Homogenization of finite metallic fibers and 3D-effective permittivity tensor

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Maxwell's equations in harmonic case

- E : total electric field
- H : total magnetic field

$$\begin{cases} \text{curl}\, E = i\omega\mu_0\ \mu(x,\omega)\ H \\ \text{curl}\, H = -i\omega\varepsilon_0\ \varepsilon(x,\omega)\ E \end{cases}$$

+ radiation conditions :

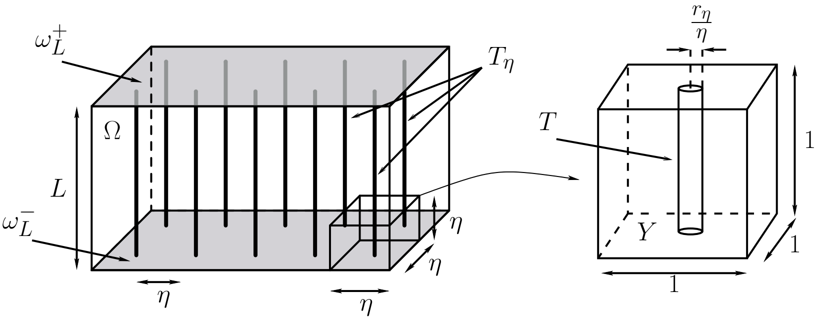
$$(E^d, H^d) = O\left(\frac{1}{|x|}\right), \quad \omega\varepsilon_0\left(\frac{x}{|x|}\wedge E^d\right) - kH^d = o\left(\frac{1}{|x|}\right).$$

$(E^d, H^d) = (E - E^i, H - H^i)$ is the diffracted field

- μ is the premeability tensor ($\simeq Id$ in nature)
- ε is the permittivity tensor (relative)
- ω waves frequency (angular)
- $k_0 := \sqrt{\varepsilon_0\mu_0}\,\omega$ wave number

First step: Homogenization of a periodic array of finite metallic fibers

We start by considering the following fibered structure (as Guy Bouchitté & Didier Felbacq in 2006).



$\Omega := \omega \times (-L/2, L/2)$, $\quad r_\eta \ll \eta \quad$ fibers with very thin section.

Scaling assumptions

We define

- θ_η the volume of fibers:

$$\theta_\eta := \frac{\pi r_\eta^2}{\eta^2} \to 0$$

- $\gamma > 0$ the limit average capacity of fibers per unit of volume:

$$\frac{1}{\eta^2 \log r_\eta} \to \gamma$$

- $\sigma_\eta \to \infty$ the fibres conductivity
- $\kappa \in [0, +\infty]$ the limit average conductivity of the structure.

$$\kappa := \lim_{\eta \to 0} \kappa_\eta \quad , \quad \kappa_\eta := \sigma_\eta \, \theta_\eta \, .$$

First problem

We have to pass to the limit when $\eta \to 0$ in :

$$\begin{cases} \operatorname{curl} E_\eta = i\omega\mu_0 H_\eta, \\ \operatorname{curl} H_\eta = -i\omega\varepsilon_0\varepsilon_\eta E_\eta \\ + \text{ radiations conditions} \end{cases}$$

with $$\varepsilon_\eta := \begin{cases} 1 & \text{on } \mathbb{R}^3 \setminus T_\eta \\ 1 + i\sigma_\eta & \text{on } T_\eta \end{cases}$$

The problem is to pass to the limit in the 2^{nd} equation.

We decompose $\varepsilon_\eta E_\eta$ in $E_\eta + iF_\eta$ where $F_\eta := \kappa_\eta \dfrac{E_\eta}{\theta_\eta} 1_{T_\eta}$

$$\operatorname{curl} H_\eta = -i\omega\varepsilon_0(E_\eta + iF_\eta)$$

$$E_\eta \rightharpoonup\!\!\rightharpoonup E_0 \quad , \quad H_\eta \rightharpoonup\!\!\rightharpoonup H_0 \quad , \quad F_\eta \rightharpoonup\!\!\rightharpoonup F_0 \text{ in mesure}$$

$E_0(x,\cdot)$, $H_0(x,\cdot)$ and $J_0(x,\cdot)$ are Y-periodic

We can prove that :

$\mathrm{div}_y F_0 = 0$, and $\mathrm{supp}\ F_0 \subset S_0$ where

$S_0 := \{(0,0)\} \times [-\frac{1}{2}, \frac{1}{2}]$

$F_0(x,y) = f(x)\,\mathbf{e_3}\,\delta_{S_0}$ (being δ_{S_0} the line ditribution along S_0).

Passing to the limit in Maxwell problem we find

$$\begin{cases} \operatorname{curl} E &= i\omega\mu_0 H \\ \operatorname{curl} H &= -i\omega\varepsilon_0 (E + i f \mathbf{e_3} 1_\Omega) \end{cases} \quad \text{dans } \mathbb{R}^3$$

To close the system we show that j satisfies the following one dimensional boundary value problem:

$$\frac{\partial^2 f}{\partial x_3^2} + \left(k_0^2 + \frac{2i\pi\gamma}{\kappa}\right) f = 2i\pi\gamma\, E_3 \quad \text{on } \Omega, \qquad \frac{\partial f}{\partial x_3} = 0 \quad \text{on } \omega_L^\pm$$

Result obtained by Bouchitté & Felbacq in 2006.

Remark

- *This limit problem is non-local.*
- *When $L = \infty$ the problem become local in the TE. case*

Second step: Reiterated homogenization

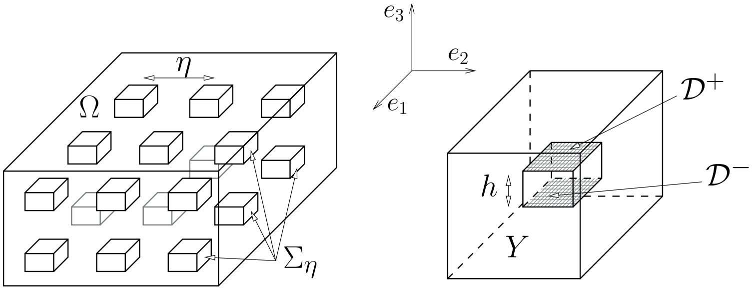


Figure: The second diffraction structure and unit cell.

Second problem

Plugging the system of effective equations of the first step on the new structure, we obtain:

$$\begin{cases} \operatorname{curl} E_\eta = i\omega\mu_0 H_\eta & \text{on } \mathbb{R}^3 \\ \operatorname{curl} H_\eta = -i\omega\varepsilon_0 (E_\eta + i\,\mathbf{1}_{\Sigma_\eta}\, f_\eta\, \mathbf{e_3}) & \text{on } \mathbb{R}^3 \\ \dfrac{\partial^2 f_\eta}{\partial x_3^2} + \left(k_0^2 + \dfrac{2i\pi\gamma}{\kappa}\right) f_\eta = 2i\pi\gamma\, E_\eta \cdot \mathbf{e_3} & \text{on } \Sigma_\eta \\ \dfrac{\partial f_\eta}{\partial x_3} = 0 & \text{on } D_\eta^\pm \\ + \text{ radiation conditions} & \end{cases}$$

Problem in the unit cell

We have

$$E_\eta \rightharpoonup E_0 \quad , \quad H_\eta \rightharpoonup H_0 \quad , \quad f_\eta \rightharpoonup f_0$$

By classical arguments, we prove :

- $\operatorname{div}_y H_0 = 0, \qquad \operatorname{curl}_y H_0 = 0$
- $\operatorname{curl}_y E_0 = 0, \qquad \operatorname{div}_y (E_0 + \mathbf{1}_\Sigma f_0) = 0$
- $f_0 = f_0(x, y_1, y_2) = \dfrac{2i\pi\gamma}{k_0^2 + \frac{2i\pi\gamma}{\kappa}} \dfrac{1}{h} \displaystyle\int_{-h/2}^{h/2} E_0(x, y_1, y_2, y_3) \cdot \mathbf{e_3} \; dy_3$

Remark

$f_0(x, \cdot)$ is supported in Σ and $\dfrac{\partial f_0}{\partial y_3} = 0$

Electrostatic problem

We have

- $H_0(x,\cdot)$ is constant, $\quad H_0(x,y) = H(x)$
- $E_0(x,\cdot)$ in term of a suitable periodic scalar potential $\Phi(x,\cdot)$:

$$E_0(x,y) = E(x) + \nabla_y \Phi(x,y)\ .$$

- Φ satisfies the following electrostatic problem

$$\Delta_y \Phi = i\ f_0\,(\delta_{D^+} - \delta_{D^-})$$

$$f_0 = \frac{2i\pi\gamma}{k_0^2 + \frac{2i\pi\gamma}{\kappa}}\,(E_3 + [\Phi]),$$

where $[\Phi](x,y_1,y_2) := \frac{1}{h}\Big(\Phi(x,y_1,y_2,\frac{h}{2}) - \Phi(x,y_1,y_2,-\frac{h}{2})\Big).$

Micro-resonator problem

We introduce the operator B define by

$$\begin{array}{rcl} B : L^2(D) & \longrightarrow & L^2(D) \\ w & \longrightarrow & [\varphi_w](y_1, y_2) \end{array}$$

where φ_w is the unique Y-periodic solution of

$$-\Delta\varphi_w \;=\; w\,(\delta_{D^+} - \delta_{D^-})$$

The previous system in f_0 can be express in term of the operator B by

$$B(f_0) - \left(\frac{k_0^2}{2\pi\gamma} + \frac{i}{\kappa}\right)\, f_0 \;=\; -i\, E_3(x)\;.$$

Spectral problem

We have

- B is a positive compact selfadjoint operator
- $\nu_0 > \nu_1 \geq \nu_2 \cdots \geq \nu_n \geq \cdots \to 0$ be the eigenvalues of B and $\{\varphi_n \ : \ n \in \mathbb{N}\}$ an associated orthonormal basis of $L^2(D)$

Then we decompose f_0 in this base so $f_0 := \sum_n c_n \varphi_n$

$$f_0(x, y_1, y_2) \ = \ i\, E_3(x) \sum_n \frac{\int_D \varphi_n}{\frac{k_0^2}{2\pi\gamma} - \nu_n + \frac{i}{\kappa}} \varphi_n$$

The limit of the term $f_\eta \mathbf{1}_{\Sigma_\eta}$ in the second equation of the global system can be identified as the average

$$\int_\Sigma f_0(x, y_1, y_2)\, dy_1 dy_2$$

We can now pass to the weak limit into the two first equations of the initial problem

$$\left\{ \begin{array}{lcll} \operatorname{curl} E_\eta & = & i\omega\mu_0 H_\eta & \text{on } \mathbb{R}^3 \\ \operatorname{curl} H_\eta & = & -i\omega\varepsilon_0(E_\eta + i\,\mathbf{1}_{\Sigma_\eta} f_\eta\, \mathbf{e_3}) & \text{on } \mathbb{R}^3 \end{array} \right.$$

Limit problem

The limit system reads

$$\left\{ \begin{array}{rcl} \mathrm{curl}\, E &=& i\omega\mu_0\, H \\ \mathrm{curl}\, H &=& -i\omega\varepsilon_0\varepsilon^{eff}(x,\omega)\; E\;. \end{array} \right.$$

With diagonal tensor $\varepsilon^{eff}(x,\omega)$ given by

$$\varepsilon^{eff}(x,\omega) = Id \quad \text{in } \mathbb{R}^3 \setminus \Omega$$

and in Ω :

$$\varepsilon^{eff}_{11}(x,\omega) = \varepsilon^{eff}_{22}(x,\omega) = 1 \quad , \quad \varepsilon^{eff}_{33}(x,\omega) = 1 - h \sum_n \frac{\left(\int_D \varphi_n\right)^2}{\frac{k_0^2}{2\pi\gamma} - \nu_n + \frac{i}{\kappa}}\,.$$

and changes of sign when $k_0^2(\omega) = \varepsilon_0\mu_0\,\omega^2$ passes through the eigenvalues (resonances) and becomes very large.

Effective permittivity and numerics

$D = (-0.25, 0.25)^2$, $h = 0.5$ and $\gamma = 1$.

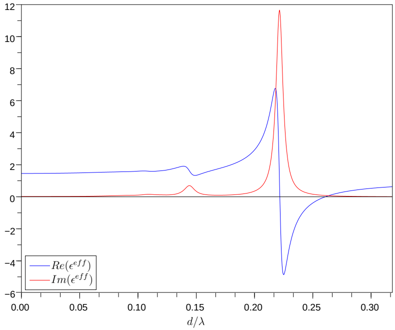


Figure: Effective permittivity of the second structure for $\kappa = 100$

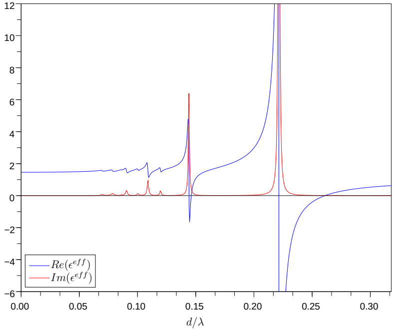
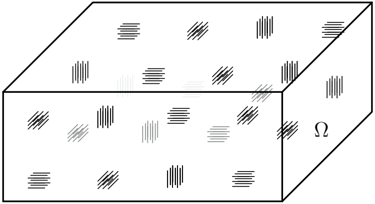


Figure: Effective permittivity of the second structure for $\kappa = 1000$

Variant with three directions of fibers



By mixing the directions of the fibers in each inclusions of the second structure, as depicted in the figure, we can reach all effective tensors of the kind

$$\varepsilon^{\text{eff}} = Id - \begin{pmatrix} h_1 \sum \frac{(\int_D \varphi_n^1)^2}{\frac{k_0^2}{2\pi\gamma_1} - \nu_n^1 + \frac{i}{\kappa_1}} & 0 & 0 \\ 0 & h_2 \sum \frac{(\int_D \varphi_n^2)^2}{\frac{k_0^2}{2\pi\gamma_2} - \nu_n^2 + \frac{i}{\kappa_2}} & 0 \\ 0 & 0 & h_3 \sum \frac{(\int_D \varphi_n^3)^2}{\frac{k_0^2}{2\pi\gamma_3} - \nu_n^3 + \frac{i}{\kappa_3}} \end{pmatrix},$$

where there is no relation between the different φ_n^i, ν_n^i and h_i.

thank you for your attention