Homogenization in elasto/visco-plasticity

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Outline

Mathematical model

Infinitesimal strains only!

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\begin{aligned}
-\mathrm{div}_x T &= b(x,t), \\
T &= \mathcal{D}(\varepsilon(\nabla_x u) - \varepsilon_p),
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$$z = (\varepsilon_p, \tilde{z}) \in \mathcal{S}^3 \times \mathbb{R}^{N-6} \simeq \mathbb{R}^N$$

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$$B : \mathbb{R}^N \to \mathcal{S}^3, \quad \varepsilon_p = Bz \in \mathcal{S}^3$$

Restrictions for f

The second law of thermodynamics (Clausius-Duhem inequality) requires

$$\nabla_\varepsilon \psi(\varepsilon, z) = T.$$
$$\nabla_z \psi(\varepsilon, z) \cdot \zeta \leq 0 \quad \zeta \in f(\varepsilon, z)$$

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Free energy can be chosen as positive semi-definite quadratic form

$$\psi(\varepsilon, z) = \frac{1}{2}\mathcal{D}(\varepsilon - Bz) \cdot (\varepsilon - Bz) + \frac{1}{2}(Lz) \cdot z$$

with symmetric, positive semi-definite $N \times N$-matrix L.

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Fulfilled if

$$f(\varepsilon, z) = g(-\nabla_z \psi(\varepsilon, z)),$$

with a function $g : \mathbb{R}^N \mapsto \mathbb{R}^N$ satisfying $g(z) \cdot z \geq 0$.

Problem of monotone type

Suppose that $g : \mathbb{R}^N \mapsto \mathbb{R}^N$ is monotone satisfying $0 \in g(0)$.

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Initial boundary value problem of monotone type [Alber, '98]

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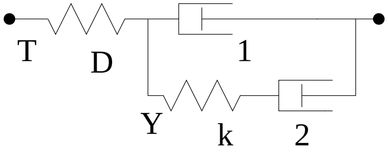
Special case: $g = \nabla\phi$, ϕ convex (Generalized standard material)

Example: nonlinear kinematic hardening



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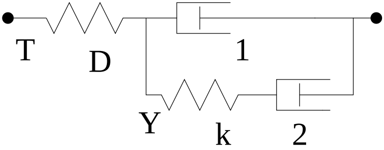

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T &= \mathcal{D}(\varepsilon(\nabla_x u) - \varepsilon_p), \\
\partial_t \varepsilon_p &= c_1 |T - Y|^r \frac{T - Y}{|T - Y|},
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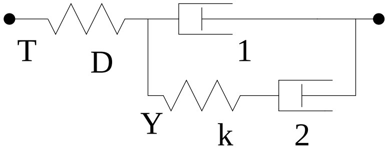

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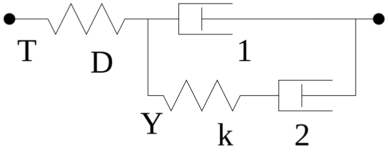

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\partial_t \varepsilon_n &= c_2 |Y|^\gamma \frac{Y}{|Y|}, \\
Y &= k(\varepsilon_p - \varepsilon_n) \quad \gamma, r \approx 5\ldots7.
\end{aligned}
$$

Y is backstress. c_1,c_2 and k are material constants.

$z = (\varepsilon_p, \varepsilon_n) \in \mathcal{S}^3 \times \mathcal{S}^3 \simeq \mathbb{R}^{12}$.

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Given: $b(x,t)$ volume force, $\gamma(x,t)$ boundary disolacement, $z_0^{(0)}(x,y)$ initial data
$y \mapsto (\mathcal{D}, f, z_0^{(0)})(y,\cdot)$ periodic with periodicity cell $Y \subseteq \mathbb{R}^3$!

Asymtotic ansatz

Assume that

$$(u_\eta, T_\eta, z_\eta) \sim (\hat{u}_\eta, \hat{T}_\eta, \hat{z}_\eta),$$

where $\hat{u}_\eta$, $\hat{T}_\eta, \hat{z}_\eta$ are given by the ansatz

$$\begin{aligned}
\hat{u}_\eta(x,t) &= u_0(x,t) + \eta u_1\Big(x, \frac{x}{\eta}, t\Big), \\
\hat{T}_\eta(x,t) &= T_0\Big(x, \frac{x}{\eta}, t\Big), \\
\hat{z}_\eta(x,t) &= z_0\Big(x, \frac{x}{\eta}, t\Big).
\end{aligned}$$

Mapping $y \mapsto (u_1, T_0, z_0)(x, y, t)$ is periodic with the periodicity cell Y.

Homogenized problem

(u_0, u_1, T_0, z_0) must satisfy

$$\begin{aligned} -\mathrm{div}_x \int_Y T_0(x,y,t)dy &= b(x,t), \\ u_0(x,t) &= \gamma(x,t), \qquad (x,t) \in \partial\Omega \times [0,\infty) \end{aligned}$$

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-\,\mathrm{div}_y\, T_0(x,y,t) &= 0, \\
T_0(x,y,t) &= \mathcal{D}[y](\varepsilon(\nabla_y u_1(x,y,t)) - Bz_0(x,y,t) \\
&\quad +\varepsilon(\nabla_x u_0(x,t))), \\
\frac{\partial}{\partial t} z_0(x,y,t) &\in g(y, B^T T_0(x,y,t) - Lz_0(x,y,t)), \\
z_0(x,y,0) &= z_0^{(0)}(x,y),
\end{aligned}
$$

Existence [Alber, 02], $L > \alpha I$

We would like to show!

After solving the homog'ed problem we define

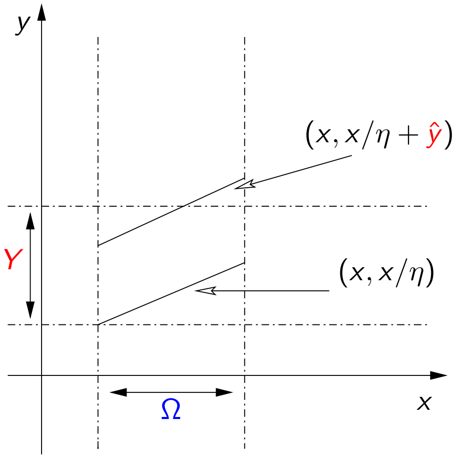
$$(\hat{u}_\eta, \hat{T}_\eta, \hat{z}_\eta)(x,t) = \left(u_0(x,t) + \eta u_1\left(x, \frac{x}{\eta}, t\right), T_0\left(x, \frac{x}{\eta}, t\right), z_0\left(x, \frac{x}{\eta}, t\right)\right)$$

and try to show that

$$\lim_{\eta \to 0} \Big(\|u_0(t) - u_\eta(t)\|_\Omega + \|\hat{T}_\eta(t) - T_\eta(t)\|_\Omega + \|\hat{z}_\eta(t) - z_\eta(t)\|_\Omega \Big) = 0, \quad 0 \le t \le T_e.$$

Problem: Regularity of u_1, T_0, z_0 is too low for that!

Low regularity of the solutions of the homogenized problem



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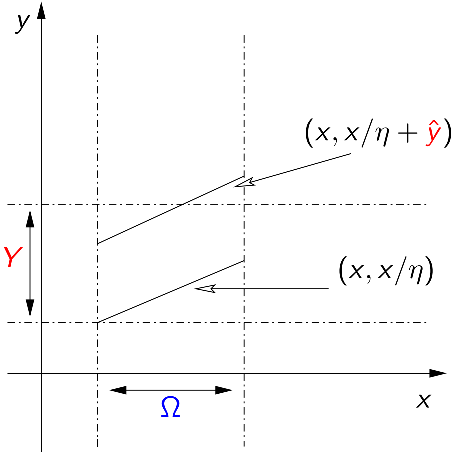

$T_0(t) \in L^2(\Omega \times Y)$ only!

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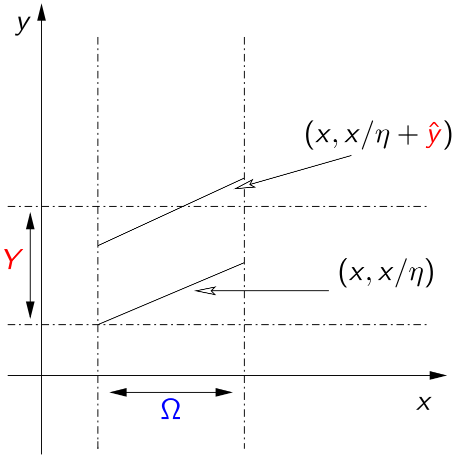

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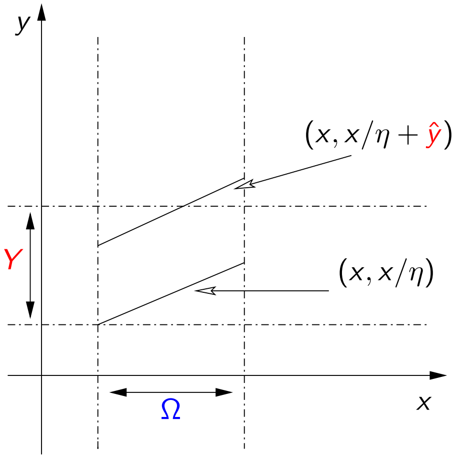

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Hence, no trace on the 3-dimentional subset $\{(x, x/\eta \mid x \in \Omega)\}$ of the 6-dimensional set $\Omega \times Y$

$x \mapsto T_0(x, x/\eta, t)$ is not defined!

Phase shift

Microscopic problem with initial data and coefficients depending on the parameter $y \in \mathbb{R}^3$:

$$\begin{aligned}
-\mathrm{div}_x \tilde{T}_\eta(x, y, t) &= b(x, t) \\
\tilde{T}_\eta(x, y, t) &= \mathcal{D}\left[\frac{x}{\eta} + y\right](\varepsilon(\nabla_x \tilde{u}_\eta(x, y, t))) - B\tilde{z}_\eta(x, y, t)) \\
\frac{\partial}{\partial t}\tilde{z}_\eta(x, y, t) &\in g\left(\frac{x}{\eta} + y, B^T \tilde{T}_\eta(x, y, t) - L\tilde{z}_\eta(x, y, t)\right) \\
\tilde{z}_\eta(x, y, 0) &= z_0^{(0)}\left(x, \frac{x}{\eta} + y\right) \\
\tilde{u}_\eta(x, y, t) &= \gamma(x, t) \qquad (x, t) \in \partial\Omega \times [0, \infty)
\end{aligned}$$

Convergence result for problem with phase shift

For a solution $(u_0, u_1, T_0, z_0)(x, y, t)$ of the homog'ed problem set

$$\hat{T}_\eta(x, y, t) = T_0\left(x, \frac{x}{\eta} + y, t\right),$$
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Theorem (N.,2006)

Let (u_0, u_1, T_0, z_0) be the solution of the homogenized problem. Then the solution $(\tilde{u}_\eta, \tilde{T}_\eta, \tilde{z}_\eta)$ of the microscopic problem with parameter y satisfies

$$\lim_{\eta \to 0} \Big(\|u_0(t) - \tilde{u}_\eta(t)\|_{\Omega \times Y} + \|\hat{T}_\eta(t) - \tilde{T}_\eta(t)\|_{\Omega \times Y} + \|\hat{z}_\eta(t) - \tilde{z}_\eta(t)\|_{\Omega \times Y} \Big) = 0, \quad 0 \le t \le T_e.$$

Convergence for a.e. $y \in Y$

Corollary

There is a subsequence such that for almost every $y \in Y$

$$\lim_{\eta \to 0} \big(\| u_0(t) - \tilde{u}_\eta(\cdot, y, t) \|_\Omega + \| \hat{T}_\eta(\cdot, y, t) - \tilde{T}_\eta(\cdot, y, t) \|_\Omega + \| \hat{z}_\eta(\cdot, y, t) - \tilde{z}_\eta(\cdot, y, t) \|_\Omega \big) = 0, \quad 0 \le t \le T_e.$$

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Does the above result hold for $y = 0$?

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Does the above result hold for $y = 0$?

If yes, then the justification is complete, since

$$(u_\eta, T_\eta, z_\eta)(x, t) = (\tilde{u}_\eta, \tilde{T}_\eta, \tilde{z}_\eta)(x, 0, t).$$

Shifting of the solution

Set $\gamma = 0$!

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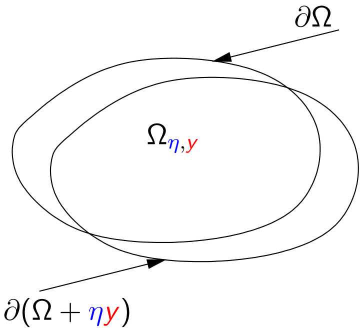

For $(x, y, t) \in \Omega \times Y \times [0, T_e)$ set

$$(u_{[\eta]}, T_{[\eta]}, z_{[\eta]})(x, y, t) = \begin{cases} (\tilde{u}_\eta, \tilde{T}_\eta, \tilde{z}_\eta)(x - \eta y, y, t), & x \in \Omega_{\eta,y} \\ 0, & \text{otherwise} \end{cases},$$

where

$$\Omega_{\eta,y} = \{x \in \Omega | x - \eta y \in \Omega\}.$$

Convergence by shifting of the solution

Theorem (Alber,N.,2008)

Let (u_η, T_η, z_η) be the solution of the micro'pic problem.

Let $(\tilde{u}_\eta, \tilde{T}_\eta, \tilde{z}_\eta)$ be the solution of the micro'pic problem with parameter y.

Let $(\tilde{u}_{[\eta]}, \tilde{T}_{[\eta]}, \tilde{z}_{[\eta]})$ be the shift of the solution of the micro'pic problem with parameter y

Convergence by shifting of the solution

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Then

$$\lim_{\eta \to 0} \left(\|(\tilde{u}_{[\eta]}, \tilde{T}_{[\eta]}, \tilde{z}_{[\eta]})(t) - (u_\eta, T_\eta, z_\eta)(t)\|_{\Omega \times Y} \right) = 0$$

for $0 \le t \le T_e$.

Averages for the shifted solution

Set

$$T_\eta^*(x,t) = \int_{Y_{\eta,x}} T_0(x-\eta y, \frac{x}{\eta}, t)dy, \qquad z_\eta^*(x,t) = \ldots,$$

where

$$Y_{\eta,x} = \{y \in Y \mid x - \eta y \in \Omega\} = Y \cap \frac{1}{\eta}(x - \Omega).$$

Averages for the shifted solution

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$$T^*_\eta(x,t) = \int_{Y_{\eta,x}} T_0(x-\eta y, \frac{x}{\eta}, t)dy, \qquad z^*_\eta(x,t) = \ldots,$$

where

$$Y_{\eta,x} = \{y \in Y \mid x - \eta y \in \Omega\} = Y \cap \frac{1}{\eta}(x - \Omega).$$

Then, by the previous theorems and the triangular inequality, we obtain that

$$\lim_{\eta\to 0} \big(\|u_0(t) - u_\eta(t)\|_\Omega + \|T^*_\eta(t) - T_\eta(t)\|_\Omega + \|z^*_\eta(t) - z_\eta(t)\|_\Omega\big) = 0, \quad 0 \le t \le T_e.$$

Rate-independent case [Mielke&Timofte, 2007]