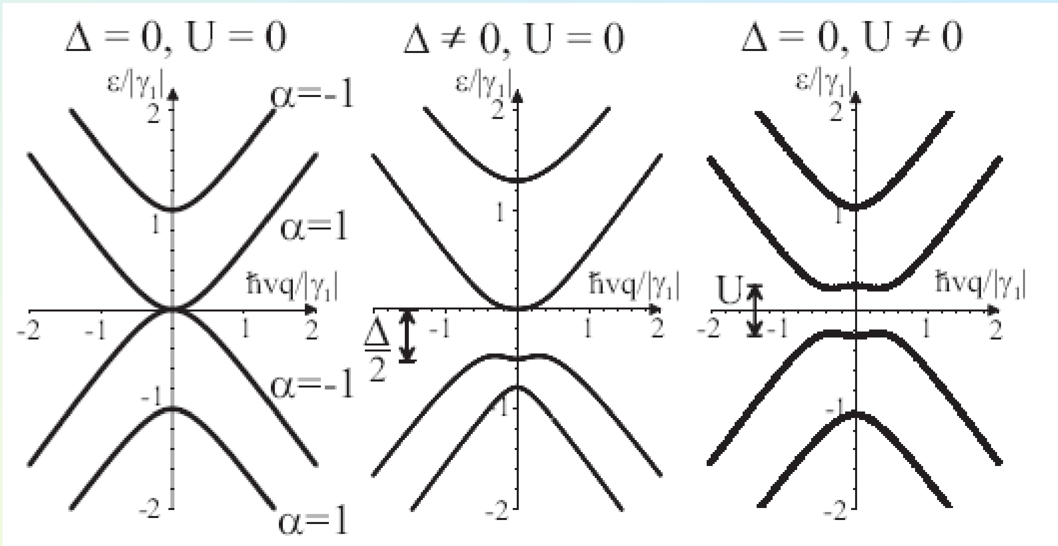
Henriksen, Jiang, Tung, Schwartz, Takita, Wang, Kim, Stormer (2008)

Interlayer asymmetry gap

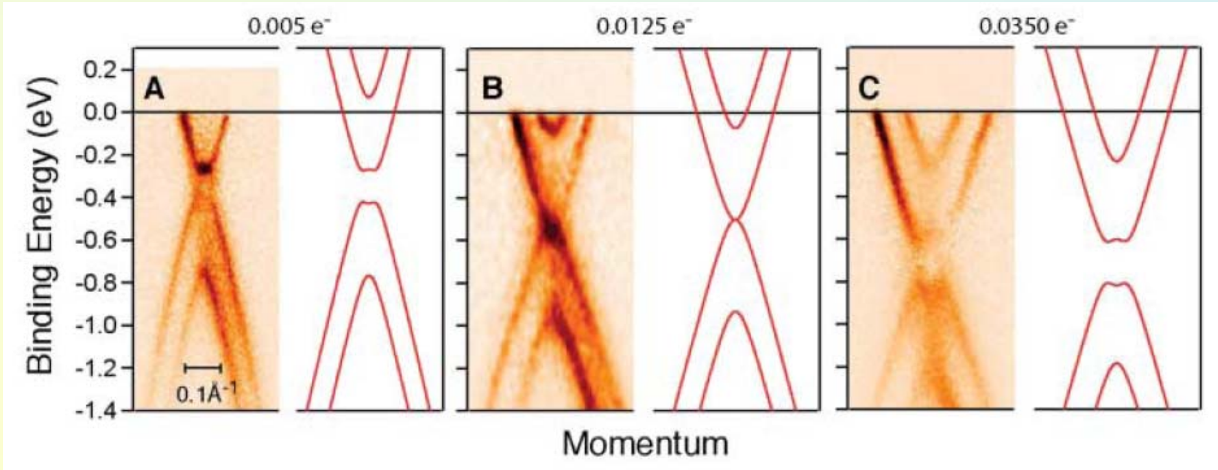


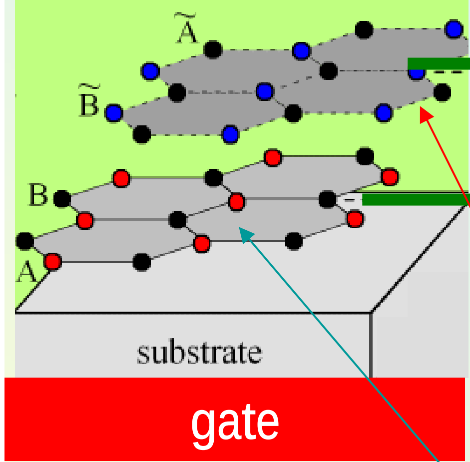
Mucha-Kruczynski, Tsyplyatyev, Grishin, McCann, VF, Boswick, Rotenberg Phys. Rev. B 77, 195403 (2008)

McCann, VF - PRL 96, 086805 (2006)
McCann - PRB 74, 161403 (2006)



T. Ohta *et al* – Science 313, 951 ('06)
(Rotenberg's group at Berkeley NL)
SiC-based highly doped
bilayer graphene



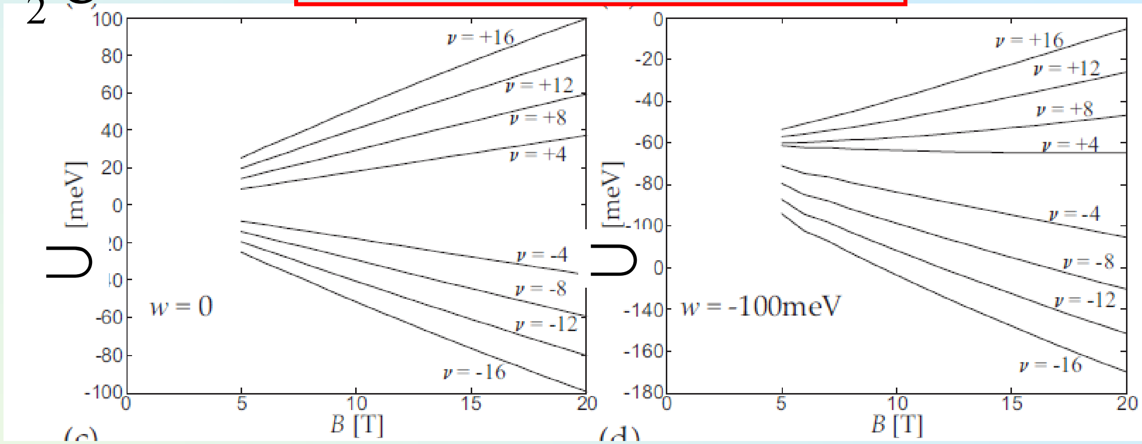


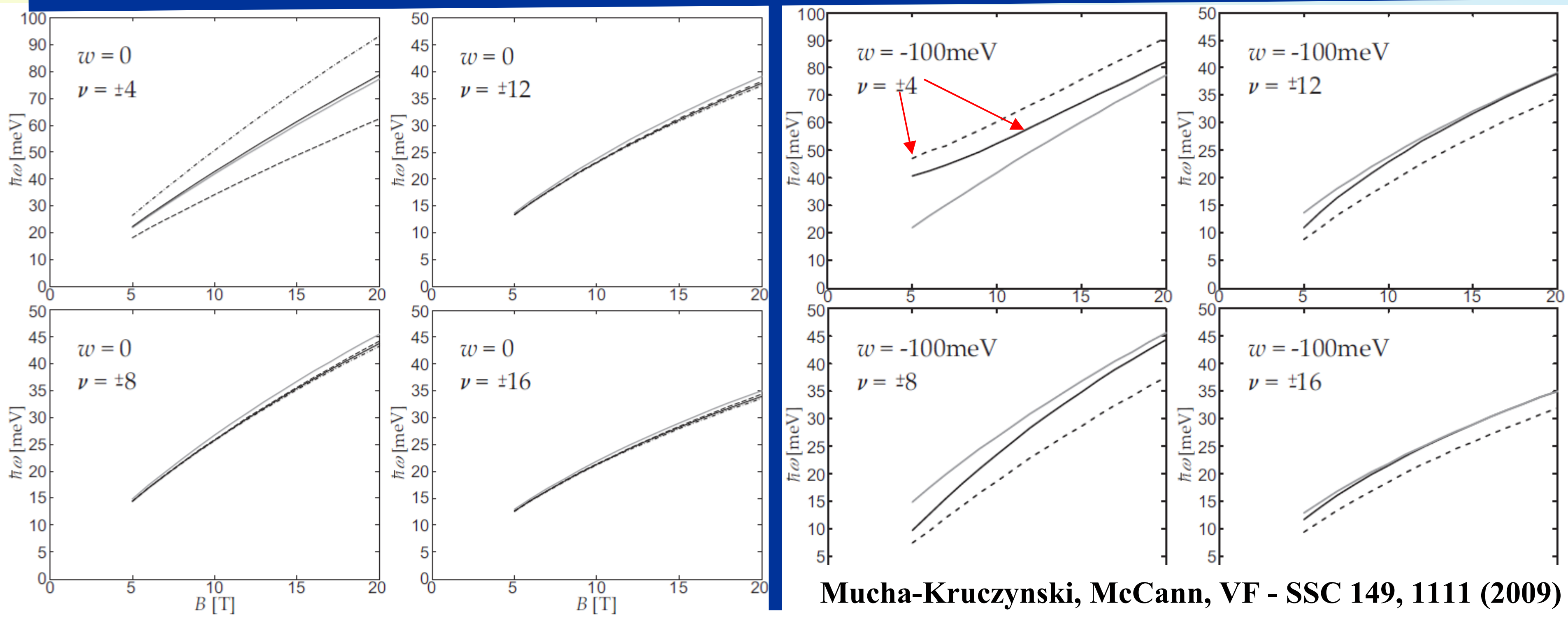
$$\frac{1}{2}U$$

$$-\frac{1}{2}U$$

$$n_e = n_1 + n_2$$

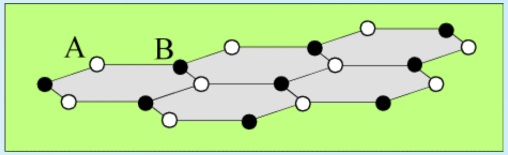
$$U = w + \frac{e^2 d}{\varepsilon_0} n_2(\nu, B)$$





Mucha-Kruczynski, McCann, VF - SSC 149, 1111 (2009)

Conclusions



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