Non-linear sigma models (tutorial)

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Plan (tentative)

- quantum interference, diagrammatics, weak localization, mesoscopic fluctuations, strong localization
- field theory: non-linear σ-model
- quasi-1D geometry: exact solution, localization
- RG, metal-insulator transition, criticality
- symmetry classification of disordered electronic systems and of corresponding σ-models
- mechanisms of delocalization and criticality in 2D systems: symmetries and topology
- disordered Dirac fermions in graphene

Evers, ADM, "Anderson transitions", Rev. Mod. Phys. 80, 1355 (2008)

Basics of disorder diagrammatics

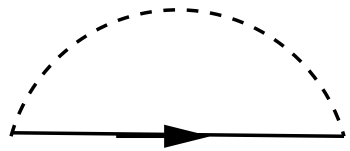
Hamiltonian $$H = H_0 + V(\mathrm{r}) \equiv \frac{(-i\nabla)^2}{2m} + V(\mathrm{r})$$

Free Green function $$G_0^{R,A}(\epsilon, p) = (\epsilon - p^2/2m \pm i0)^{-1}$$

Disorder $$\langle V(\mathrm{r})V(\mathrm{r}')\rangle = W(\mathrm{r}-\mathrm{r}')$$

simplest model: white noise $$W(\mathrm{r}-\mathrm{r}') = \Gamma\delta(\mathrm{r}-\mathrm{r}')$$

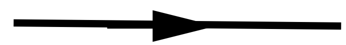
self-energy $\Sigma(\epsilon, p)$



$$\mathrm{Im}\,\Sigma_R = \Gamma\int(dp)\mathrm{Im}\frac{1}{\epsilon - p^2/2m + i0} = \pi\nu\Gamma \equiv -\frac{1}{2\tau}\ ,$$

τ – **mean free time**

disorder-averaged Green function $G(\epsilon, p)$



$$G^{R,A}(\epsilon, p) = \frac{1}{\epsilon - p^2/2m - \Sigma_{R,A}} \simeq \frac{1}{\epsilon - p^2/2m \pm i/2\tau}$$

$$G^{R,A}(\epsilon, r) \simeq G_0^{R,A}(\epsilon, r)e^{-r/2l}\ ,$$

$l = v_F\tau$ – **mean free path**

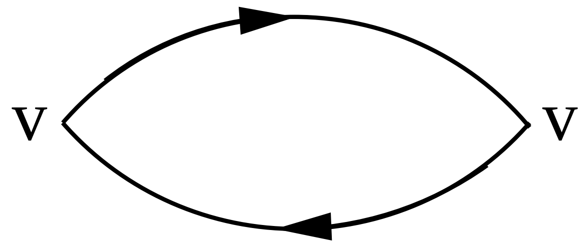
Conductivity

Kubo formula $$\sigma_{\mu\nu}(\omega) = \frac{1}{i\omega}\left\{\frac{i}{\hbar}\int_0^\infty dt \int dr e^{i\omega t}\langle[j_\mu(r,t), j_\nu(0,0)]\rangle - \frac{ne^2}{m}\delta_{\mu\nu}\right\}$$

Non-interacting electrons, $T, \omega \ll \epsilon_F$:

$$\sigma_{xx}(\omega) \simeq \frac{e^2}{2\pi V}\mathrm{Tr}\, \hat{v}_x G^R_{\epsilon+\omega}\hat{v}_x(G^A_\epsilon - G^R_\epsilon) \qquad \epsilon \equiv \epsilon_F$$

Drude conductivity:

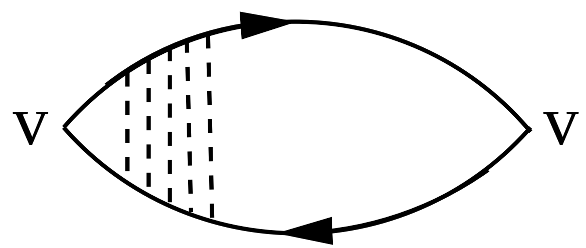


$$\sigma_{xx} = \frac{e^2}{2\pi}\int (dp)\frac{1}{m^2}p_x^2 G^R_{\epsilon+\omega}(p)[G^A_\epsilon(p) - G^R_\epsilon(p)]$$

$$\simeq \frac{e^2}{2\pi}\nu\frac{v_F^2}{d}\int d\xi_p \frac{1}{(\omega - \xi_p + \frac{i}{2\tau})(-\xi_p - \frac{i}{2\tau})} = e^2\frac{\nu v_F^2}{d}\frac{\tau}{1 - i\omega\tau}\,, \qquad \xi_p = \frac{p^2}{2m} - \epsilon$$

Finite-range disorder $\longrightarrow$ **anisotropic scattering**

$\longrightarrow$ **vertex correction** , $\tau \longrightarrow \tau_{\mathrm{tr}}$

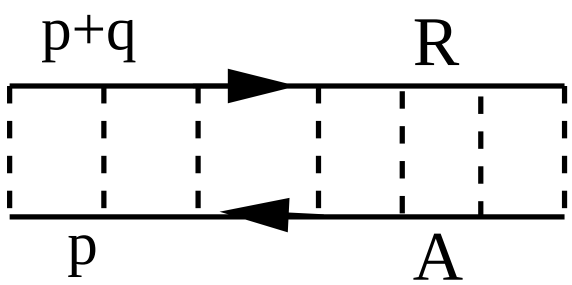


$$\frac{1}{\tau} = \nu\int\frac{d\phi}{2\pi}w(\phi) \qquad \frac{1}{\tau_{\mathrm{tr}}} = \nu\int\frac{d\phi}{2\pi}w(\phi)(1 - \cos\phi)$$

Diffuson and Cooperon

$$\mathcal{D}(q,\omega) = (2\pi\nu\tau)^{-2} \int d(r-r')\langle G^R_\epsilon(r',r)G^A_{\epsilon+\omega}(r,r')\rangle e^{-iq(r-r')}$$

Ladder diagrams (diffuson)

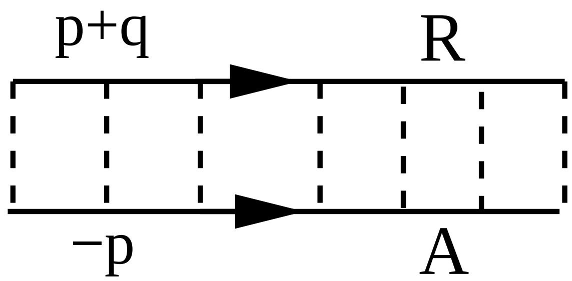


$$\frac{1}{2\pi\nu\tau}\sum_{n=0}^{\infty}\left[\frac{1}{2\pi\nu\tau}\int (dp) G^R_{\epsilon+\omega}(p+q)G^A_\epsilon(p)\right]^n$$

$$\int G^R G^A \simeq \int d\xi_p \frac{d\phi}{2\pi}\frac{1}{(\omega-\xi_p - v_F q\cos\phi + \frac{i}{2\tau})(-\xi_p - \frac{i}{2\tau})} = 2\pi\nu\tau[1-\tau(Dq^2 - i\omega)]$$

$$\mathcal{D}(q,\omega) = \frac{1}{2\pi\nu\tau^2}\,\frac{1}{Dq^2 - i\omega} \qquad \text{diffusion pole} \qquad ql,\omega\tau \ll 1$$

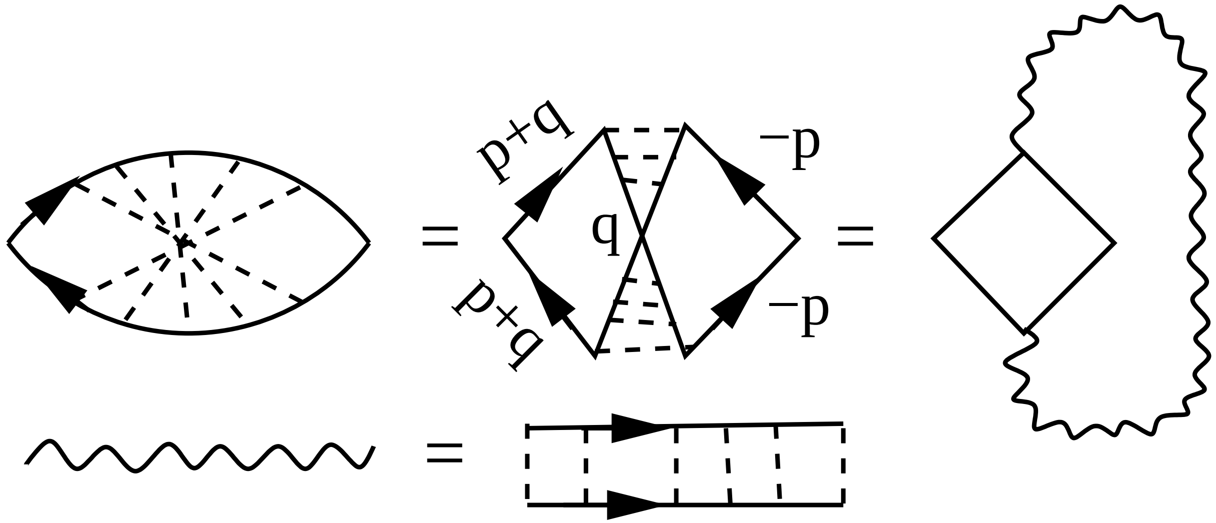
$$\left(\partial/\partial t - D\nabla^2_r\right)\mathcal{D}(r-r',t-t') = 2\pi\nu\delta(r-r')\delta(t-t')$$

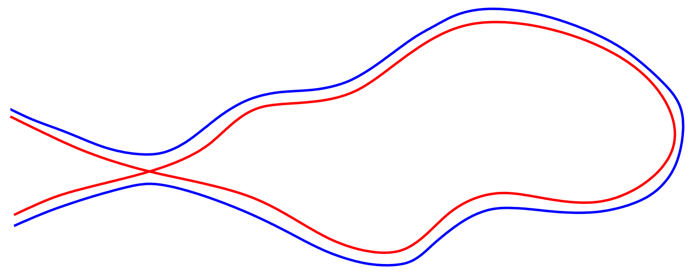


Weak. loc. correction: Cooperon $\mathcal{C}(q,\omega)$

Time-reversal symmetry preserved, no interaction $\longrightarrow$ $\mathcal{C}(q,\omega) = \mathcal{D}(q,\omega)$

Weak localization (orthogonal symmetry class)





Cooperon loop (interference of time-reversed paths)

$$\Delta\sigma_{\rm WL} \simeq -\frac{e^2}{2\pi}\frac{v_F^2}{d}\nu\int d\xi_p G_R^2 G_A^2 \int (dq)\frac{1}{2\pi\nu\tau^2}\,\frac{1}{Dq^2 - i\omega} = -\sigma_0\frac{1}{\pi\nu}\int\frac{(dq)}{Dq^2 - i\omega}$$

$$\Delta\sigma_{\rm WL} = -\frac{e^2}{(2\pi)^2}\left(\frac{\sim 1}{l} - \frac{1}{L_\omega}\right) , \qquad \textbf{3D}$$

$$L_\omega = \left(\frac{D}{-i\omega}\right)^{1/2}$$

$$\Delta\sigma_{\rm WL} = -\frac{e^2}{2\pi^2}\ln\frac{L_\omega}{l} , \qquad \textbf{2D}$$

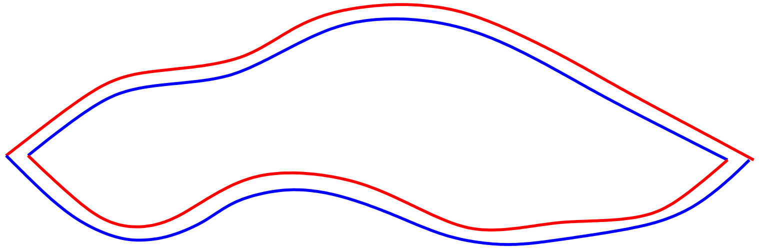
$$\Delta\sigma_{\rm WL} = -\frac{e^2}{2\pi}L_\omega , \qquad \textbf{quasi-1D}$$

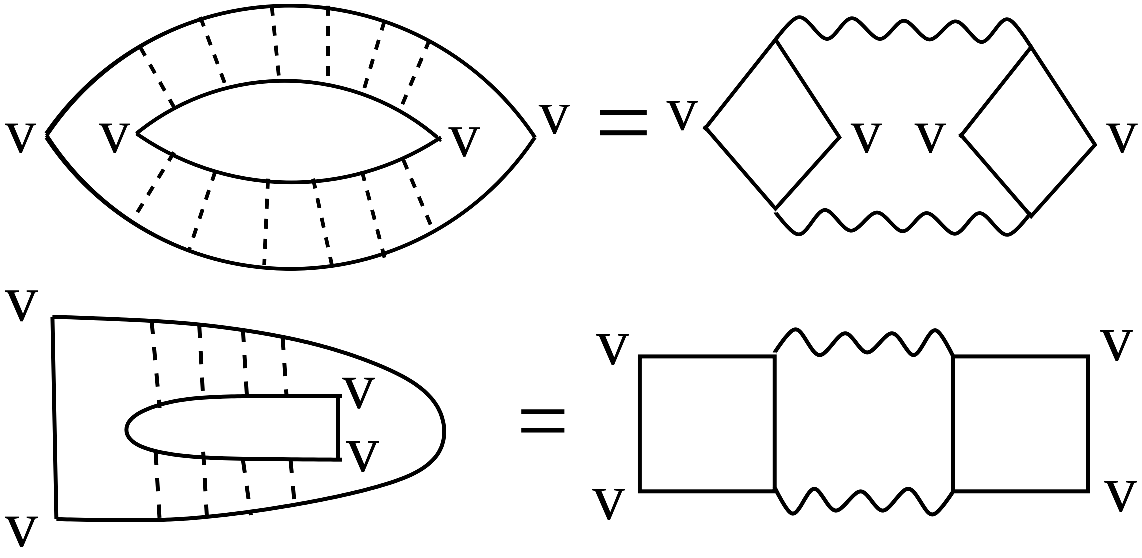
Generally: IR cutoff

$$L_\omega \longrightarrow \min\{L_\omega, L_\phi, L, L_H\}$$

Mesoscopic conductance fluctuations

$\langle(\delta G)^2\rangle \sim \langle(\sum_{i\neq j} A_i^* A_j)^2\rangle \sim \sum_{i\neq j}\langle|A_i|^2\rangle\langle|A_j|^2\rangle$





$$\langle(\delta\sigma)^2\rangle = 3\left(\frac{e^2}{2\pi V}\right)^2 (4\pi\nu\tau^2 D)^2 \sum_{\mathrm{q}} \left(\frac{1}{2\pi\nu\tau^2 D\mathrm{q}^2}\right)^2 = 12\left(\frac{e^2}{2\pi V}\right)^2 \sum_{\mathrm{q}} \left(\frac{1}{\mathrm{q}^2}\right)^2$$

$$\langle(\delta g)^2\rangle = \frac{12}{\pi^4}\sum_{\mathrm{n}}\left(\frac{1}{\mathrm{n}^2}\right)^2 \qquad n_x = 1,2,3,\ldots\ , \qquad n_{y,z} = 0,1,2,\ldots$$

$\langle(\delta g)^2\rangle \sim 1$ independent of system size; depends only on geometry!

$\longrightarrow$ universal conductance fluctuations (UCF)

quasi-1D geometry: $\langle(\delta g)^2\rangle = 8/15$

Mesoscopic conductance fluctuations (cont'd)

Additional comments:

- UCF are anomalously strong from classical point of view:

  $\langle(\delta g)^2\rangle/g^2 \sim L^{4-2d} \gg L^{-d}$

  reason: quantum coherence

- UCF are universal for $L \ll L_T, L_\phi$; otherwise fluctuations suppressed
- symmetry dependent: $8 = 2$ (Cooperons) $\times$ 4 (spin)
- autocorrelation function $\langle\delta g(B)\delta g(B+\delta B)\rangle$; magnetofingerprints
- mesoscopic fluctuations of various observables

Strong localization

WL correction is IR-divergent in quasi-1D and 2D; becomes $\sim \sigma_0$ at a scale

$$\xi \sim 2\pi\nu D \,, \qquad \text{quasi-1D}$$

$$\xi \sim l \exp(2\pi^2\nu D) = l \exp(\pi g) \,, \qquad \text{2D}$$

indicates strong localization, ξ – localization length

confirmed by exact solution in quasi-1D and renormalization group in 2D

Philip W. Anderson

1958 "Absence of diffusion in certain random lattices"

Disorder-induced localization

$\longrightarrow$ Anderson insulator

The Nobel Prize in Physics 1977

Metal vs Anderson insulator

Localization transition $\longrightarrow$ change in behavior of diffusion propagator,

$$\Pi(\mathrm{r}_1, \mathrm{r}_2; \omega) = \langle G^R_{\epsilon+\omega/2}(\mathrm{r}_1, \mathrm{r}_2) G^A_{\epsilon-\omega/2}(\mathrm{r}_2, \mathrm{r}_1)\rangle,$$

Delocalized regime: Π has the diffusion form:

$$\Pi(\mathrm{q}, \omega) = 2\pi\nu(\epsilon)/(Dq^2 - i\omega),$$

Insulating phase: propagator ceases to have Goldstone form, becomes massive,

$$\Pi(\mathrm{r}_1, \mathrm{r}_2; \omega) \simeq \frac{2\pi\nu(\epsilon)}{-i\omega}\mathcal{F}(|\mathrm{r}_1 - \mathrm{r}_2|/\xi),$$

$\mathcal{F}(\mathrm{r})$ decays on the scale of the localization length, $\mathcal{F}(r/\xi) \sim \exp(-r/\xi)$.

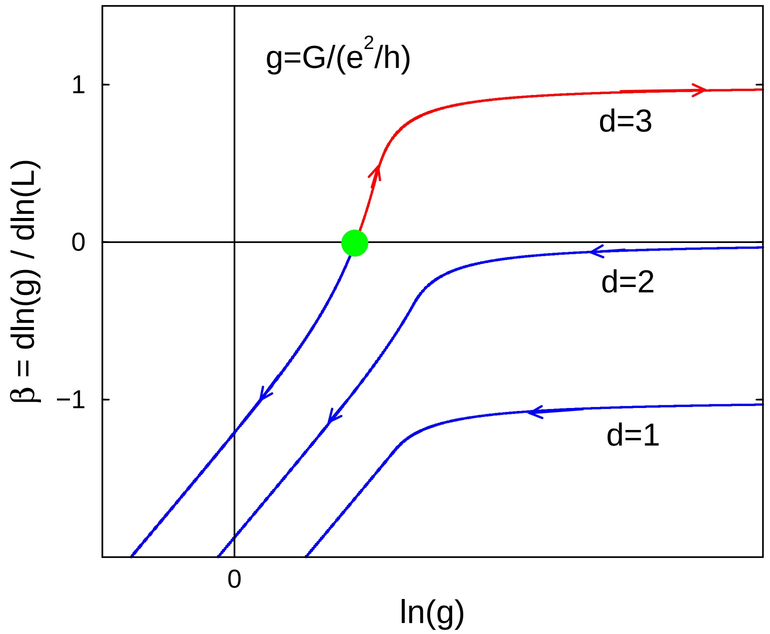
Comment:

Localization length ξ obtained from the averaged correlation function $\Pi = \langle G^R G^A\rangle$ is in general different from the one governing the exponential decay of the typical value $\Pi_{\mathrm{typ}} = \exp\langle \ln G^R G^A\rangle$.

E.g., in quasi-1D systems: $\xi_{\mathrm{av}} = 4\xi_{\mathrm{typ}}$

This is usually not important for the definition of the critical index ν.

Anderson transition

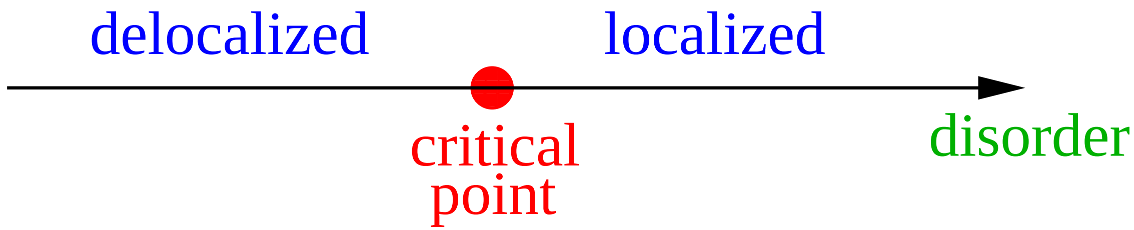


Scaling theory of localization:
Abrahams, Anderson, Licciardello, Ramakrishnan '79

Modern approach:
RG for field theory (σ-model)

quasi-1D, 2D : metallic $\longrightarrow$ localized crossover with decreasing g

$d > 2$: Anderson metal-insulator transition (sometimes also in $d = 2$)



Continuous phase transition with highly unconventional properties!

Field theory: non-linear σ-model

$$S[Q] = \frac{\pi\nu}{4}\int d^d r \,\mathrm{Str}\,[-D(\nabla Q)^2 - 2i\omega\Lambda Q], \qquad Q^2(r) = 1$$

Wegner'79 (replicas); Efetov'83 (supersymmetry)

(non-equilibrium: Keldysh σ-model, will not discuss here)

σ-model manifold:

- unitary class:
  - fermionic replicas: $\mathrm{U}(2n)/\mathrm{U}(n) \times \mathrm{U}(n)$, $n \to 0$
  - bosonic replicas: $\mathrm{U}(n,n)/\mathrm{U}(n) \times \mathrm{U}(n)$, $n \to 0$
  - supersymmetry: $\mathrm{U}(1,1|2)/\mathrm{U}(1|1) \times \mathrm{U}(1|1)$
- orthogonal class:
  - fermionic replicas: $\mathrm{Sp}(4n)/\mathrm{Sp}(2n) \times \mathrm{Sp}(2n)$, $n \to 0$
  - bosonic replicas: $\mathrm{O}(2n,2n)/\mathrm{O}(2n) \times \mathrm{O}(2n)$, $n \to 0$
  - supersymmetry: $\mathrm{OSp}(2,2|4)/\mathrm{OSp}(2|2) \times \mathrm{OSp}(2|2)$

in general, in supersymmetry:

$Q \in \{$"sphere" $\times$ "hyperboloid"$\}$ "dressed" by anticommuting variables

Non-linear σ-model: Sketch of derivation

Consider unitary class for simplicity

- introduce supervector field $\Phi = (S_1, \chi_1, S_2, \chi_2)$:

$$G^R_{E+\omega/2}(\mathrm{r}_1, \mathrm{r}_2) G^A_{E-\omega/2}(\mathrm{r}_2, \mathrm{r}_1) = \int D\Phi\, D\Phi^\dagger S_1(\mathrm{r}_1) S_1^*(\mathrm{r}_2) S_2(\mathrm{r}_2) S_2^*(\mathrm{r}_1)$$

$$\times \exp\left\{ i \int d\mathrm{r} \Phi^\dagger(\mathrm{r})[(E - \hat{H})\Lambda + \frac{\omega}{2} + i\eta]\Phi(\mathrm{r}) \right\},$$

where $\Lambda = \mathrm{diag}\{1, 1, -1, -1\}$. No denominator! $Z = 1$

- disorder averaging $\longrightarrow$ quartic term $(\Phi^\dagger \Phi)^2$
- Hubbard-Stratonovich transformation:

quartic term decoupled by a Gaussian integral over a 4×4 supermatrix variable $\mathcal{R}_{\mu\nu}(\mathrm{r})$ conjugate to the tensor product $\Phi_\mu(\mathrm{r})\Phi^\dagger_\nu(\mathrm{r})$

- integrate out Φ fields $\longrightarrow$ action in terms of the $\mathcal{R}$ fields:

$$S[\mathcal{R}] = \pi\nu\tau \int d^d\mathrm{r}\, \mathrm{Str}\mathcal{R}^2 + \mathrm{Str}\ln[E + (\frac{\omega}{2} + i\eta)\Lambda - \hat{H}_0 - \mathcal{R}]$$

- saddle-point approximation $\longrightarrow$ equation for $\mathcal{R}$:

$$\mathcal{R}(\mathrm{r}) = (2\pi\nu\tau)^{-1} \langle \mathrm{r} | (E - \hat{H}_0 - \mathcal{R})^{-1} | \mathrm{r} \rangle$$

Non-linear σ-model: Sketch of derivation (cont'd)

The relevant set of the solutions (the saddle-point manifold) has the form:

$$\mathcal{R} = \Sigma \cdot I - (i/2\tau) Q \ , \qquad Q = T^{-1} \Lambda T \ , \qquad Q^2 = 1$$

Q – 4×4 supermatrix on the σ-model target space

- gradient expansion with a slowly varying $Q(\mathrm{r})$ $\longrightarrow$

$$\Pi(\mathrm{r}_1, \mathrm{r}_2; \omega) = \int DQ \, Q_{12}^{bb}(\mathrm{r}_1) Q_{21}^{bb}(\mathrm{r}_2) e^{-S[Q]},$$

where $S[Q]$ is the σ-model action

$$S[Q] = \frac{\pi \nu}{4} \int d^d \mathrm{r} \, \mathrm{Str} \left[-D(\nabla Q)^2 - 2i\omega \Lambda Q \right],$$

- size of Q-matrix: $4 = 2$ (Adv.–Ret.) $\times$ 2 (Bose–Fermi)
- orthogonal & symplectic classes (preserved time-reversal)

$\longrightarrow$ $8 = 2$ (Adv.–Ret.) $\times$ 2 (Bose–Fermi) $\times$ 2 (Diff.-Coop.)

- product of N retarded and N advanced Green functions

$\longrightarrow$ σ-model defined on a larger manifold, with the base being a product of $\mathrm{U}(N,N)/\mathrm{U}(N) \times \mathrm{U}(N)$ and $\mathrm{U}(2N)/\mathrm{U}(N) \times \mathrm{U}(N)$

σ model: Perturbative treatment

For comparison, consider a ferromagnet model in an external magnetic field:

$$H[\mathrm{S}] = \int d^d r \left[\frac{\kappa}{2}(\nabla \mathrm{S}(r))^2 - \mathrm{BS}(r)\right] , \qquad \mathrm{S}^2(r) = 1$$

n-component vector σ-model. Target manifold: sphere $\mathrm{S}^{n-1} = \mathrm{O}(n)/\mathrm{O}(n-1)$

Independent degrees of freedom: transverse part $\mathrm{S}_\perp$; $\quad S_1 = (1 - \mathrm{S}_\perp^2)^{1/2}$

$$H[\mathrm{S}_\perp] = \frac{1}{2}\int d^d r \left[\kappa[\nabla \mathrm{S}_\perp(r)]^2 + B\mathrm{S}_\perp^2(r) + O(\mathrm{S}_\perp^4(r))\right]$$

Ferromagnetic phase: symmetry is broken; Goldstone modes – spin waves:

$$\langle \mathrm{S}_\perp \mathrm{S}_\perp \rangle_q \propto \frac{1}{\kappa q^2 + B}$$

$$Q = \left(1 - \frac{W}{2}\right)\Lambda\left(1 - \frac{W}{2}\right)^{-1} = \Lambda\left(1 + W + \frac{W^2}{2} + \ldots\right) ; \qquad W = \begin{pmatrix} 0 & W_{12} \\ W_{21} & 0 \end{pmatrix}$$

$$S[W] = \frac{\pi\nu}{4}\int d^d r \,\mathrm{Str}[D(\nabla W)^2 - i\omega W^2 + O(W^3)]$$

theory of “interacting” diffusion modes. Goldstone mode: diffusion propagator

$$\langle W_{12} W_{21} \rangle_{q,\omega} \sim \frac{1}{\pi\nu(Dq^2 - i\omega)}$$

σ-models: What are they good for?

- reproduce diffuson-cooperon diagrammatics . . .

  . . . and go beyond it:

- metallic samples ($g \gg 1$):

  level & wavefunction statistics: random matrix theory + deviations

- quasi-1D samples:

  exact solution, crossover from weak to strong localization

- Anderson transitions: RG treatment, phase diagrams, critical exponents

- non-trivial saddle-points:

  nonperturbative effects, asymptotic tails of distributions

- generalizations: interaction, non-equilibrium (Keldysh)

Quasi-1D geometry: Exact solution of the σ-model

quasi-1D geometry (many-channel wire) $\longrightarrow$ 1D σ-model

$\longrightarrow$ "quantum mechanics" , longitudinal coordinate – (imaginary) "time"

$\longrightarrow$ "Schroedinger equation" of the type $\partial_t W = \Delta_Q W \ , \quad t = x/\xi$

- Localization length, diffusion propagator — Efetov, Larkin '83
- Exact solution for the statistics of eigenfunctions — Fyodorov, ADM '92-94
- Exact $\langle g\rangle(L/\xi)$ and $\mathrm{var}(g)(L/\xi)$ — Zirnbauer, ADM, Müller-Groeling '92-94

e.g. for orthogonal symmetry class:

$$\langle g^n\rangle(L) = \frac{\pi}{2}\int_0^\infty d\lambda \tanh^2(\pi\lambda/2)(\lambda^2+1)^{-1}p_n(1,\lambda,\lambda)\exp\left[-\frac{L}{2\xi}(1+\lambda^2)\right]$$
$$+\ 2^4\sum_{l\in 2N+1}\int_0^\infty d\lambda_1 d\lambda_2 l(l^2-1)\lambda_1\tanh(\pi\lambda_1/2)\lambda_2\tanh(\pi\lambda_2/2)$$
$$\times\, p_n(l,\lambda_1,\lambda_2)\prod_{\sigma,\sigma_1,\sigma_2=\pm1}(-1+\sigma l+i\sigma_1\lambda_1+i\sigma_2\lambda_2)^{-1}\exp\left[-\frac{L}{4\xi}(l^2+\lambda_1^2+\lambda_2^2+1)\right]$$

$$p_1(l,\lambda_1,\lambda_2) = l^2+\lambda_1^2+\lambda_2^2+1,$$
$$p_2(l,\lambda_1,\lambda_2) = \frac{1}{2}(\lambda_1^4+\lambda_2^4+2l^4+3l^2(\lambda_1^2+\lambda_2^2)+2l^2-\lambda_1^2-\lambda_2^2-2)$$

Quasi-1D geometry: Exact solution of the σ-model (cont'd)

$L \ll \xi$ asymptotics:

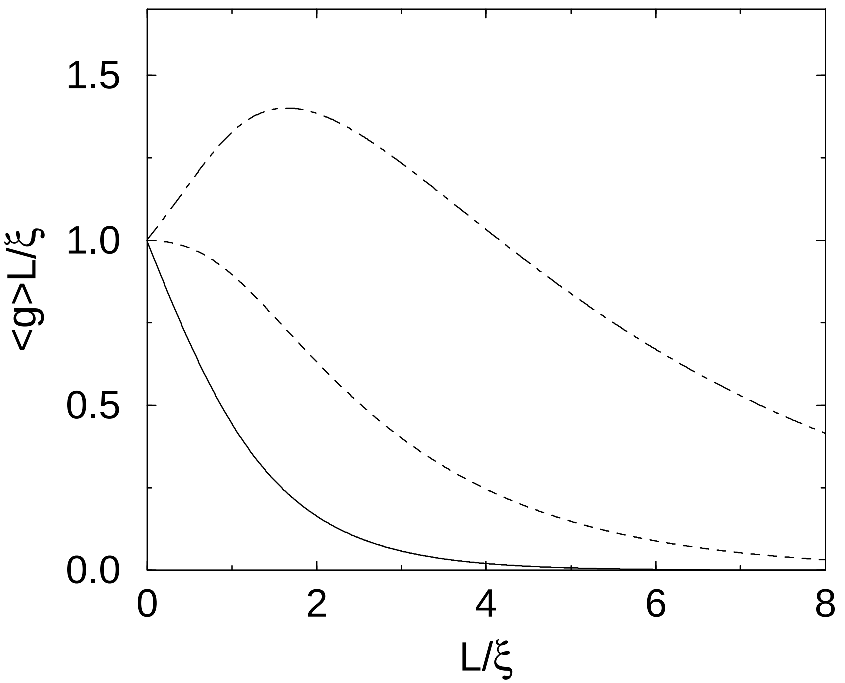
$$\langle g \rangle(L) = \frac{2\xi}{L} - \frac{2}{3} + \frac{2}{45}\frac{L}{\xi} + \frac{4}{945}\left(\frac{L}{\xi}\right)^2 + O\left(\frac{L}{\xi}\right)^3$$

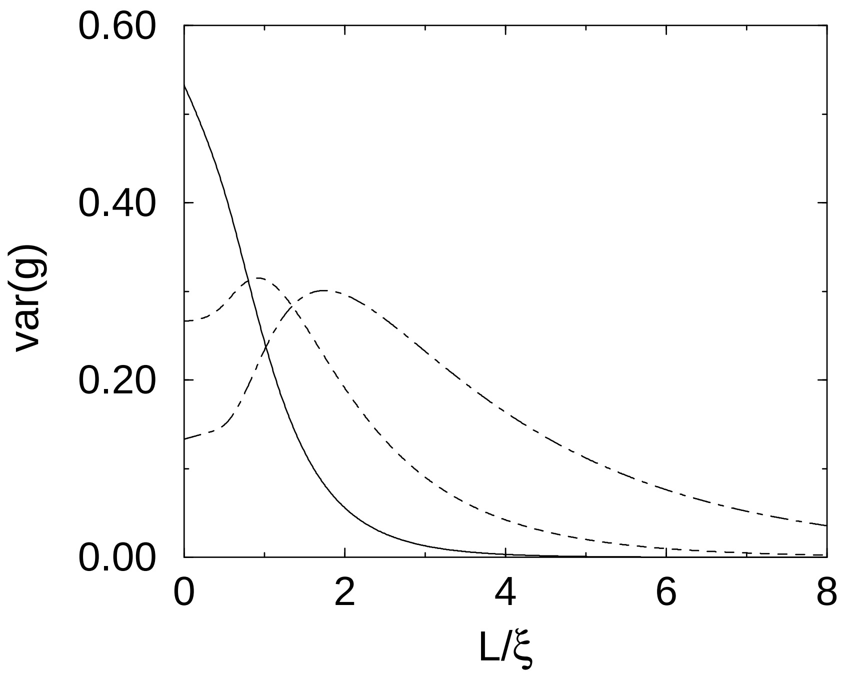
and

$$\mathrm{var}(g(L)) = \frac{8}{15} - \frac{32}{315}\frac{L}{\xi} + O\left(\frac{L}{\xi}\right)^2 .$$

$L \gg \xi$ asymptotics:

$$\langle g^n \rangle = 2^{-3/2-n}\pi^{7/2}(\xi/L)^{3/2}e^{-L/2\xi}$$





orthogonal (full), unitary (dashed), symplectic (dot-dashed)

Renormalization group and ϵ–expansion

analytical treatment of Anderson transition:

RG and ϵ-expansion for $d = 2 + \epsilon$ dimensions

β-function $\quad \beta(t) = -\dfrac{dt}{d\ln L}; \quad t = 1/2\pi g\ ,\qquad g$ – dimensionless conductance

orthogonal class (preserved spin and time reversal symmetry):

$$\beta(t) = \epsilon t - 2t^2 - 12\zeta(3)t^5 + O(t^6) \qquad \text{beta-function}$$

$$t_* = \frac{\epsilon}{2} - \frac{3}{8}\zeta(3)\epsilon^4 + O(\epsilon^5) \qquad \text{transition point}$$

$$\nu = -1/\beta'(t_*) = \epsilon^{-1} - \frac{9}{4}\zeta(3)\epsilon^2 + O(\epsilon^3) \qquad \text{localization length exponent}$$

$$s = \nu\epsilon = 1 - \frac{9}{4}\zeta(3)\epsilon^3 + O(\epsilon^4) \qquad \text{conductivity exponent}$$

Numerics for 3D: $\quad \nu \simeq 1.57 \pm 0.02 \qquad$ Slevin, Ohtsuki ’99

RG for σ-models of all Wigner-Dyson classes

- orthogonal symmetry class (preserved T and S): $t = 1/2\pi g$

$$\beta(t) = \epsilon t - 2t^2 - 12\zeta(3)t^5 + O(t^6) \; ; \qquad t_* = \frac{\epsilon}{2} - \frac{3}{8}\zeta(3)\epsilon^4 + O(\epsilon^5)$$

$$\nu = -1/\beta'(t_*) = \epsilon^{-1} - \frac{9}{4}\zeta(3)\epsilon^2 + O(\epsilon^3) \; ; \qquad s = \nu\epsilon = 1 - \frac{9}{4}\zeta(3)\epsilon^3 + O(\epsilon^4)$$

- unitary class (broken T):

$$\beta(t) = \epsilon t - 2t^3 - 6t^5 + O(t^7) \; ; \qquad t_* = \left(\frac{\epsilon}{2}\right)^{1/2} - \frac{3}{2}\left(\frac{\epsilon}{2}\right)^{3/2} + O(\epsilon^{5/2});$$

$$\nu = \frac{1}{2\epsilon} - \frac{3}{4} + O(\epsilon) \; ; \qquad s = \frac{1}{2} - \frac{3}{4}\epsilon + O(\epsilon^2).$$

- symplectic class (preserved T, broken S):

$$\beta(t) = \epsilon t + t^2 - \frac{3}{4}\zeta(3)t^5 + O(t^6)$$

$\longrightarrow$ metal insulator transition in 2D at $t_* \sim 1$

Multifractality at the Anderson transition

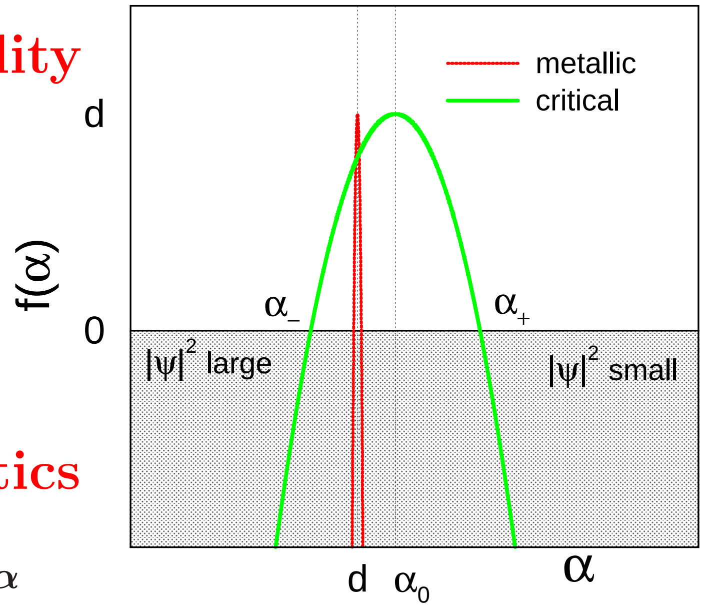
$P_q = \int d^d r |\psi(\mathrm{r})|^{2\mathrm{q}}$ inverse participation ratio

$$\langle P_q \rangle \sim \begin{cases} L^0 & \text{insulator} \\ L^{-\tau_q} & \text{critical} \\ L^{-d(q-1)} & \text{metal} \end{cases}$$

$$\tau_q = \underbrace{d(q-1)}_{\text{normal}} + \underbrace{\Delta_q}_{\text{anomalous}} \equiv D_q(q-1) \quad \text{multifractality}$$

$\tau_q \longrightarrow$ Legendre transformation

$\longrightarrow$ singularity spectrum $f(\alpha)$



$\mathcal{P}(\ln|\psi^2|) \sim L^{-d+f(\ln|\psi^2|/\ln L)}$ wave function statistics

$L^{f(\alpha)}$ – measure of the set of points where $|\psi|^2 \sim L^{-\alpha}$

Multifractality is characteristic for a variety of complex systems:

turbulence, strange attractors, diffusion-limited aggregation, . . .

Statistical ensemble $\longrightarrow$ $f(\alpha)$ may become negative

Multifractality and the field theory

$\boldsymbol{\Delta_q}$ – scaling dimensions of operators $\mathcal{O}^{(q)} \sim (Q\Lambda)^q$

$d = 2 + \epsilon$: $\Delta_q = -q(q-1)\epsilon + O(\epsilon^4)$ Wegner '80

- Infinitely many operators with negative scaling dimensions
- $\Delta_1 = 0 \longleftrightarrow \langle Q \rangle = \Lambda$ naive order parameter uncritical

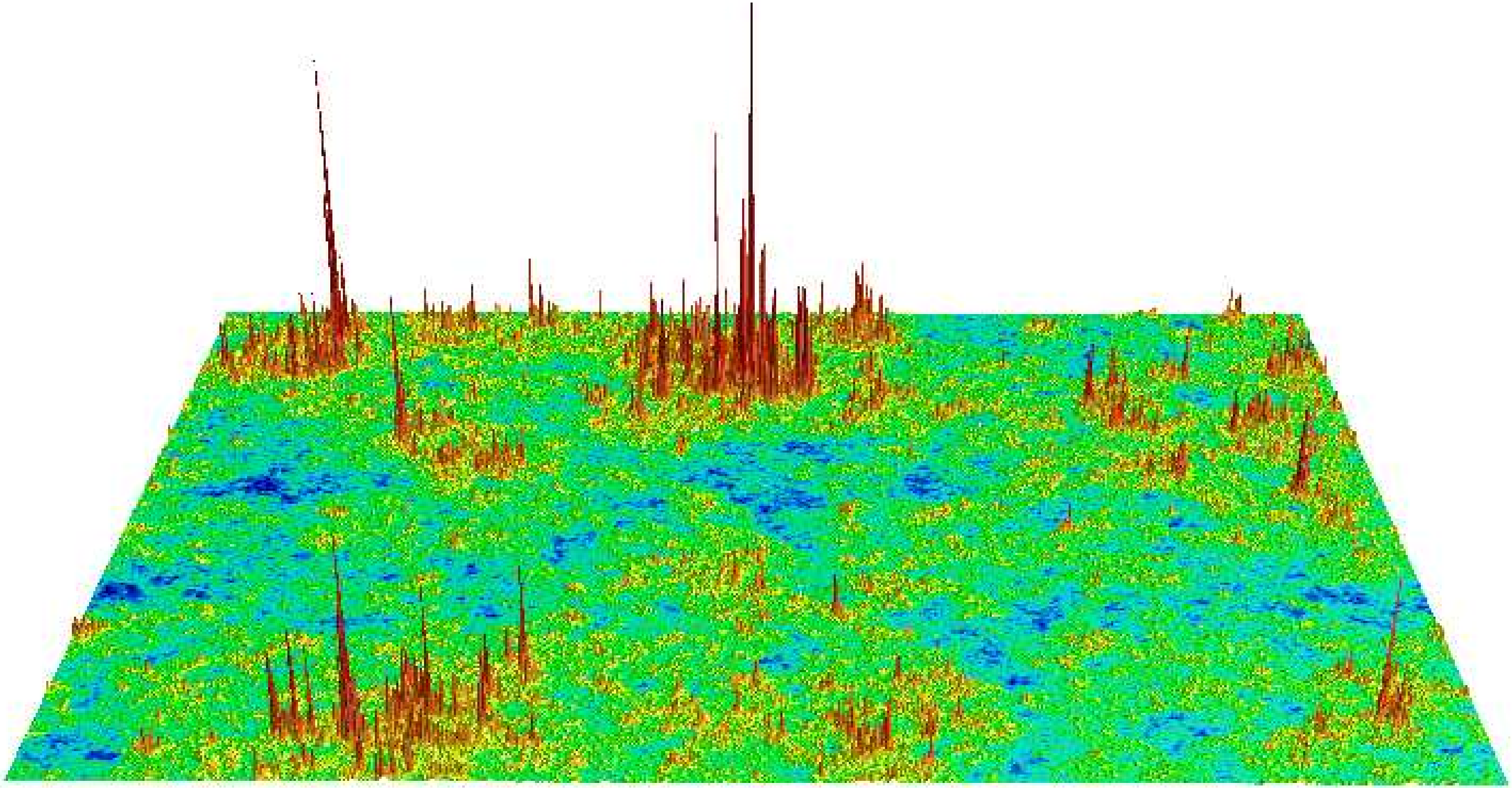
Transition described by an order parameter function $F(Q)$

Zirnbauer 86, Efetov 87

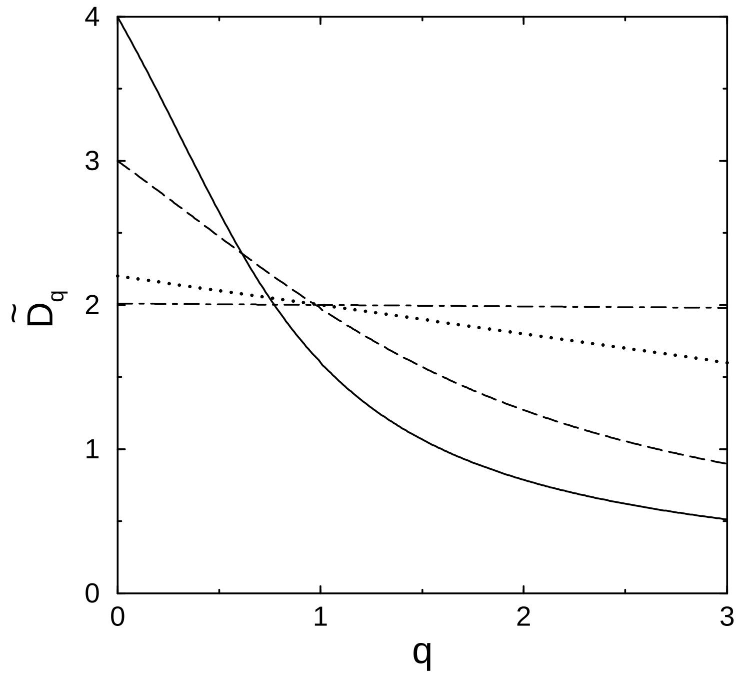
$\longleftrightarrow$ distribution of local Green functions and wave function amplitudes

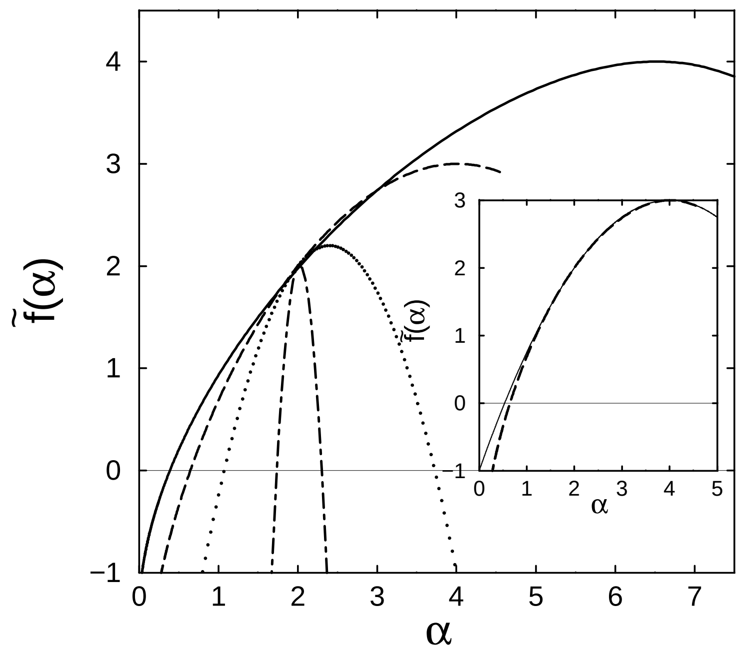
ADM, Fyodorov '91

Multifractal wave functions at the Quantum Hall transition



Dimensionality dependence of multifractality





Analytics ($2+\epsilon$, one-loop) and numerics

$$\tau_q = (q-1)d - q(q-1)\epsilon + O(\epsilon^4)$$

$$f(\alpha) = d - (d+\epsilon-\alpha)^2/4\epsilon + O(\epsilon^4)$$

$d = 4$ (full)
$d = 3$ (dashed)
$d = 2+\epsilon, \quad \epsilon = 0.2$ (dotted)
$d = 2+\epsilon, \quad \epsilon = 0.01$ (dot-dashed)

Inset: $d = 3$ (dashed)
vs. $d = 2+\epsilon, \quad \epsilon = 1$ (full)

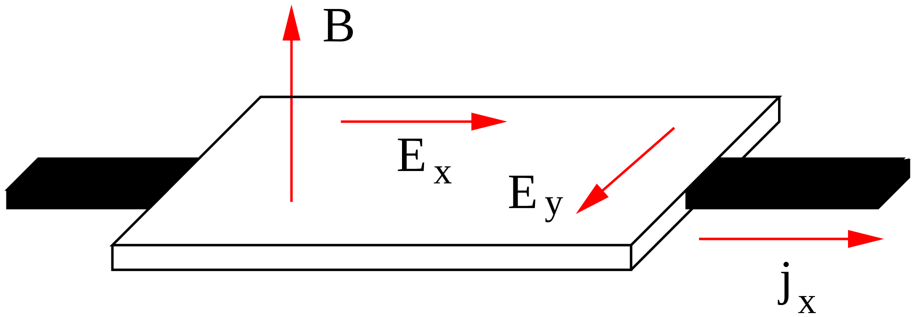
Mildenberger, Evers, ADM ’02

Magnetotransport

resistivity tensor:

$$\rho_{xx} = E_x / j_x$$

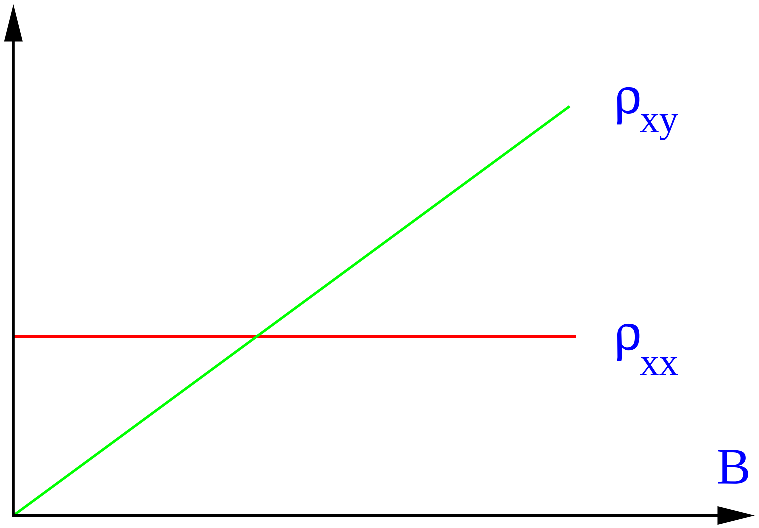
$$\rho_{yx} = E_y / j_x \quad \text{Hall resistivity}$$



classically (Drude–Boltzmann theory):

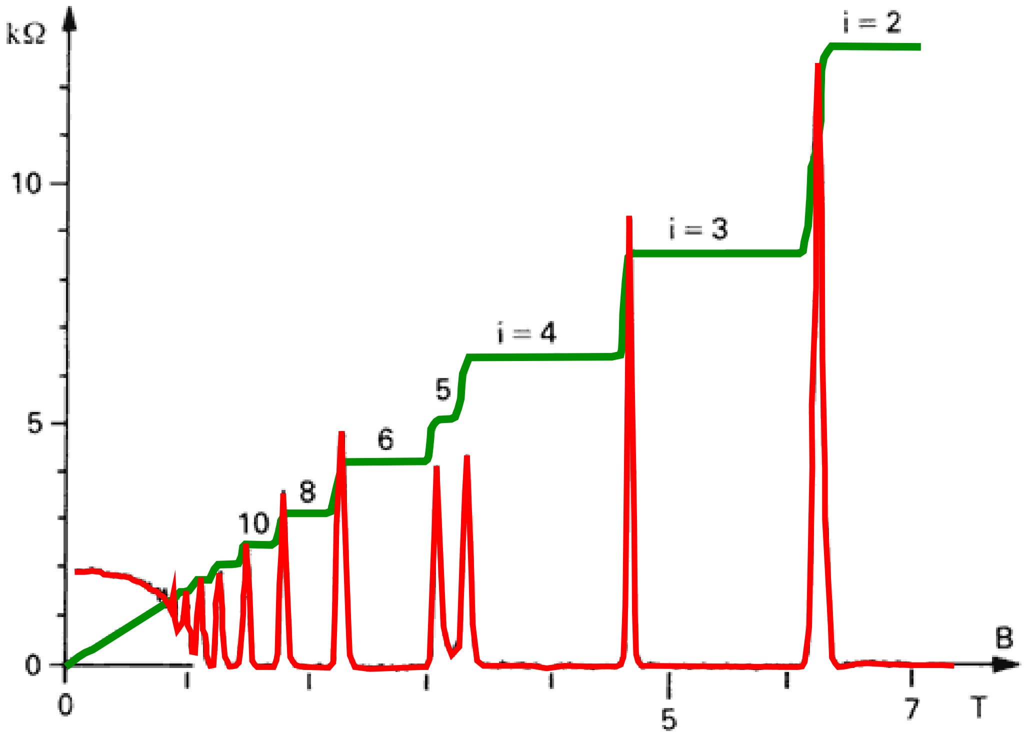
$$\rho_{xx} = \frac{m}{e^2 n_e \tau} \quad \text{independent of } B$$

$$\rho_{yx} = -\frac{B}{n_e e c}$$

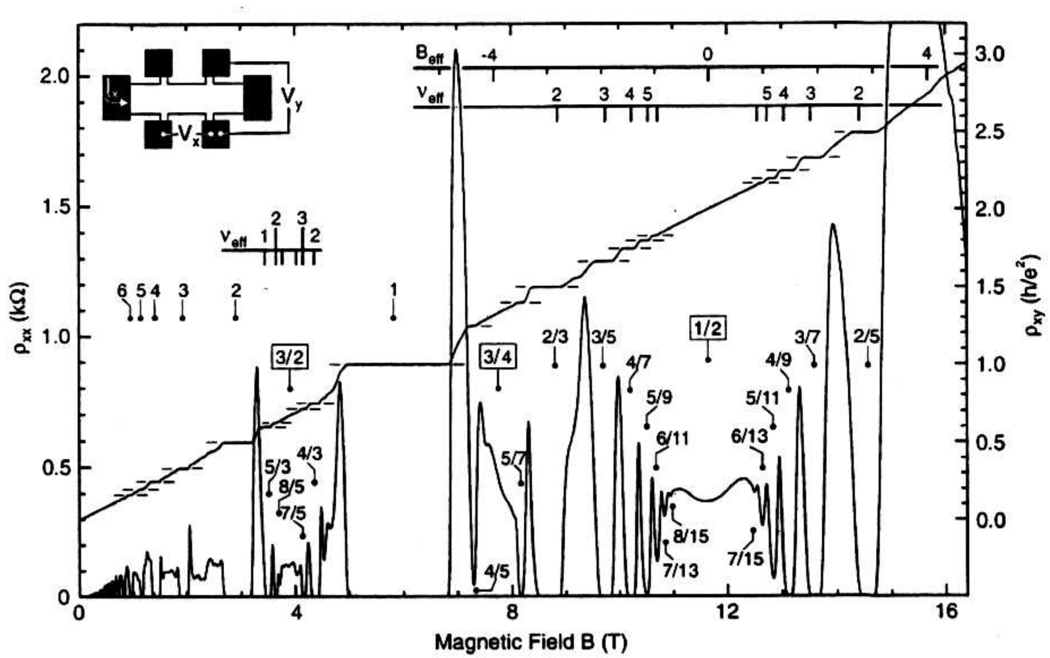


Quantum transport in strong magnetic fields

Integer Quantum Hall Effect (IQHE)



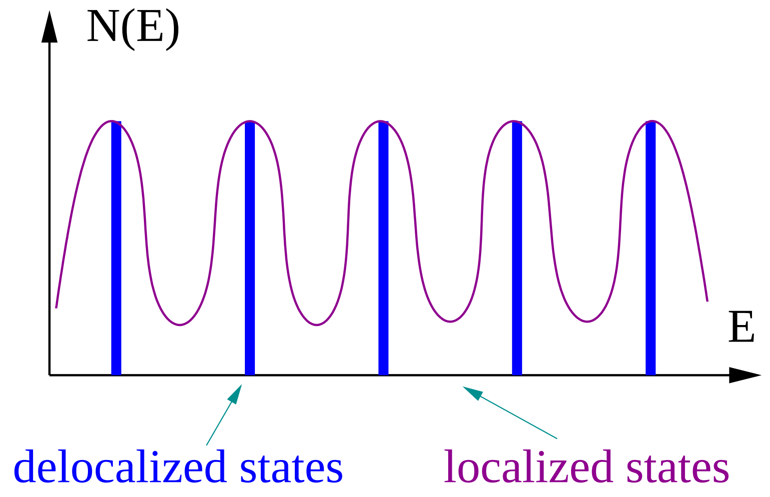
Fractional Quantum Hall Effect (FQHE)



Basics of IQHE

2D Electron in transverse magnetic field

$\longrightarrow$ Landau levels $E_n = \hbar\omega_c(n + 1/2)$



ω_c - cyclotron frequency

$\nu = \Phi_0 \frac{n}{B} = \frac{N_e}{N_\Phi}$ - filling factor

$\Phi_0 = \frac{hc}{e}$ - flux quantum

disorder $\longrightarrow$ Landau levels broadened

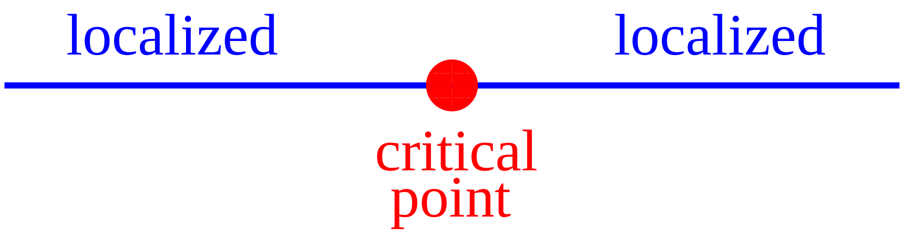
Anderson localization $\longrightarrow$ only states in the band center delocalized

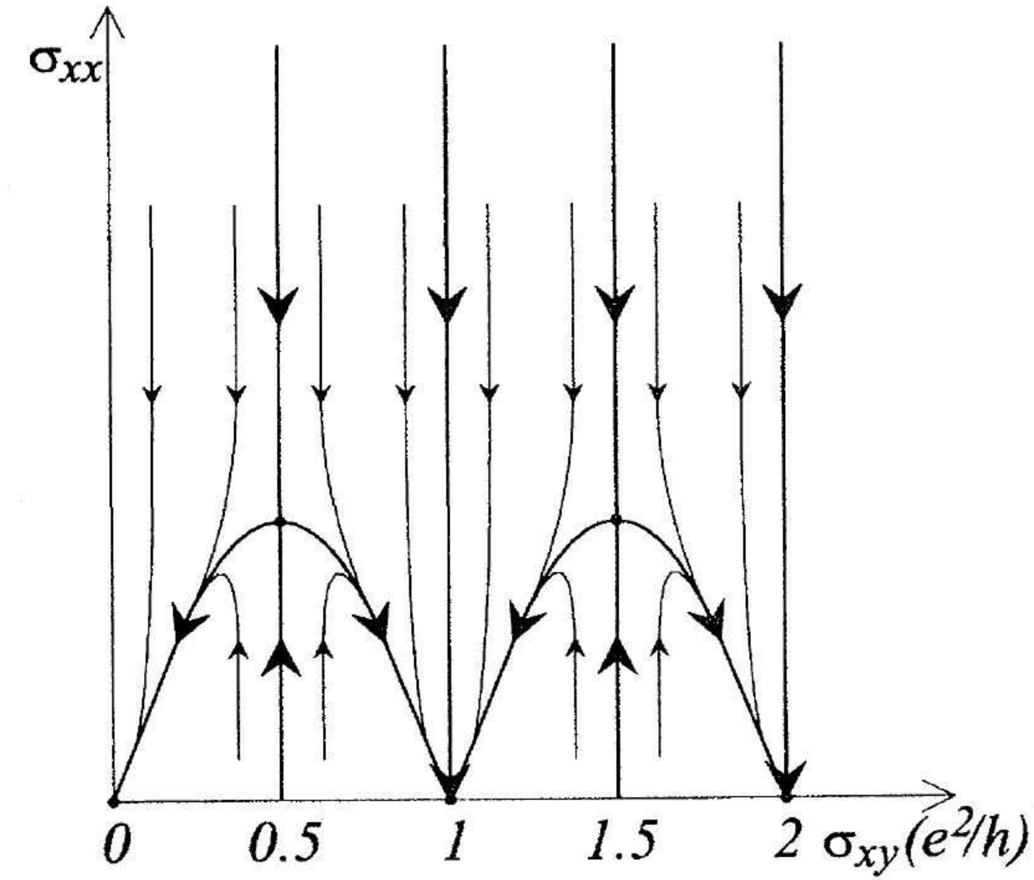
E_F in the range of localized states $\longrightarrow$ $\begin{cases} \text{quantized plateau in } \sigma_{xy} \\ \sigma_{xx} = 0 \end{cases}$

IQH transition

IQHE flow diagram

Khmelnitskii' 83, Pruisken' 84





Field theory (Pruisken):

σ-model with topological term (θ-term) $\qquad \theta = 2\pi\sigma_{xy}$

$$S = \int d^2r \left\{ -\frac{\sigma_{xx}}{8} \mathrm{Tr}(\partial_\mu Q)^2 + \frac{\sigma_{xy}}{8} \mathrm{Tr}\,\epsilon_{\mu\nu} Q \partial_\mu Q \partial_\nu Q \right\}$$

Disordered electronic systems: Symmetry classification

Altland, Zirnbauer '97

Conventional (Wigner-Dyson) classes

	T	spin rot.	chiral	p-h	symbol
GOE	+	+	−	−	AI
GUE	−	+/−	−	−	A
GSE	+	−	−	−	AII

Chiral classes

	T	spin rot.	chiral	p-h	symbol
ChOE	+	+	+	−	BDI
ChUE	−	+/−	+	−	AIII
ChSE	+	−	+	−	CII

$$\boldsymbol{H} = \begin{pmatrix} \mathbf{0} & \mathrm{t} \\ \mathrm{t}^\dagger & \mathbf{0} \end{pmatrix}$$

Bogoliubov-de Gennes classes

	T	spin rot.	chiral	p-h	symbol
	+	+	−	+	CI
	−	+	−	+	C
	+	−	−	+	DIII
	−	−	−	+	D

$$\boldsymbol{H} = \begin{pmatrix} \mathrm{h} & \boldsymbol{\Delta} \\ -\boldsymbol{\Delta}^* & -\mathrm{h}^T \end{pmatrix}$$

Disordered electronic systems: Symmetry classification

Ham. class	RMT	T	S	compact symmetric space	non-compact symmetric space	σ-model B|F	σ-model compact sector $\mathcal{M}_F$
Wigner-Dyson classes							
A	GUE	$-$	$\pm$	$\mathrm{U}(N)$	$\mathrm{GL}(N,\mathbb{C})/\mathrm{U}(N)$	AIII|AIII	$\mathrm{U}(2n)/\mathrm{U}(n)\times\mathrm{U}(n)$
AI	GOE	$+$	$+$	$\mathrm{U}(N)/\mathrm{O}(N)$	$\mathrm{GL}(N,\mathbb{R})/\mathrm{O}(N)$	BDI|CII	$\mathrm{Sp}(4n)/\mathrm{Sp}(2n)\times\mathrm{Sp}(2n)$
AII	GSE	$+$	$-$	$\mathrm{U}(2N)/\mathrm{Sp}(2N)$	$\mathrm{U}^*(2N)/\mathrm{Sp}(2N)$	CII|BDI	$\mathrm{O}(2n)/\mathrm{O}(n)\times\mathrm{O}(n)$
chiral classes							
AIII	chGUE	$-$	$\pm$	$\mathrm{U}(p+q)/\mathrm{U}(p)\times\mathrm{U}(q)$	$\mathrm{U}(p,q)/\mathrm{U}(p)\times\mathrm{U}(q)$	A|A	$\mathrm{U}(n)$
BDI	chGOE	$+$	$+$	$\mathrm{SO}(p+q)/\mathrm{SO}(p)\times\mathrm{SO}(q)$	$\mathrm{SO}(p,q)/\mathrm{SO}(p)\times\mathrm{SO}(q)$	AI|AII	$\mathrm{U}(2n)/\mathrm{Sp}(2n)$
CII	chGSE	$+$	$-$	$\mathrm{Sp}(2p+2q)/\mathrm{Sp}(2p)\times\mathrm{Sp}(2q)$	$\mathrm{Sp}(2p,2q)/\mathrm{Sp}(2p)\times\mathrm{Sp}(2q)$	AII|AI	$\mathrm{U}(n)/\mathrm{O}(n)$
Bogoliubov - de Gennes classes							
C		$-$	$+$	$\mathrm{Sp}(2N)$	$\mathrm{Sp}(2N,\mathbb{C})/\mathrm{Sp}(2N)$	DIII|CI	$\mathrm{Sp}(2n)/\mathrm{U}(n)$
CI		$+$	$+$	$\mathrm{Sp}(2N)/\mathrm{U}(N)$	$\mathrm{Sp}(2N,\mathbb{R})/\mathrm{U}(N)$	D|C	$\mathrm{Sp}(2n)$
BD		$-$	$-$	$\mathrm{SO}(N)$	$\mathrm{SO}(N,\mathbb{C})/\mathrm{SO}(N)$	CI|DIII	$\mathrm{O}(2n)/\mathrm{U}(n)$
DIII		$+$	$-$	$\mathrm{SO}(2N)/\mathrm{U}(N)$	$\mathrm{SO}^*(2N)/\mathrm{U}(N)$	C|D	$\mathrm{O}(n)$

Mechanisms of Anderson criticality in 2D

"Common wisdom": all states are localized in 2D

In fact, in 9 out of 10 symmetry classes the system can escape localization!

$\longrightarrow$ variety of critical points

Mechanisms of delocalization & criticality in 2D:

- broken spin-rotation invariance $\longrightarrow$ antilocalization, metallic phase, MIT
classes AII, D, DIII

- topological term $\pi_2(\mathcal{M}) = \mathbb{Z}$ (quantum-Hall-type)
classes A, C, D : IQHE, SQHE, TQHE

- topological term $\pi_2(\mathcal{M}) = \mathbb{Z}_2$
classes AII, CII

- chiral classes: vanishing β-function, line of fixed points
classes AIII, BDI, CII

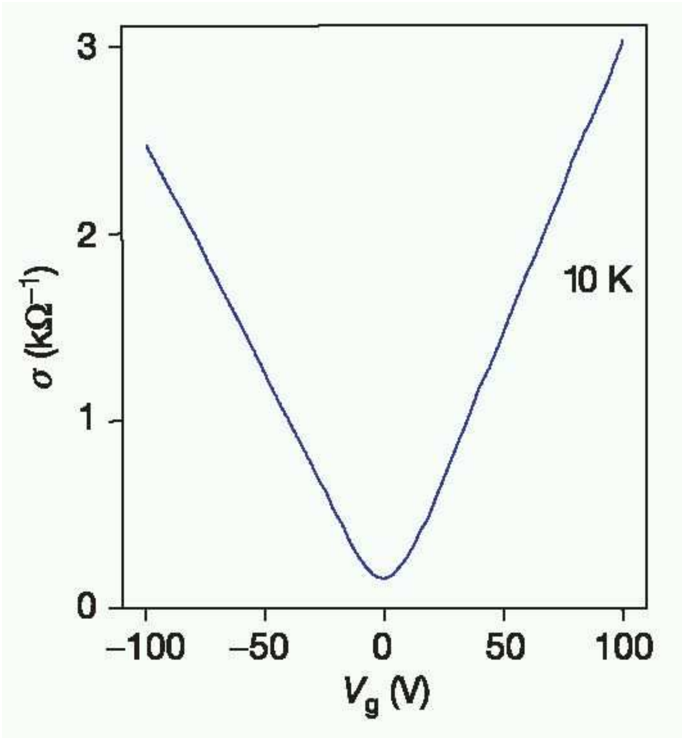
- Wess-Zumino term (random Dirac fermions, related to chiral anomaly)
classes AIII, CI, DIII

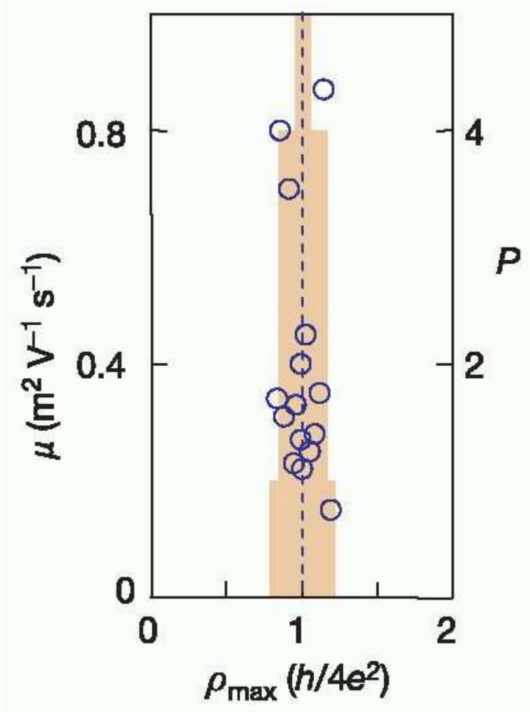
Electron transport in disordered graphene

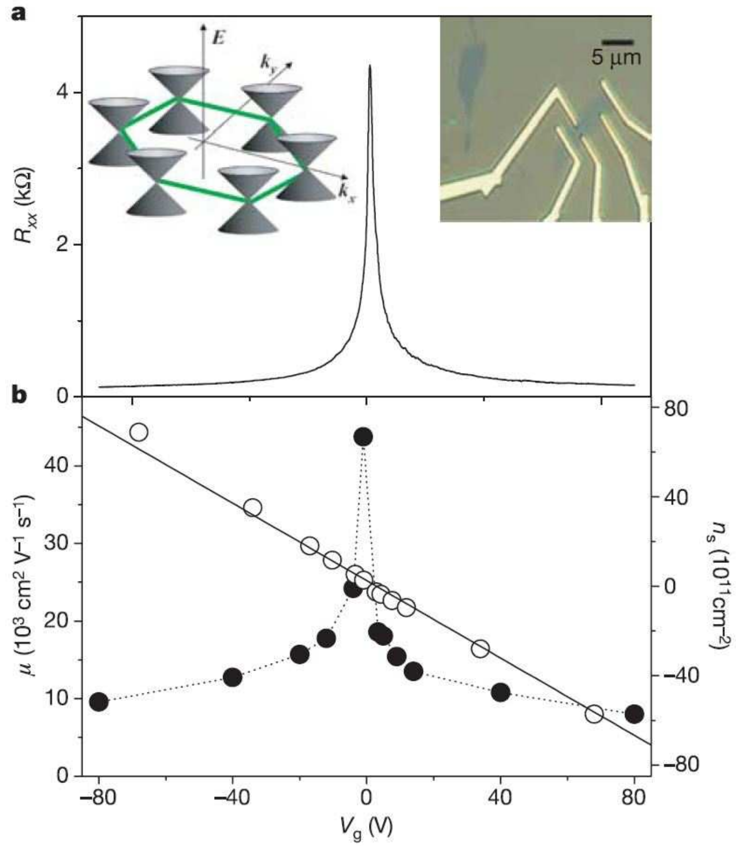
Ostrovsky, Gornyi, ADM, *Phys. Rev. B* **74, 235443 (2006)**
Phys. Rev. Lett. **98, 256801 (2007)**
Eur. Phys. J. Special Topics **148, 63 (2007)**
Phys. Rev. B **77, 195430 (2008)**

Experiments on transport in graphene

Novoselov, Geim et al; Zhang, Tan, Stormer, and Kim; Nature 2005





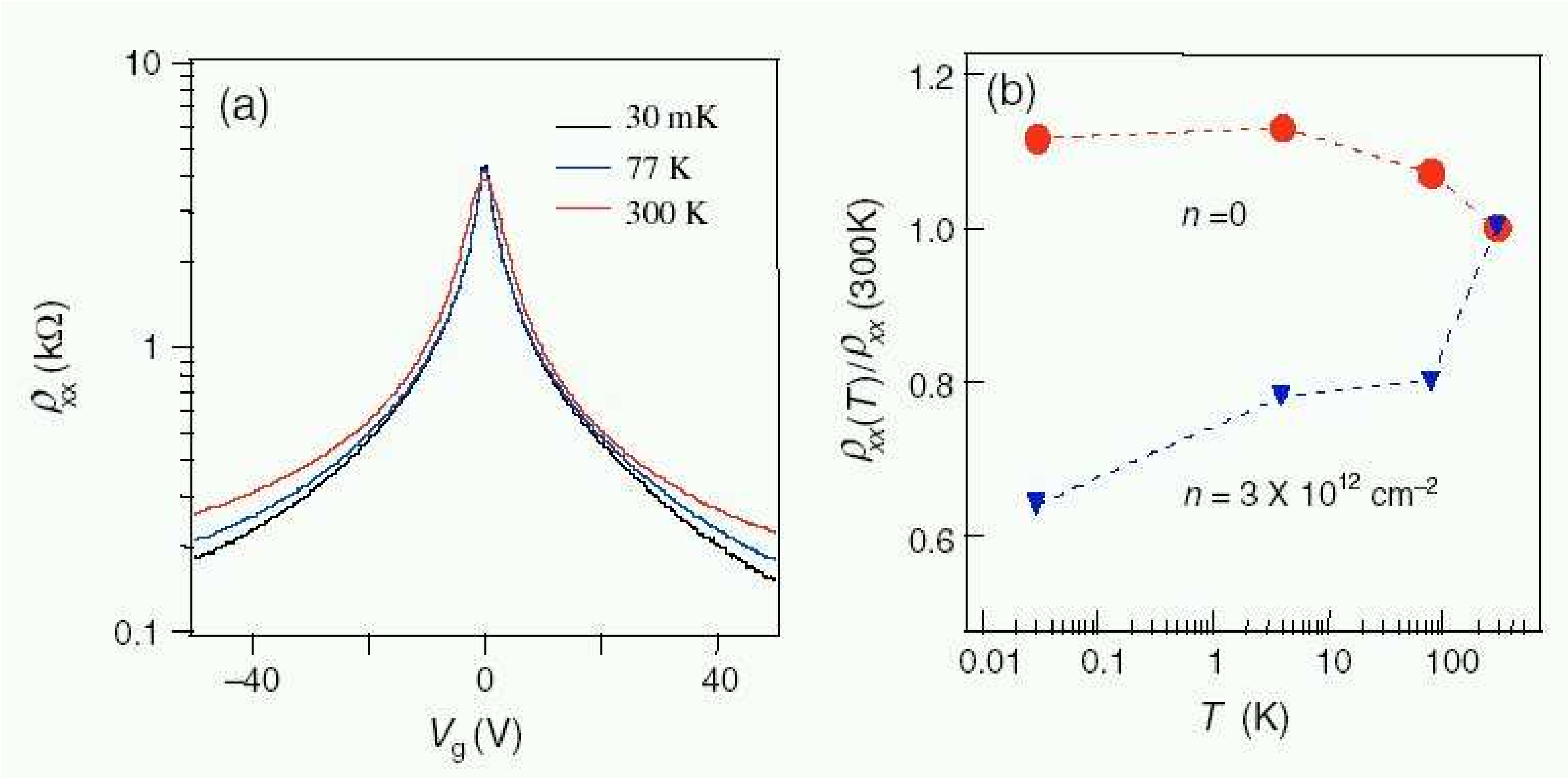


- linear dependence of conductivity on electron density ($\propto V_g$)
- minimal conductivity $\sigma \approx 4e^2/h$ ($\approx e^2/h$ per spin per valley)

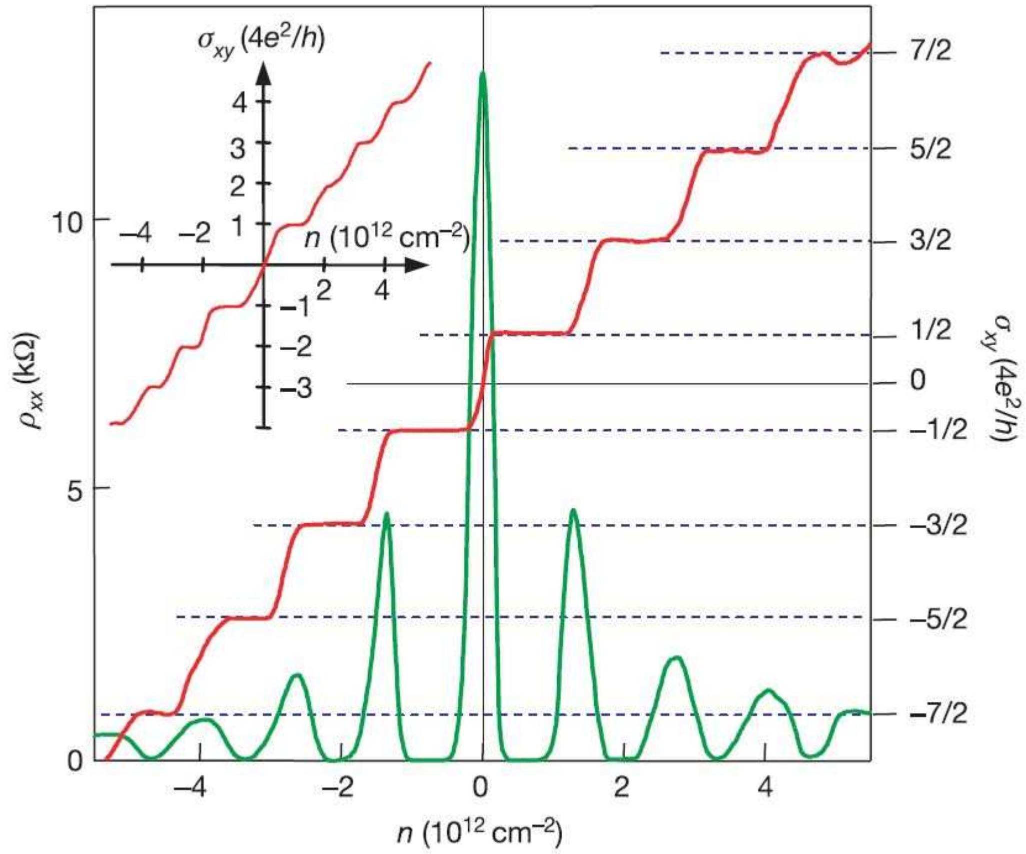
  T–independent in the range $T = 30$ mK $\div$ 300 K

T-independent minimal conductivity in graphene

Tan, Zhang, Stormer, Kim '07 $T = 30$ mK $\div$ 300 K

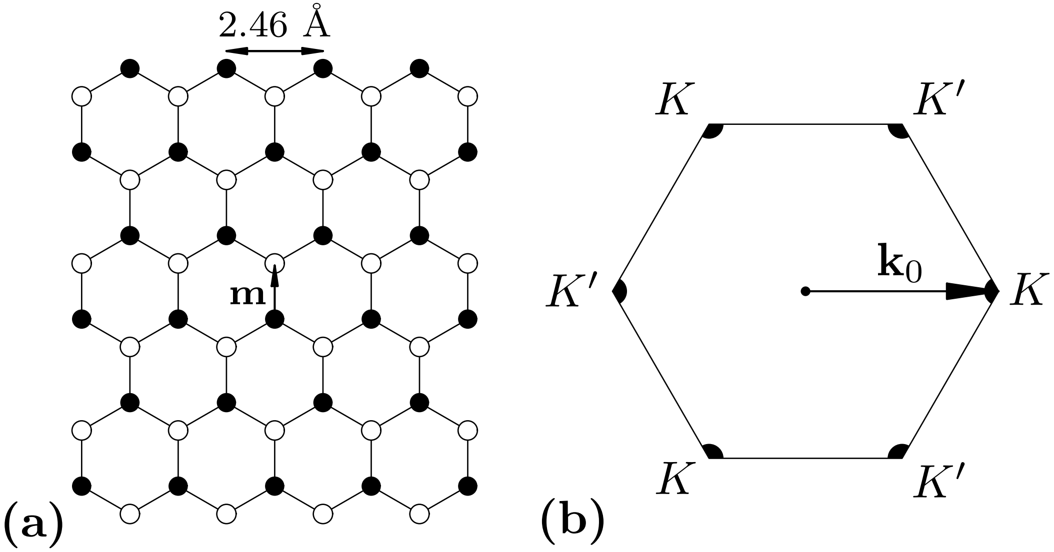


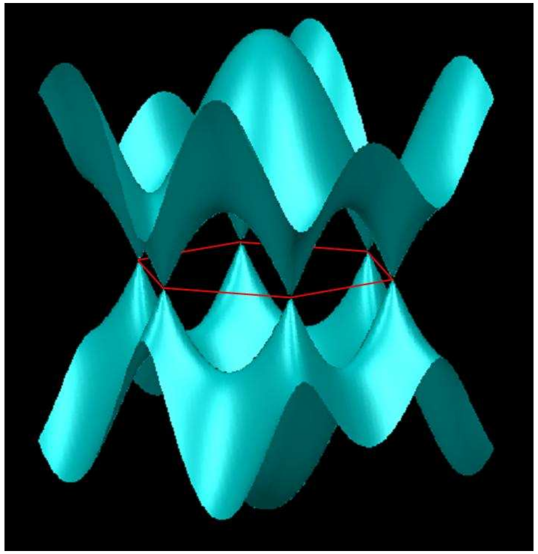
Graphene in transverse magnetic field



Anomalous, odd-integer IQHE: $\boldsymbol{\sigma_{xy} = (2n+1) \times (2e^2/h)}$

Graphene dispersion: 2D massless Dirac fermions





Two sublattices: A and B **Hamiltonian:** $H = \begin{pmatrix} 0 & t_{\mathrm{k}} \\ t_{\mathrm{k}}^* & 0 \end{pmatrix}$

$t_{\mathrm{k}} = t\left[1 + 2e^{i(\sqrt{3}/2)k_y a}\cos(k_x a/2)\right]$ **Spectrum** $\varepsilon_{\mathrm{k}}^2 = |t_{\mathrm{k}}|^2$

The gap vanishes at 2 points, $K, K' = (\pm k_0, 0)$, where $k_0 = 4\pi/3a$.

In the vicinity of K, K' the spectrum is of massless Dirac-fermion type:

$$H_K = v_0(k_x\sigma_x + k_y\sigma_y), \qquad H_{K'} = v_0(-k_x\sigma_x + k_y\sigma_y)$$

$v_0 \simeq 10^8$ cm/s – effective “light velocity”, sublattice space $\longrightarrow$ isospin

Graphene: Disordered Dirac-fermion Hamiltonian

Hamiltonian $\longrightarrow$ 4×4 matrix operating in:

AB space of the two sublattices ($\boldsymbol{\sigma}$ Pauli matrices),

$\boldsymbol{K}$–$\boldsymbol{K'}$ space of the valleys ($\boldsymbol{\tau}$ Pauli matrices).

Four-component wave function:

$$\Psi = \{\phi_{AK}, \phi_{BK}, \phi_{BK'}, \phi_{AK'}\}^T$$

Hamiltonian:

$$H = -iv_0\tau_z(\sigma_x\nabla_x + \sigma_y\nabla_y) + V(x,y)$$

Disorder:

$$V(x,y) = \sum_{\mu,\nu=0,x,y,z} \sigma_\mu\tau_\nu V_{\mu\nu}(x,y)$$

Clean graphene: symmetries

Space of valleys K–K': Isospin $\Lambda_x = \sigma_3\tau_1$, $\Lambda_y = \sigma_3\tau_2$, $\Lambda_z = \sigma_0\tau_3$.

Time inversion	Chirality
$T_0: \quad H = \sigma_1\tau_1 H^T \sigma_1\tau_1$	$C_0: \quad H = -\sigma_3\tau_0 H \sigma_3\tau_0$

Combinations with $\Lambda_{x,y,z}$

$T_x: \quad H = \sigma_2\tau_0 H^T \sigma_2\tau_0$	$C_x: \quad H = -\sigma_0\tau_1 H \sigma_0\tau_1$
$T_y: \quad H = \sigma_2\tau_3 H^T \sigma_2\tau_3$	$C_y: \quad H = -\sigma_0\tau_2 H \sigma_0\tau_2$
$T_z: \quad H = \sigma_1\tau_2 H^T \sigma_1\tau_2$	$C_z: \quad H = -\sigma_3\tau_3 H \sigma_3\tau_3$

Spatial isotropy $\Rightarrow$ $T_{x,y}$ and $C_{x,y}$ occur simultaneously $\Rightarrow$ $T_\perp$ and $C_\perp$

Symmetries of various types of disorder in graphene

		$\Lambda_\perp$	Λ_z	T_0	$T_\perp$	T_z	C_0	$C_\perp$	C_z	CT_0	$CT_\perp$	CT_z
$\sigma_0\tau_0$	α_0	+	+	+	+	+	−	−	−	−	−	−
$\sigma_{\{1,2\}}\tau_{\{1,2\}}$	$\beta_\perp$	−	−	+	−	−	+	−	−	+	−	−
$\sigma_{1,2}\tau_0$	$\gamma_\perp$	−	+	+	−	+	+	−	+	+	−	+
$\sigma_0\tau_{1,2}$	β_z	−	−	+	−	−	−	−	+	−	−	+
$\sigma_3\tau_3$	γ_z	−	+	+	−	+	−	+	−	−	+	−
$\sigma_3\tau_{1,2}$	β_0	−	−	−	−	+	−	−	+	+	−	−
$\sigma_0\tau_3$	γ_0	−	+	−	+	−	−	+	−	+	−	+
$\sigma_{1,2}\tau_3$	$\alpha_\perp$	+	+	−	−	−	+	+	+	−	−	−
$\sigma_3\tau_0$	α_z	+	+	−	−	−	−	−	−	+	+	+

Related works:

S. Guruswamy, A. LeClair, and A.W.W. Ludwig, Nucl. Phys. B 583, 475 (2000)

E. McCann, K. Kechedzhi, V.I. Fal'ko, H. Suzuura, T. Ando, and B.L. Altshuler, PRL 97, 146805 (2006)

I.L. Aleiner and K.B. Efetov, PRL 97, 236801 (2006)

Conductivity at $\mu = 0$

Drude conductivity (SCBA = self-consistent Born approximation):

$$\sigma = -\frac{8e^2 v_0^2}{\pi\hbar}\int \frac{d^2k}{(2\pi)^2}\frac{(1/2\tau)^2}{[(1/2\tau)^2 + v_0^2k^2]^2} = \frac{2e^2}{\pi^2\hbar} = \frac{4}{\pi}\frac{e^2}{h}$$

BUT: For generic disorder, the Drude result $\sigma = 4 \times e^2/\pi h$ at $\mu = 0$ does not make much sense: Anderson localization will drive $\sigma \to 0$.

Experiment: $\sigma \approx 4 \times e^2/h$ independent of T

Quantum criticality ?

Can one have non-zero σ ?

Yes, if disorder either

(i) preserves one of chiral symmetries

or

(ii) is of long-range character (does not mix the valleys)

Realizations of chiral disorder

(i) **bond disorder: randomness in hopping elements** t_{ij}

or

infinitely strong on-site impurities – unitary limit:
all bonds adjacent to the impurity are effectively cut **(C_z-symmetry)**

(ii) **dislocations: random non-Abelian gauge field** **(C_0-symmetry)**

(iii) **random magnetic field, ripples** **(both C_0 and C_z symmetries)**

Realizations of long-range disorder

(i) **smooth random potential: correlation length** $\gg$ **lattice spacing**

(ii) **charged impurities**

(iii) **ripples: smooth random magnetic field**

Absence of localization of Dirac fermions in graphene with chiral or long-range disorder

Disorder	Symmetries	Class	Conductivity	QHE
Vacancies	C_z, T_0	BDI	$\approx 4e^2/\pi h$	normal
Vacancies + RMF	C_z	AIII	$\approx 4e^2/\pi h$	normal
$\sigma_z\tau_{x,y}$ disorder	C_z, T_z	CII	$\approx 4e^2/\pi h$	normal
Dislocations	C_0, T_0	CI	$4e^2/\pi h$	chiral
Dislocations + RMF	C_0	AIII	$4e^2/\pi h$	chiral
Ripples, RMF	C_0, Λ_z	2×AIII	$4e^2/\pi h$	odd-chiral
Charged impurities	Λ_z, $T_\perp$	2×AII	$(4e^2/\pi h)\ln L$	odd
random Dirac mass: $\sigma_z\tau_{0,z}$	Λ_z, $CT_\perp$	2×D	$4e^2/\pi h$	odd
Charged imp. + RMF/ripples	Λ_z	2×A	$4\sigma_U^*$	odd

C_z-chirality $\longrightarrow$ Gade-Wegner phase

C_0-chirality $\longrightarrow$ Wess-Zumino-Witten term

Λ_z-symmetry $\equiv$ decoupled valleys $\longrightarrow$ $\theta = \pi$ topological term

Conductivity at $\mu = 0$: C_0-chiral disorder

Current operator $\mathbf{j} = e v_0 \boldsymbol{\tau}_3 \boldsymbol{\sigma}$

relation between G^R and G^A & $\boldsymbol{\sigma}_3 j^x = i j^y, \ \boldsymbol{\sigma}_3 j^y = -i j^x,$

$\longrightarrow$ transform the conductivity at $\mu = 0$ to $RR + AA$ form:

$$\sigma^{xx} = -\frac{1}{\pi} \sum_{\alpha = x,y} \int d^2(r - r') \, \mathrm{Tr} \Big[j^\alpha G^R(0; \mathbf{r}, \mathbf{r}') j^\alpha G^R(0; \mathbf{r}', \mathbf{r}) \Big] \equiv \sigma_{RR}.$$

Gauge invariance: $\mathbf{p} \rightarrow \mathbf{p} + e\mathbf{A}$ constant vector potential

$$\sigma_{RR} = -\frac{2}{\pi} \frac{\partial^2}{\partial A^2} \mathrm{Tr} \ \ln G^R \qquad \longrightarrow \qquad \sigma_{RR} = 0 \quad (?!)$$

But: contribution with no impurity lines $\longrightarrow$ anomaly:

UV divergence $\Rightarrow$ shift of p is not legitimate *(cf. Schwinger model '62)*.

Universal conductivity at $\mu = 0$ for C_0-chiral disorder

Calculate explicitly (δ – infinitesimal ImΣ)

$$\sigma = -\frac{8e^2 v_0^2}{\pi\hbar}\int \frac{d^2k}{(2\pi)^2}\frac{\delta^2}{(\delta^2 + v_0^2 k^2)^2} = \frac{2e^2}{\pi^2\hbar} = \frac{4}{\pi}\frac{e^2}{h}$$

for C_0-chiral disorder $\sigma(\mu = 0)$ does not depend on disorder strength

Alternative derivation: use Ward identity

$$-ie(\mathrm{r} - \mathrm{r}')G^R(0; \mathrm{r}, \mathrm{r}') = [G^R \mathrm{j} G^R](0; \mathrm{r}, \mathrm{r}')$$

and integrate by parts $\longrightarrow$ only surface contribution remains:

$$\sigma = -\frac{ev_0}{4\pi^3}\oint d\mathrm{k}_n \,\mathrm{Tr}\big[\mathrm{j}\, G^R(\mathrm{k})\big] = \frac{e^2}{\pi^3\hbar}\oint \frac{d\mathrm{k}_n \mathrm{k}}{k^2} = \frac{4}{\pi}\frac{e^2}{h}$$

Related works: *Ludwig, Fisher, Shankar, Grinstein '94; Tsvelik '95*

Long-range disorder

Smooth random potential does not scatter between valleys

Reduced Hamiltonian: $H = v_0 \boldsymbol{\sigma}\mathbf{k} + \sigma_\mu V_\mu(\mathbf{r})$

Ludwig, Fisher, Shankar, Grinstein '94; Ostrovsky, Gornyi, ADM '06-07

Disorder couplings: $\alpha_0 = \frac{\langle V_0^2 \rangle}{2\pi v_0^2}, \qquad \alpha_\perp = \frac{\langle V_x^2 + V_y^2 \rangle}{2\pi v_0^2}, \qquad \alpha_z = \frac{\langle V_z^2 \rangle}{2\pi v_0^2}$

Random scalar potential, vector potential, mass

Symmetries:

- α_0 disorder $\Rightarrow$ T-invariance $H = \sigma_y H^T \sigma_y$ $\Rightarrow$ AII (GSE)
- $\alpha_\perp$ disorder $\Rightarrow$ C-invariance $H = -\sigma_z H \sigma_z$ $\Rightarrow$ AIII (ChUE)
- α_z disorder $\Rightarrow$ CT-invariance $H = -\sigma_x H^T \sigma_x$ $\Rightarrow$ D (BdG)
- generic long-range disorder $\Rightarrow$ A (GUE)

σ-model topologies:

A, AII, D: θ-term with $\theta = \pi$ AIII: WZW term

Long-range disorder (cont'd)

- Class D (random mass):

Disorder is marginally irrelevant $\Longrightarrow$ diffusion never occurs

DoS: $\rho(\varepsilon) = \frac{\varepsilon}{\pi v_0^2} 2\alpha_z \log \frac{\Delta}{\varepsilon}$ Conductivity: $\sigma = \frac{4e^2}{\pi h}$

- Class AIII (random vector potential):

C_0/C_z chiral disorder; considered above

DOS: $\rho(\varepsilon) \propto |\varepsilon|^{(1-\alpha_\perp)/(1+\alpha_\perp)}$ Conductivity: $\sigma = \frac{4e^2}{\pi h}$

Long-range disorder: unitary symmetry (ripples + charged imp.)

Generic long-range disorder (no symmetries) $\Longrightarrow$ class A (GUE)

Effective infrared theory is Pruisken's unitary σ-model with topological θ-term:

$$S[Q] = \frac{1}{4}\,\mathrm{Str}\left[-\frac{\sigma_{xx}}{2}(\nabla Q)^2 + \left(\sigma_{xy} + \frac{1}{2}\right) Q\nabla_x Q \nabla_y Q\right] \Rightarrow -\frac{\sigma_{xx}}{8}\,\mathrm{Str}(\nabla Q)^2 + i\pi N[Q]$$

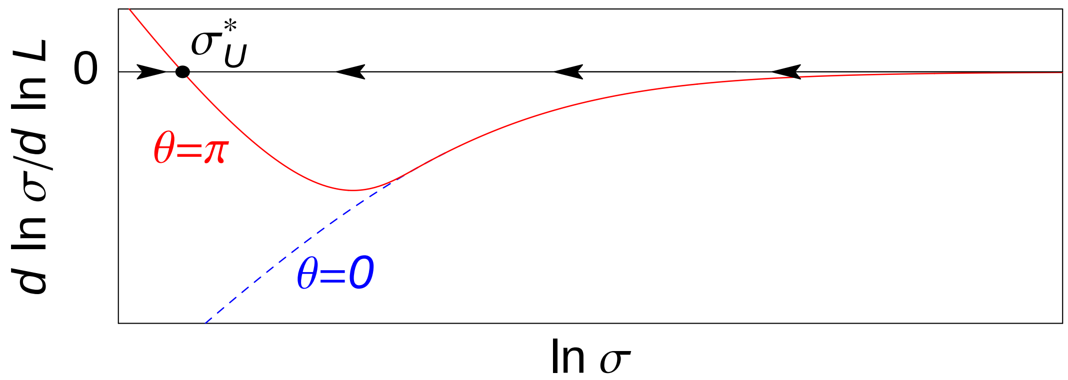
Compact (FF) sector of the model: $Q_{\mathrm{FF}} \in \mathcal{M}_{\mathrm{F}} = \dfrac{U(2n)}{U(n)\times U(n)}$

Topological term takes values $N[Q] \in \pi_2(\mathcal{M}_{\mathrm{F}}) = \mathbb{Z}$

Vacuum angle $\theta = \pi$ in the absence of magnetic field due to anomaly

$\Longrightarrow$ **Quantum Hall critical point**

$$\sigma = 4\sigma_U^* \simeq 4\times(0.5 \div 0.6)\frac{e^2}{h}$$



Long-range disorder: symplectic symmetry (charged imp.)

Random scalar potential α_0 preserves T-inversion symmetry

$\Longrightarrow$ class AII (GSE)

Partition function is real $\Longrightarrow$ $\mathrm{Im}\, S = 0$ or π

Compact sector: $Q_{\mathrm{FF}} \in \mathcal{M}_{\mathrm{F}} = \dfrac{O(4n)}{O(2n) \times O(2n)}$

$$\Longrightarrow \qquad \pi_2(\mathcal{M}_{\mathrm{F}}) = \begin{cases} \mathbb{Z} \times \mathbb{Z}, & n = 1; \\ \mathbb{Z}_2, & n \geq 2 \end{cases}$$

At $n = 1$ $\mathcal{M}_{\mathrm{F}} = S^2 \times S^2/\mathbb{Z}_2 \approx$ [Cooperons] $\times$ [diffusons]

$$\Longrightarrow \qquad \mathrm{Im}\, S = \theta_c N_c[Q] + \theta_d N_d[Q]$$

T-invariance $\longrightarrow$ $\theta_c = \theta_d = 0$ or π $\longrightarrow$ $\mathbb{Z}_2$ subgroup

Explicit calculation $\longrightarrow$ Anomaly $\longrightarrow$ $\theta_{c,d} = \pi$

At $n \geq 2$ we use $\mathcal{M}_{\mathrm{F}}|_{n=1} \subset \mathcal{M}_{\mathrm{F}}|_{n\geq 2}$ $\Longrightarrow$ $\mathrm{Im}\, S = \pi N[Q]$

possibility of $\mathbb{Z}_2$ topological term: Fendley '01

Long-range potential disorder (cont'd): symplectic σ-model with $\mathbb{Z}_2$ topological term

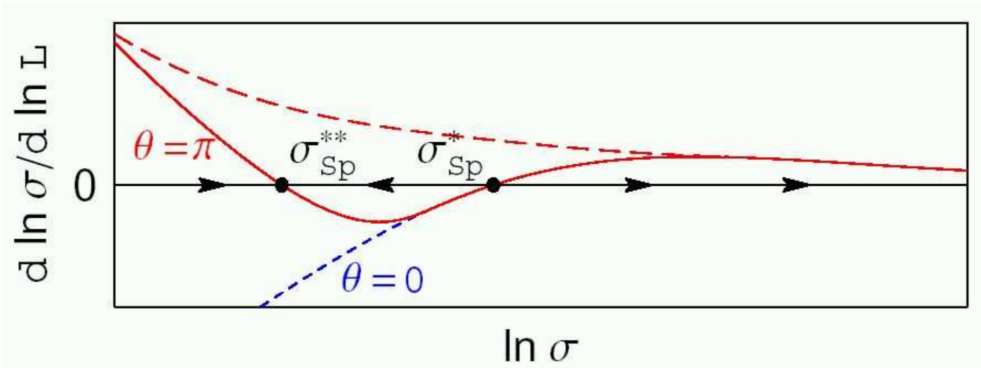
Symplectic sigma-model with $\theta = \pi$ term:

$$S[Q] = -\frac{\sigma_{xx}}{16} \mathrm{Str}(\nabla Q)^2 + i\pi N[Q]$$

"Topological delocalization":

as for Pruisken σ-model of QHE at criticality, instantons suppress localization

$\Longrightarrow$ possible scenarios:

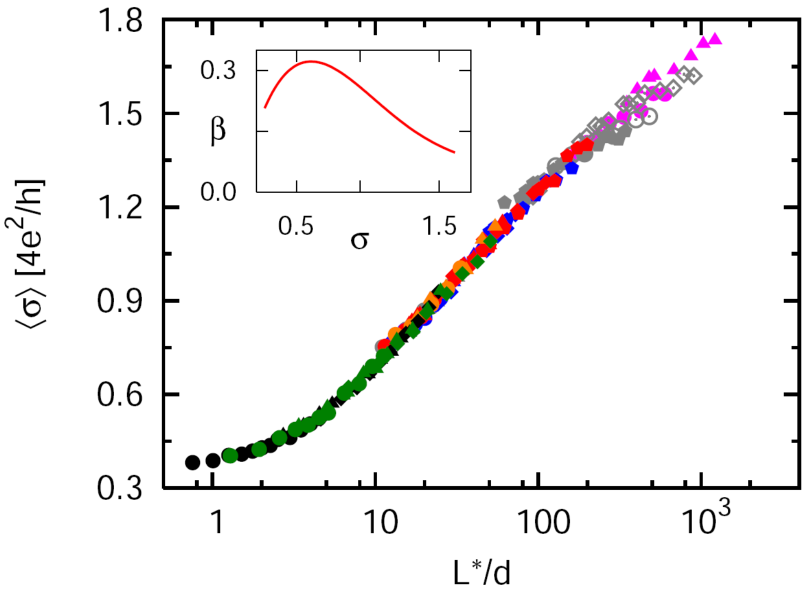


- β function everywhere positive, conductivity $\sigma \to \infty$
- intermediate attractive fixed point, $\sigma = 4\sigma^{**}_{Sp}$

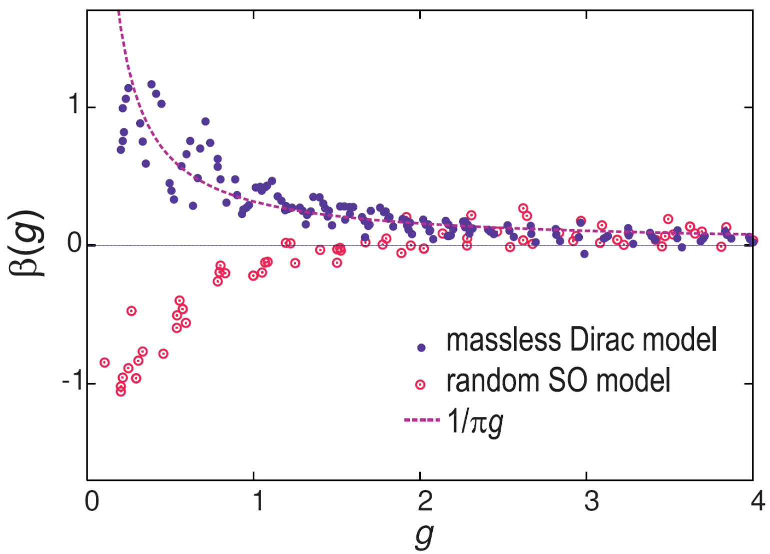
Numerics needed !

Long-range potential disorder: numerics

Bardarson, Tworzydło, Brouwer, Beenakker, PRL '07



Nomura, Koshino, Ryu, PRL '07



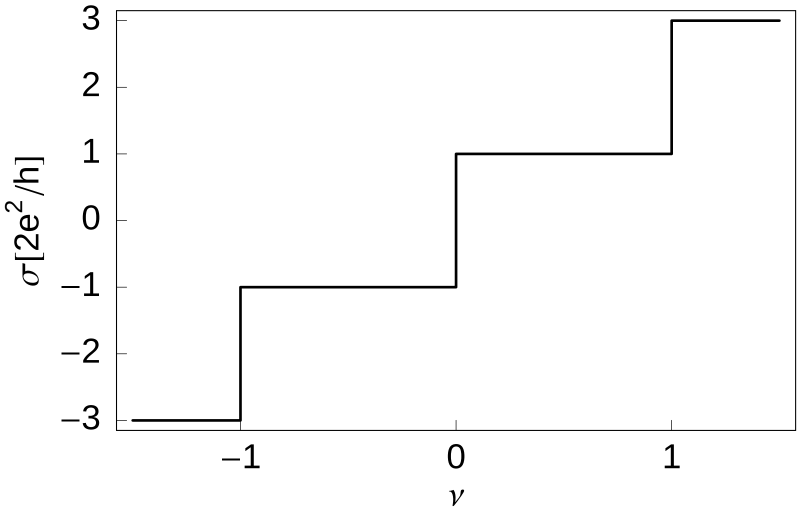
- **absence of localization confirmed**
- **scaling towards the perfect-metal fixed point** $\sigma \to \infty$

Odd quantum Hall effect

Decoupled valleys + magnetic field $\Longrightarrow$

unitary sigma model with anomalous topological term:

$$S[Q] = \frac{1}{4}\mathrm{Str}\left[-\frac{\sigma_{xx}}{2}(\nabla Q)^2 + \left(\sigma_{xy} + \frac{1}{2}\right) Q\nabla_x Q \nabla_y Q\right] \quad \Longrightarrow \quad \text{odd-integer QHE}$$

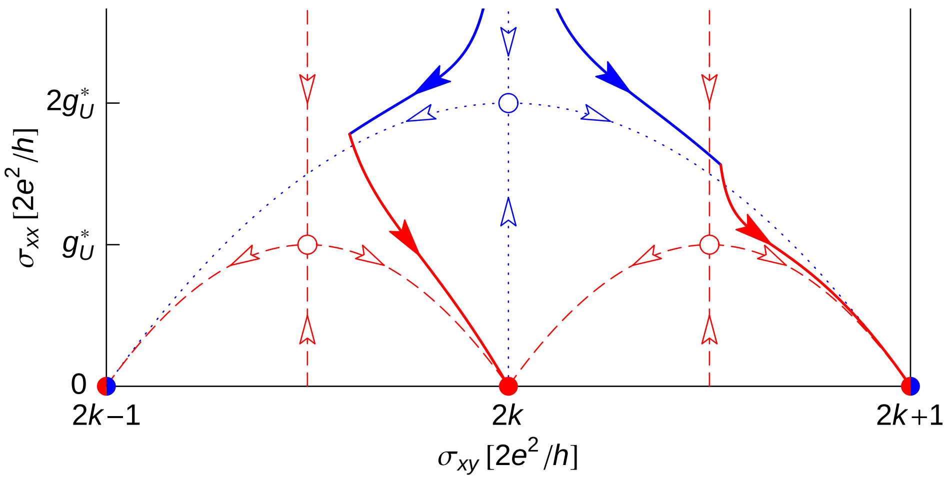


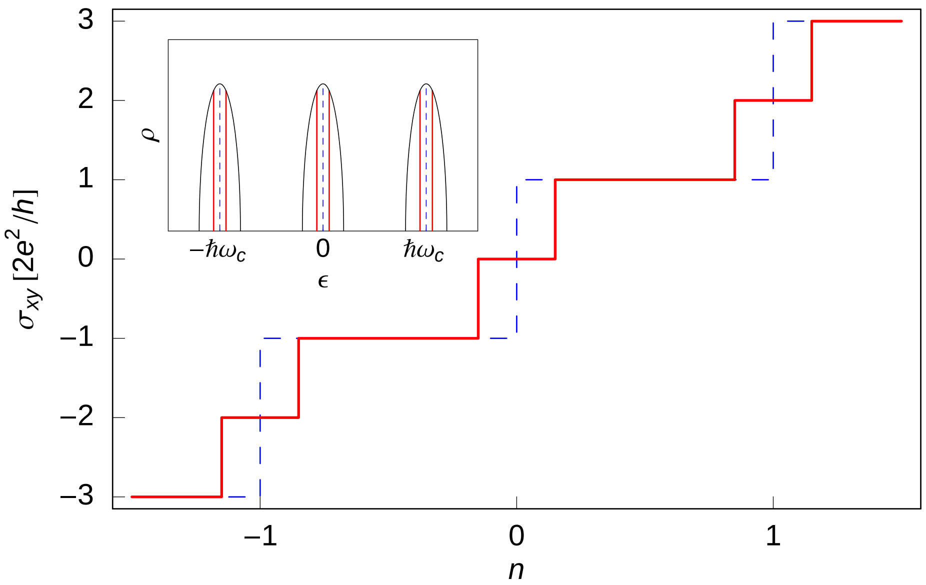
generic (valley-mixing) disorder $\Longrightarrow$ conventional IQHE

weakly valley-mixing disorder $\Longrightarrow$ even plateaus narrow, emerge at low T

Quantum Hall effect: Weak valley mixing

$$S[Q_K, Q_{K'}] = S[Q_K] + S[Q_{K'}] + \frac{\hbar\rho}{\tau_{\rm mix}} \operatorname{Str} Q_K Q_{K'}$$





Even plateau width $\sim (\tau/\tau_{\rm mix})^{0.45}$, visible at $T < T_{\rm mix} \sim \hbar/\tau_{\rm mix}$

Estimate for Coulomb scatterers:

$T_{\rm mix} \sim 100$ mK ; even plateau width $\delta n_{\rm even} \sim 0.05$

Cf. splitting of delocalized states in ordinary QHE by spin-orbit / spin-flip scattering, Khmelnitskii '92; Lee, Chalker '94

Chiral quantum Hall effect

C_0-chiral disorder $\iff$ random vector potential

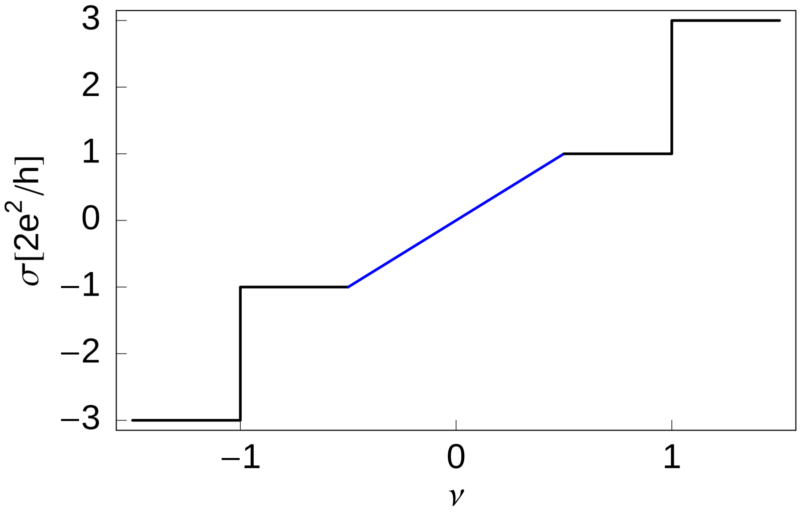
Atiyah-Singer theorem:

In magnetic field, zeroth Landau level remains degenerate!!!

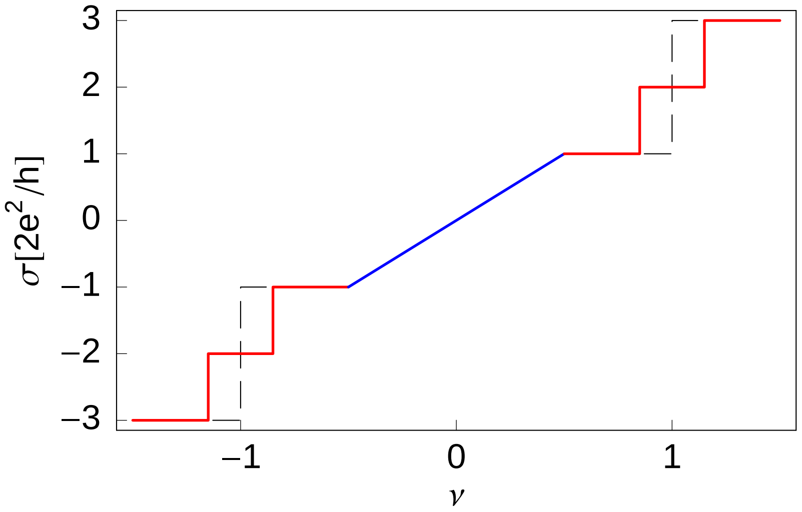
(Aharonov and Casher '79)

Within zeroth Landau level Hall effect is classical

Decoupled valleys (ripples)



Weakly mixed valleys (dislocations)



Plan (tentative)

- quantum interference, diagrammatics, weak localization, mesoscopic fluctuations, strong localization
- field theory: non-linear σ-model
- quasi-1D geometry: exact solution, localization
- RG, metal-insulator transition, criticality
- symmetry classification of disordered electronic systems and of corresponding σ-models
- mechanisms of delocalization and criticality in 2D systems: symmetries and topology
- disordered Dirac fermions in graphene

Evers, ADM, "Anderson transitions", Rev. Mod. Phys. 80, 1355 (2008)