Boundary Velocity Suboptimal Control of Incompressible Flow in Cylindrically Perforated Domain

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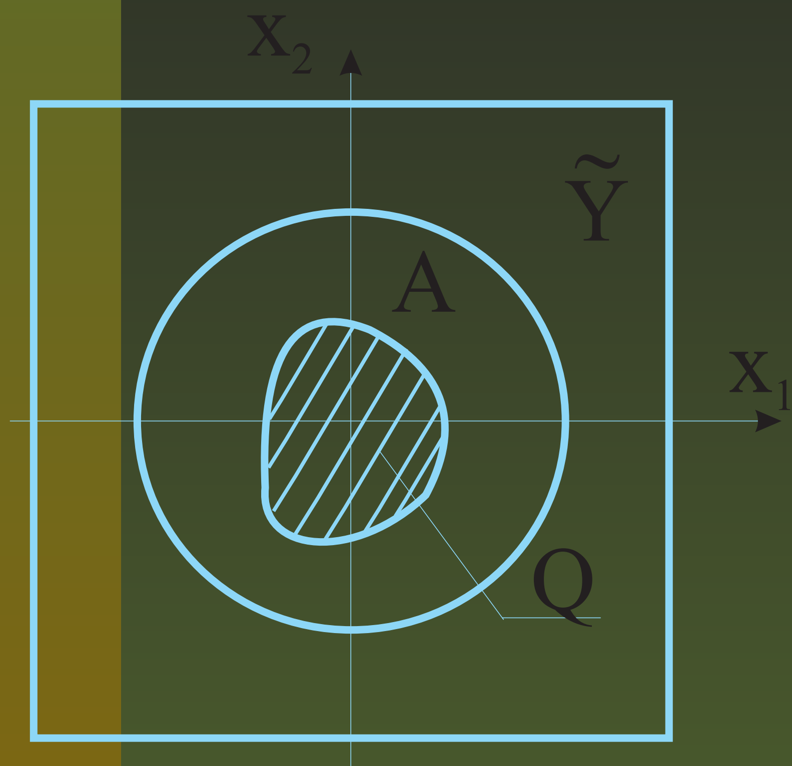
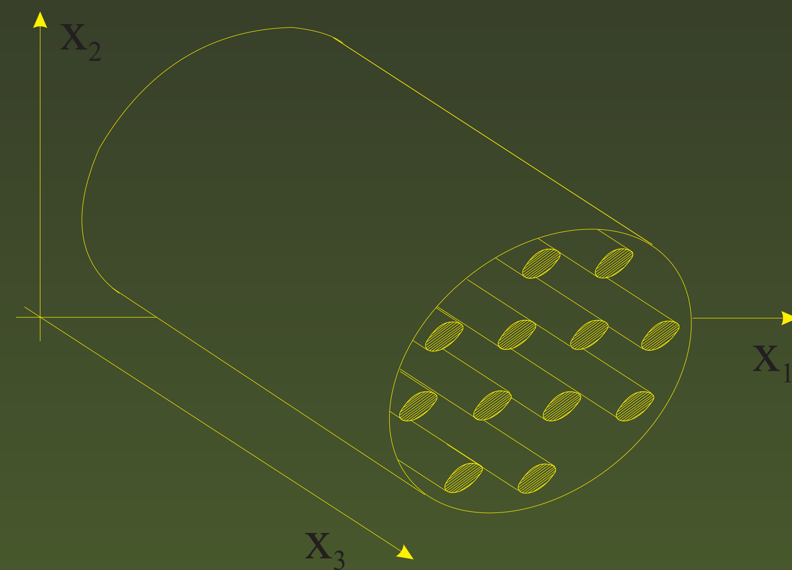
The Definition of a cylindrically perforated domain Ω_ε



(a) The cell of perforation



(b) The domain Ω_ε

Classification of the thin cylinders

The domain Ω_ε is defined by removing the thin cylinders $T_\varepsilon^{\mathbf{k}}$ from Ω. We use the following decomposition for the boundary of this domain:

$$\partial\Omega_\varepsilon = \Gamma_\varepsilon^1 \cup \Gamma_\varepsilon^2 \cup \Gamma^3 \cup \partial T_\varepsilon.$$

We consider three types of possible cross-sizes of thin cylinders. If the limit of

$$\sigma_\varepsilon = \varepsilon^2 \left(\log 1/r_\varepsilon\right). \tag{1}$$

as ε tends to zero, is positive and finite then the cross-size of the cylinders is called critical.
If $\lim\limits_{\varepsilon\to 0} \sigma_\varepsilon = +\infty$, the cross-size of cylinders is smaller and if $\lim\limits_{\varepsilon\to 0} \sigma_\varepsilon = 0$, the cross-size is larger.

The statement of optimal control problem

Find a boundary velocity field $\overline{\alpha}_\varepsilon = \left(\alpha_{\mathbf{k}_1}, \alpha_{\mathbf{k}_2}, \ldots, \alpha_{\mathbf{k}_{J_\varepsilon}}\right)$ and a corresponding velocity-pressure pair $(\mathbf{y}_\varepsilon, p_\varepsilon)$ such that the functional

$$\mathcal{J}_\varepsilon(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon) = \lambda \int_{\Omega_\varepsilon} |\nabla \mathbf{y}_\varepsilon|^2 \, dx + \frac{\beta\varepsilon}{r_\varepsilon} \sum_{j=1}^{J_\varepsilon} \int_{\partial T_\varepsilon^{\mathbf{k}_j}} \left|\alpha_{\mathbf{k}_j}\right|^2 d\mathcal{H}^2 \tag{2}$$

is minimized subject to the steady-state Navier-Stokes equations

$$-\nu \triangle \mathbf{y}_\varepsilon + (\mathbf{y}_\varepsilon \cdot \nabla)\mathbf{y}_\varepsilon + \nabla p_\varepsilon = \mathbf{f}_\varepsilon \quad \text{in } \Omega_\varepsilon, \tag{3}$$

$$\operatorname{div} \mathbf{y}_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon, \tag{4}$$

$$\mathbf{y}_\varepsilon|_{\Gamma_\varepsilon^1} = \mathbf{y}_\varepsilon^1, \; \mathbf{y}_\varepsilon|_{\Gamma_\varepsilon^2} = \mathbf{y}_\varepsilon^2, \; \mathbf{y}_\varepsilon|_{\Gamma^3} = 0, \tag{5}$$

$$\mathbf{y}_\varepsilon\big|_{\partial T_\varepsilon^{\mathbf{k}_j}} = \alpha_{\mathbf{k}_j}, \quad \forall\, j = 1, \ldots, J_\varepsilon. \tag{6}$$

The set of admissible solutions

We say that a triplet $(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon)$ is admissible to the optimal control problem, if $\overline{\alpha}_\varepsilon \in \mathbf{U}_\varepsilon$, where

$$\mathbf{U}_\varepsilon = \left\{ \overline{\alpha}_\varepsilon = \left(\alpha_{\mathbf{k}_1}, \alpha_{\mathbf{k}_2}, \ldots, \alpha_{\mathbf{k}_{J_\varepsilon}}\right) \;\middle|\; \begin{array}{c} \alpha_{\mathbf{k}_j} = \mathbf{u}\big|_{\partial T_\varepsilon^{\mathbf{k}_j}}, \ \forall j = 1, \ldots, J_\varepsilon \\ \forall \mathbf{u} \in \mathbf{H}^1_{sol}(\Omega_\varepsilon) \cap \mathbf{H}^2(\Omega) \ : \ \|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} \leqslant \gamma, \\ \mathbf{u}\big|_{\Gamma^1_\varepsilon} = \mathbf{y}^1_\varepsilon, \ \mathbf{u}\big|_{\Gamma^2_\varepsilon} = \mathbf{y}^2_\varepsilon, \ \mathbf{u}\big|_{\Gamma^3_\varepsilon} = 0. \end{array} \right\}$$

and the pair $(\mathbf{y}_\varepsilon, p_\varepsilon) \in \mathbf{H}^1(\Omega_\varepsilon) \times L^2_0(\Omega_\varepsilon)$ is a corresponding solution of the variational problem

$$\nu a_\varepsilon(\mathbf{y}_\varepsilon, \mathbf{v}) + c_\varepsilon(\mathbf{y}_\varepsilon, \mathbf{y}_\varepsilon, \mathbf{v}) + b_\varepsilon(\mathbf{v}, p_\varepsilon) = \int_{\Omega_\varepsilon} \mathbf{f}_\varepsilon \cdot \mathbf{v}\, dx, \quad \forall\, \mathbf{v} \in \mathbf{H}^1_0(\Omega_\varepsilon),$$

$$b_\varepsilon(\mathbf{y}_\varepsilon, q) = 0, \quad \forall\, q \in L^2_0(\Omega_\varepsilon),$$

$$\mathbf{y}_\varepsilon\big|_{\Gamma^1_\varepsilon} = \mathbf{y}^1_\varepsilon, \ \mathbf{y}_\varepsilon\big|_{\Gamma^2_\varepsilon} = \mathbf{y}^2_\varepsilon, \ \mathbf{y}_\varepsilon\big|_{\Gamma^3_\varepsilon} = 0, \ \mathbf{y}_\varepsilon\big|_{\partial T_\varepsilon^{\mathbf{k}_j}} = \alpha_{\mathbf{k}_j}, \quad \forall j = 1, \ldots, J_\varepsilon.$$

Solvability result

Theorem 1. Let $\overline{\alpha}_\varepsilon$ be an admissible control ($\overline{\alpha}_\varepsilon \in \mathbf{U}_\varepsilon$), and let $\mathbf{u}_\varepsilon \in \mathbf{H}^1_{sol}(\Omega_\varepsilon) \cap \mathbf{H}^2(\Omega)$ be its prototype. Then there exists a corresponding velocity-pressure pair $(\mathbf{y}_\varepsilon, p_\varepsilon) \in \mathbf{H}^2_{sol}(\Omega_\varepsilon) \times \left[H^1(\Omega_\varepsilon) \cap L^2_0(\Omega_\varepsilon)\right]$ satisfying the original boundary value problem in the following variational sense:

$$\mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon \in \mathbf{H}^1_{0,sol}(\Omega_\varepsilon), \tag{7}$$

$$a_\varepsilon(\mathbf{y}_\varepsilon, \mathbf{v}) + c_\varepsilon(\mathbf{y}_\varepsilon, \mathbf{y}_\varepsilon, \mathbf{v}) = \int_{\Omega_\varepsilon} \mathbf{f}_\varepsilon \cdot \mathbf{v}\, dx, \quad \forall\, \mathbf{v} \in \mathbf{H}^1_{0,sol}(\Omega_\varepsilon), \tag{8}$$

$$\nabla p_\varepsilon = \nu \triangle \mathbf{y}_\varepsilon - (\mathbf{y}_\varepsilon \cdot \nabla)\mathbf{y}_\varepsilon + \mathbf{f}_\varepsilon \quad \text{in } \mathcal{D}'(\Omega_\varepsilon). \tag{9}$$

Theorem 2. The optimal control problem ($\mathbb{P}_\varepsilon$) has a solution iff this problem is regular, that is, $\Xi_\varepsilon \neq \varnothing$ for every fixed $\varepsilon > 0$.

General settings

The object of our consideration is the following the parameterized optimal control problem (OCP_ε):

$$(\mathrm{OCP}_\varepsilon): \qquad \min\left\{I_\varepsilon(u,y)\ :\ (u,y)\in\Xi_\varepsilon\right\}, \tag{10}$$

where

(B1) $(\mathrm{CF}_\varepsilon)\ I_\varepsilon : \mathbb{U}_\varepsilon \times \mathbb{Y}_\varepsilon \to \overline{\mathbb{R}}$ is a cost functional;

(B2) $\mathbb{Y}_\varepsilon$ is a space of states ;

(B3) $\mathbb{U}_\varepsilon$ is a space of controls;

(B4) $\Xi_\varepsilon \subset \{(u_\varepsilon, y_\varepsilon)\in \mathbb{U}_\varepsilon \times \mathbb{Y}_\varepsilon\ :\ u \in U_\varepsilon, I_\varepsilon(u,y) < +\infty\}$ is a set of all admissible pairs linked by some state equation (SE_ε), and control and state constraints $(\mathrm{CSC}_\varepsilon)$.

Main goal

Any OCP that can be described as follows

$$
(\mathrm{OCP}_\varepsilon) \; : \; \left\{ \begin{array}{ccc} (\mathrm{CF}_\varepsilon) & : & I_\varepsilon(u,y) \to \inf\,, \\ & \text{subject to} & \\ (\mathrm{CSC}_\varepsilon) & : & (u,y) \in \mathbb{U}_\varepsilon \times \mathbb{Y}_\varepsilon, u \in U_\varepsilon \subset \mathbb{U}_\varepsilon\,, \\ (\mathrm{SE}_\varepsilon) & : & L_\varepsilon(u,y) + F_\varepsilon(y) = 0. \end{array} \right. \tag{11}
$$

The question is: what does it mean the ”behaviour” of an optimal control problem $(\mathrm{OCP}_\varepsilon)$ under various values of the parameter ε?

Convergence formalism

Definition 1. Let $(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon)$ be any admissible solution to the problem $(\mathbb{P}_\varepsilon)$. Then we say that a triplet $(\mathbf{u}_\varepsilon, \breve{\mathbf{y}}_\varepsilon, \breve{p}_\varepsilon) \in \mathbb{X}_\varepsilon$ is a prototype to $(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon)$, if

$$\mathbb{X}_\varepsilon = \left[\mathbf{H}^1_{sol}(\Omega_\varepsilon) \cap \mathbf{H}^2(\Omega) \cap \mathbf{L}^2(\Omega, d\eta_\varepsilon^{r(\varepsilon)})\right] \times \left[\mathbf{H}^1_{sol}(\Omega_\varepsilon) \cap \mathbf{H}^1(\Omega)\right] \times L^2_0(\Omega),$$

$\mathbf{u}_\varepsilon$ is a control prototype, and $(\breve{\mathbf{y}}_\varepsilon, \breve{p}_\varepsilon)$ are some extensions of the functions $(\mathbf{y}_\varepsilon, p_\varepsilon)$ on the whole Ω.

Definition 2. We say that a bounded sequence $\{(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ is w-convergent to a triplet $(\mathbf{u}, \mathbf{y}, p) \in \mathbf{H}^2(\Omega) \times \mathbf{H}^1(\Omega) \times L^2_0(\Omega)$ in the variable space $\mathbb{X}_\varepsilon$ as ε tends to zero (in symbols, $(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon) \xrightarrow{w} (\mathbf{u}, \mathbf{y}, p)$), if some bounded sequence of its prototypes $\left\{(\mathbf{u}_\varepsilon, \breve{\mathbf{y}}_\varepsilon, \breve{p}_\varepsilon) \in \widehat{\Xi}_\varepsilon\right\}_{\varepsilon>0}$ converges to $(\mathbf{u}, \mathbf{y}, p)$ in the following sense:

(i) $\mathbf{u}_\varepsilon \rightharpoonup \mathbf{u}$ in $\mathbf{H}^2(\Omega)$; (ii) $\breve{p}_\varepsilon \rightharpoonup p$ in $L^2_0(\Omega)$; (iii) $\breve{\mathbf{y}}_\varepsilon \rightharpoonup \mathbf{y}$ in $\mathbf{H}^1(\Omega)$.

Definition of suboptimal controls

Definition 3. We say that a function $\overline{\alpha}_\varepsilon^{sub} = \left(\alpha_{\mathbf{k}_1}^{sub}, \alpha_{\mathbf{k}_2}^{sub}, ..., \alpha_{\mathbf{k}_{\mathrm{J}_\varepsilon}}^{sub}\right)$ is an asymptotically suboptimal control for the problem $(\mathbb{P}_\varepsilon)$ if

$$\alpha_{\mathbf{k}_j}^{sub} \in \mathbf{H}^{1/2}(\partial T_\varepsilon^{\mathbf{k}_j}), \quad \int_{\partial T_\varepsilon^{\mathbf{k}_j}} \mathbf{n} \cdot \alpha_{\mathbf{k}_j}^{sub} \, d\mathcal{H}^2 = 0, \quad \forall j = 1, \ldots, \mathrm{J}_\varepsilon,$$

and for every $\delta > 0$ there is $\varepsilon_0 > 0$ such that

$$\left| \inf_{(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon) \in \Xi_\varepsilon} \mathcal{J}_\varepsilon(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon) - \mathcal{J}_\varepsilon(\overline{\alpha}_\varepsilon^{sub}, \mathbf{y}_\varepsilon^{sub}) \right| < \delta, \quad \forall \, \varepsilon < \varepsilon_0,$$

where $\mathbf{y}_\varepsilon^{sub} = \mathbf{y}_\varepsilon(\overline{\alpha}_\varepsilon^{sub})$ denotes the corresponding solution of the original boundary value problem.

Main result. Theorem 1.

Assume that the origin belongs to a smooth part of the boundary ∂Q and condition $C_0 = \lim_{\varepsilon \to 0} \varepsilon^2 (\log 1/r_\varepsilon) = +\infty$ holds true. Then the boundary velocity field

$$\overline{\alpha}_\varepsilon^{sub} = \left(\alpha_{\mathbf{k}_1}^{sub}, \alpha_{\mathbf{k}_2}^{sub}, ..., \alpha_{\mathbf{k}_{J_\varepsilon}}^{sub} \right) = \Lambda_\varepsilon(\mathbf{u}^0)\big|_{\partial T_\varepsilon}$$

can be taken as the suboptimal control, where $\Lambda_\varepsilon : \mathbf{H}^1_{sol}(\Omega) \mapsto \mathbf{H}^1_{sol}(\Omega_\varepsilon)$ is some linear bounded operator, and $\mathbf{u}^0$ is a solution to the following problem: the functional $\int_\Omega |\mathbf{u}(x)|^2 \, dx$ is minimized subject to the constraints

$$\left\{ \mathbf{u}(x) \in \mathbf{H}^2(\Omega) \;\middle|\; \begin{array}{c} \|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} \leqslant \gamma, \quad \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega, \\ \mathbf{u}|_{\Gamma^1} = \mathbf{y}^*|_{\Gamma^1}, \; \mathbf{u}|_{\Gamma^2} = \mathbf{y}^*|_{\Gamma^2}, \; \mathbf{u}|_{\Gamma^3} = 0, \\ \mathbf{y}^* \cdot \mathbf{n} = 0 \text{ on } \Gamma^1 \cup \Gamma^2, \end{array} \right\} \qquad (12)$$

Main result. Theorem 2.

Assume that the origin belongs to a smooth part of the boundary ∂Q and condition $0 < C_0 = \lim_{\varepsilon\to 0} \varepsilon^2 \left(\log 1/r_\varepsilon\right) < +\infty$ holds true. Then any optimal control to the problem (for Brinkman-type law)

$$\mathcal{J}_0(\mathbf{u},\mathbf{y}) = \lambda \int_\Omega |\nabla \mathbf{y}|^2\, dx + \frac{2\pi\lambda}{C_0} \int_\Omega |\mathbf{y}-\mathbf{u}|^2\, dx + \beta |\partial Q|_H \int_\Omega |\mathbf{u}|^2\, dx \longrightarrow \inf,$$

$$-\nu\triangle\mathbf{y} + \frac{2\pi\nu}{C_0}(\mathbf{y}-\mathbf{u}) + (\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{f} \quad \text{in } \Omega;$$

$$\operatorname{div}\mathbf{y} = 0 \quad \text{in } \Omega, \quad \mathbf{y}|_{\partial\Omega} = \mathbf{u}|_{\partial\Omega};$$

$$p \in L^2_0(\Omega),\ \mathbf{u} \in \mathbf{H}^2(\Omega),\ \mathbf{y}-\mathbf{u} \in \mathbf{H}^1_{0,sol}(\Omega),\ \|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} \leqslant \gamma.$$

can be taken as the suboptimal one to the original problem.