(Microlocal) Techniques for Boundary and Interface Problems

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We set $D = \partial / i$

Let $p(x,\xi)$ be a polynomial, i.e., $p(x,\xi) = \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha$

Then

$$p(x, D_x)u(x) = \sum_{|\alpha|\leq m} a_\alpha(x) D_x^\alpha u(x)$$

which we write

$$p(x, D_x)u(x) = \sum_{|\alpha|\leq m} \frac{1}{(2\pi)^n} \iint e^{i\langle x-y|\xi\rangle} a_\alpha(x)\xi^\alpha u(y)\ dyd\xi$$

Kernel:

$$K(x,y) = \sum_{|\alpha|\leq m} \frac{1}{(2\pi)^n} \int e^{i\langle x-y|\xi\rangle} a_\alpha(x)\xi^\alpha\, d\xi$$

Oscillatory integral

Basic symbol class:

$\sigma(x,\xi) \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$, if $\forall \alpha, \beta$:

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-|\beta|},$$

$$x \in \mathbb{R}^n, \xi \in \mathbb{R}^n, \ \langle \xi \rangle = (1+|\xi|^2)^{\frac{1}{2}}$$

We shall consider operator with kernel of the form

$$\frac{1}{(2\pi)^n} \int e^{i\langle x-y|\xi\rangle} \sigma(x,\xi)\, d\xi,$$

to yield $\sigma(x, D_x) \in \Psi^m$ with

$$\sigma(x, D_x) u(x)$$

$$= \mathrm{Op}(\sigma) u(x) = \frac{1}{(2\pi)^n} \iint e^{i\langle x-y|\xi\rangle} \sigma(x,\xi) u(y) \ dy d\xi$$

Basic properties

- If $\sigma \in S^m$ then $Op(\sigma) : H^s \to H^{s-m}$ cont.
- If $a \in S^m$, $b \in S^{m'}$ then

  $\mathrm{Op}(a) \circ \mathrm{Op}(b) = \mathrm{Op}(c) \in \Psi^{m+m'}$

$$c(x,\xi) = \frac{1}{(2\pi)^n} \iint e^{-i\langle y|\eta\rangle} a(x,\xi+\eta) b(x+y,\xi)\, dy d\eta$$

- If $a \in S^m$, then $\mathrm{Op}(a)^* = \mathrm{Op}(a^*) \in \Psi^m$

$$a^*(x,\xi) = \frac{1}{(2\pi)^n} \iint e^{-i\langle y|\eta\rangle} \overline{a}(x+y,\xi+\eta)\, dy d\eta$$

Basic properties

- if $a \in S^m$ elliptic, there exists $b \in S^{-m}$ such that

$$\mathrm{Op}(a) \circ \mathrm{Op}(b) = \mathrm{Id} + R, \quad \mathrm{Op}(b) \circ \mathrm{Op}(a) = \mathrm{Id} + R',$$

with R and R' regularizing.

$$b(x,\xi) \sim \sum_{j \in \mathbb{N}} b_j, \quad b_j \in S^{m-j}, \quad b_0 = \chi / a$$

$\chi \in \mathscr{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, $\chi = 0$ in a neighborhood of $\xi = 0$,
$\chi = 1$ for $|\xi|$ large

References: **Treves (80), Hörmander (Vol3, 85), Alinhac – Gérard (91)**, etc...

Tangential operators

$x = (x', x_n) \quad \xi = (\xi', \xi_n)$.

Basic symbol class:

$\sigma(x, \xi') \in S^m_{\mathcal{T}}(\mathbb{R}^n \times \mathbb{R}^{n-1})$, if $\forall \alpha, \beta$:

$$|\partial_x^\alpha \partial_{\xi'}^\beta \sigma(x, \xi')| \leq C_{\alpha,\beta} \langle \xi' \rangle^{m-|\beta|}, \quad x \in \mathbb{R}^n, \xi' \in \mathbb{R}^{n-1},$$

We shall consider operator of the form $\sigma(x, D_{x'}) \in \Psi^m$ with

$$\sigma(x, D_{x'}) u(x) = \mathrm{Op}_{\mathcal{T}}(\sigma) u(x) = \frac{1}{(2\pi)^{n-1}} \iint e^{i\langle x'-y' | \xi' \rangle} \sigma(x, \xi') u(y', x_n) \, dy' d\xi'$$

We have similar properties and composition formulae

Model boundary problem: $A = -\Delta$ formal approach

"Formal" solution of $\Delta u = 0$ in $x_n > 0$ + boundary conditions

Fourier transformation in $x' = (x_1, \cdots, x_{n-1})$, we obtain

$$(\partial_{x_n}^2 - |\xi'|^2)\hat{u}(\xi', x_n) = 0.$$

The solution $\hat{u}$ is given by $\hat{u}(\xi', x_n) = A(\xi')e^{-x_n|\xi'|} + B(\xi')e^{x_n|\xi'|}$.

As $e^{x_n|\xi'|} \notin \mathscr{S}'$ if $x_n > 0 \rightarrow$ we exclude this "bad" solution

$$\hat{u}(\xi', x_n) = A(\xi')e^{-x_n|\xi'|}.$$

In particular $\hat{u}(\xi', 0) = A(\xi')$, $\partial_{x_n}\hat{u}(\xi', 0) = -|\xi'|A(\xi')$ then

$$|\xi'|\hat{u}(\xi', 0) + \partial_{x_n}\hat{u}(\xi', 0) = 0$$

Conclusion : **One** of the traces determines the **two** traces.
Dirichlet-to-Neumann, Neumann-to-Dirichlet etc.

Model boundary problem: $A = -\Delta$

$A = -\partial_{x_n}^2 - \Delta_{x'}$, $\quad Au = f$ in $x_n > 0$

Symbol:

$$\sigma(A) = \xi_n^2 + |\xi'|^2 = (\xi_n - \rho^+)(\xi_n - \rho^-), \quad \rho^\pm = \pm i|\xi'|.$$

Introduce

$$\underline{\psi}(x) = \begin{cases} \psi(x) & \text{if } x_n \geq 0, \\ 0 & \text{if } x_n < 0. \end{cases}$$

Then

$$\partial_{x_n} \underline{u} = \underline{\partial_{x_n} u} + (u|_{x_n=0^+})\delta_{x_n},$$
$$\partial_{x_n}^2 \underline{u} = \underline{\partial_{x_n}^2 u} + (u|_{x_n=0^+})\delta'_{x_n} + (\partial_{x_n} u|_{x_n=0^+})\delta_{x_n}.$$

It follows that we have

$$A\underline{u} = \underline{f} - (u|_{x_n=0^+})\delta'_{x_n} - (\partial_{x_n} u|_{x_n=0^+})\delta_{x_n}$$

Model boundary problem: $A = -\Delta$ 2/8

There exists $b \in S^{-2}$, $b_0 = \chi/|\xi|^2$, such that $\mathrm{Op}(b) \circ A = I + R$

This yields

$$\underline{u} = \mathrm{Op}(b)\underline{f} - R\underline{u} - \mathrm{Op}(b)\Big((u|_{x_n=0^+})\delta'_{x_n}\Big) - \mathrm{Op}(b)\Big((\partial_{x_n} u|_{x_n=0^+})\delta_{x_n}\Big)$$

- Computation of $\mathrm{Op}(b)\Big((\partial_{x_n} u|_{x_n=0^+})\delta_{x_n}\Big)$:

$$= \frac{1}{(2\pi)^{n-1}} \iint e^{i\langle x'-y'|\xi'\rangle}\sigma(x,\xi')\,(\partial_{x_n} u|_{x_n=0^+})(y')\ dy'\,d\xi',$$

with

$$\sigma(x,\xi') = \frac{1}{2\pi}\int e^{i\langle x_n|\xi_n\rangle} b(x,\xi)\,d\xi_n$$

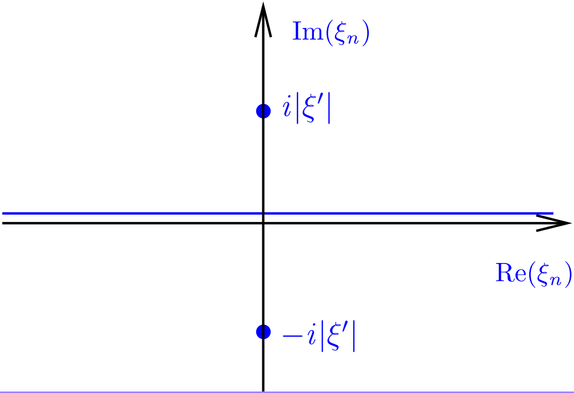
A similar formula is obtained for $\mathrm{Op}(b)\Big((u|_{x_n=0^+})\delta'_{x_n}\Big)$

Nature of σ ?

$b \sim \sum_j b_j$, with

$$b_0 = \chi \frac{1}{(\xi_n + i|\xi'|)(\xi_n - i|\xi'|)}.$$

$i|\xi'|$ and $-i|\xi'|$ are the only poles. (higher order for b_j, $j \geq 1$)



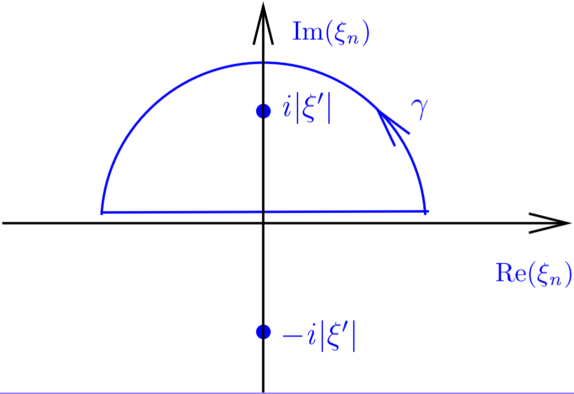
Model boundary problem: $A = -\Delta$

Nature of σ ?

$b \sim \sum_j b_j$, with

$$b_0 = \chi(x,\xi)\frac{1}{(\xi_n + i|\xi'|)(\xi_n - i|\xi'|)}.$$

b holomorphic for large $|\xi_n|$



Nature of σ ?

Choose $\chi_{\mathcal{T}}(x,\xi')$ such that

$$\begin{cases} \chi_{\mathcal{T}}(x,\xi') = 1 & \text{for } |\xi'| \text{ large} \\ \chi_{\mathcal{T}}(x,\xi') = 0 & \text{for } |\xi'| \text{ small} \end{cases}$$

AND $\chi = 1$ in the support of $\chi_{\mathcal{T}}$

Set

$$\tilde{\sigma}(x,\xi') = \chi_{\mathcal{T}}(x,\xi')\sigma(x,\xi'), \quad \underbrace{\underline{\sigma}(x,\xi') = (1-\chi_{\mathcal{T}}(x,\xi'))\sigma(x,\xi')}_{\to \text{ regularizing operator}},$$

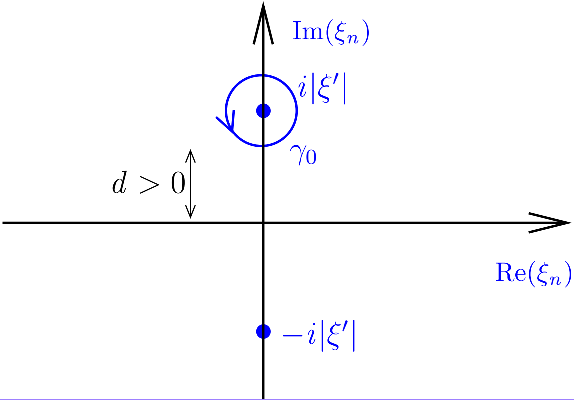
$$\tilde{\sigma}(x,\xi') = \frac{1}{2\pi}\int_{\gamma} e^{i\langle x_n|\xi_n\rangle}\chi_{\mathcal{T}}(x,\xi')b(x,\xi)\,d\xi_n$$

$\xi_n \longrightarrow b(x,\xi)$ holomorphic in the support of $\tilde{\sigma}(x,\xi')$

Nature of σ ?

$$\tilde{\sigma}(x,\xi') = \frac{1}{2\pi}\int_\gamma e^{i\langle x_n|\xi_n\rangle}\chi_{\mathcal{T}}(x,\xi')b(x,\xi)\,d\xi_n$$

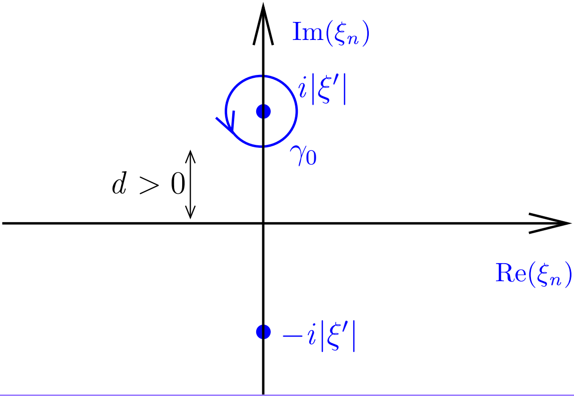
$\xi_n \to b(x,\xi)$ holomorphic in the support of $\tilde{\sigma}(x,\xi')$



Nature of σ ?

$$\tilde{\sigma}(x,\xi') = \frac{1}{2\pi} \int_{\gamma_0} e^{i\langle x_n|\xi_n\rangle} \chi_{\mathcal{T}}(x,\xi') b(x,\xi)\, d\xi_n$$

Residue formula $\rightarrow \tilde{\sigma}(x,\xi') \in S_{\mathcal{T}}^{-1}$ with exponential decay w.r.t. x_n.



We have

$$\underline{u} = \mathrm{Op}(b)\underline{f} + G + \mathrm{Op}_{\mathcal{T}}(\alpha)(u|_{x_n=0^+}) + \mathrm{Op}_{\mathcal{T}}(\beta)(\partial_{x_n} u|_{x_n=0^+})$$

with $\alpha \in S^0_{\mathcal{T}}$ and $\beta \in S^{-1}_{\mathcal{T}}$

Trace at $x_n = 0^+$:

$$u|_{x_n=0^+} = (\mathrm{Op}(b)\underline{f} + G)|_{x_n=0^+} + \mathrm{Op}_{\mathcal{T}}(\alpha|_{x_n=0^+})(u|_{x_n=0^+}) + \mathrm{Op}_{\mathcal{T}}(\beta|_{x_n=0^+})(\partial_{x_n} u|_{x_n=0^+})$$

Calderón projectors: $\mathrm{Op}_{\mathcal{T}}(\alpha|_{x_n=0^+})$, $\mathrm{Op}_{\mathcal{T}}(\beta|_{x_n=0^+})$
Residue formula: the symbols of α and β exhibit the difference
$\rho^+ - \rho^- = 2i|\xi'|$ in the denominator

Moreover $1 - \alpha|_{x_n=0^+} \in S^0_{\mathcal{T}}$ is elliptic.

One trace determines the two traces

$h =$ small parameter: $0 < h \le h_0$
We set $D = h\partial/i$

Basic symbol class:

$\sigma(x,\xi,h) \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$, if $\forall \alpha, \beta$:

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi,h)| \le C_{\alpha,\beta} \langle \xi \rangle^{m-|\beta|}, \qquad x \in \mathbb{R}^n, \xi \in \mathbb{R}^n,\ 0 < h \le h_0$$

Asymptotic series

$$\sigma(x,\xi,h) \sim \sum_{j \in \mathbb{N}} h^j \sigma_j(x,\xi,h), \qquad \sigma_j \in S^{m-j}.$$

We shall consider operator of the form

$$\mathrm{Op}(\sigma)u(x) = \frac{1}{(2\pi h)^n} \iint e^{i\langle x-y|\xi\rangle/h}\sigma(x,\xi,h)u(y)\ dyd\xi$$

Examples:
$A = -h^2\Delta + V(x)$
Symbol: $\sigma(A) = |\xi|^2 + V(x)$

Basic properties

- If $a \in S^m$, $b \in S^{m'}$ then

  $\mathrm{Op}(a) \circ \mathrm{Op}(b) = \mathrm{Op}(c) \in \Psi^{m+m'}$

$$c(x,\xi) = \frac{1}{(2\pi h)^n} \iint e^{-i\langle y|\eta\rangle/h} a(x,\xi+\eta,h) \times b(x+y,\xi,h)\, dy d\eta$$

- If $a \in S^m$, then $\mathrm{Op}(a)^* = \mathrm{Op}(a^*) \in \Psi^m$

$$a^*(x,\xi) = \frac{1}{(2\pi h)^n} \iint e^{-i\langle y|\eta\rangle/h} \overline{a}(x+y,\xi+\eta,h)\, dy d\eta$$

Basic properties

- if $a \in S^m$ elliptic, for all $N \in \mathbb{N}$ there exists $b \in S^{-m}$ such that

$$\mathrm{Op}(a) \circ \mathrm{Op}(b) = \mathrm{Id} + h^N R,$$
$$\mathrm{Op}(b) \circ \mathrm{Op}(a) = \mathrm{Id} + h^N R',$$

with R and $R' \in \Psi^{-N}$.

$$b(x,\xi) \sim \sum_{j \in \mathbb{N}} h^j b_j, \quad b_j \in S^{m-j}, \quad b_0 = \chi / a$$

$\chi \in \mathscr{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, $\chi = 0$ in a neighborhood of $\xi = 0$, $\chi = 1$ for $|\xi|$ large

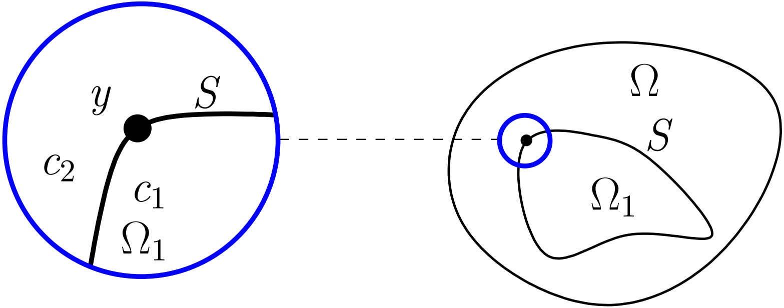
References: **Dimassi – Sjöstrand (99), Martinez (02)**

Application - Carleman estimates for coefiicients with jump at an interface

$A = \nabla(\gamma\nabla)$ or $A = \partial_t - \nabla(\gamma\nabla)$

- **Doubova – Osses – Puel, 02**: γ is piecewise $\mathscr{C}^1$
  Parabolic Carleman estimate - yet observation region is where the value of γ is the lowest.
- **LR – Robbiano, 08**: γ is piecewise $\mathscr{C}^\infty$
  Elliptic Carleman estimate
- **LR – Robbiano, 09**: γ is piecewise $\mathscr{C}^\infty$
  Parabolic estimate

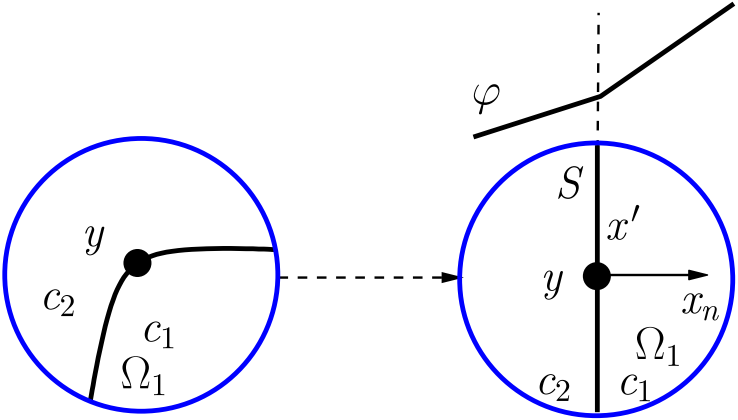
Global geometry



Transmission condition at interface

$$w|_{x_n=0^-} = w|_{x_n=0^+}, \quad c\partial_{x_n} w|_{x_n=0^-} = c\partial_{x_n} w|_{x_n=0^+}, \qquad \text{(CT)}$$

Normal geodesic coordinnates



$x = (x', x_n)$

The **principal part** of the elliptic operator is:
$-\partial_{x_n} \gamma(x) \partial_{x_n} - \gamma(x) r(x, \partial_{x'})$ on both sides of $S = \{x_n = 0\}$.

Carleman estimate

$$(S^*) \begin{cases} -\nabla \cdot \gamma \nabla q = f & \text{dans } Q, \\ q = 0 & \text{sur } \Sigma, \end{cases}$$

$\exists K > 0$, $\tau_0 > 0$, that depend on ω, Ω and γ, s.t.

$$\tau^3 \| e^{\tau\varphi} q \|^2_{L^2(\Omega)} + \tau \| e^{\tau\varphi} \nabla q \|^2_{L^2(\Omega)} \leq K \left(\| e^{\tau\varphi} f \|^2_{L^2(\Omega)} + \tau^3 \| e^{\tau\varphi} q \|^2_{L^2(\omega)} \right),$$

for $\tau \geq \tau_0$.

The weight function is chosen of the following form

$$\varphi(x) = e^{\lambda \psi(x)}$$

References: **Carleman (39), Hörmander (63, 85), Zuily (83), Lebeau – Robbiano (95,97), Fursikov – Imanuvilov (96)**, etc...

Carleman estimate

$$(S^*)\begin{cases} -\nabla\cdot\gamma\nabla q = f & \text{dans } Q, \\ q = 0 & \text{sur } \Sigma, \end{cases}$$

$\exists K>0$, $h_0>0$, that depend on ω, Ω and γ, s.t.

$$h\left\|e^{\varphi/h}q\right\|^2_{L^2(\Omega)} + h^3\left\|e^{\varphi/h}\nabla q\right\|^2_{L^2(\Omega)} \\ \leq K\left(h^4\left\|e^{\varphi/h}f\right\|^2_{L^2(\Omega)} + h\left\|e^{\varphi/h}q\right\|^2_{L^2(\omega)}\right),$$

for $0<h\leq h_0$.

The weight function is chosen of the following form

$$\varphi(x) = e^{\lambda\psi(x)}$$

Applications

- Control and stabilization of PDEs:

  controllability of classes of semi-linear parabolic equations

- Inverse problems:

  identification of coefficients including stability results

Conjugated operator

We introduce $P_\varphi = h^2 e^{\varphi/h} P e^{-\varphi/h}$: semi-classical operator of order 2 (on both sides of the interface)

We set $v = e^{\varphi/h} w$

$$Pw = f \quad \Longleftrightarrow \quad P_\varphi v = h^2 e^{\varphi/h} f$$

The transmission conditions become

$$\begin{aligned} &v^l|_{x_n=0^+} = v^r br, \\ &c^l(D_{x_n} + i\partial_{x_n}\varphi^l)v^l|_{x_n=0^+} + c^r(D_{x_n} + i\partial_{x_n}\varphi^r)v^r|_{x_n=0^+} = 0 \end{aligned} \qquad (\mathrm{CT}_\varphi)$$

with $D = \frac{h}{i}\partial$

The "Carleman" approach

- $P_\varphi = A + iB$, (A, B selfadjoint).

$$\|P_\varphi v\|^2 = \|Av\|^2 + \|Bv\|^2 + 2\mathrm{Re}(Av, iBv)$$

- Computation of $\mathrm{Re}(Av, iBv)$

$$\|P_\varphi v^l\|^2 + \|P_\varphi v^r\|^2 = U(v) + \mathcal{B}(v)$$

- where $U(v)$ are "usual" terms: simple.
- $\mathcal{B}(v)$ boundary terms of the form:

$$(L_0 D_{x_n} v, D_{x_n} v)_0, \quad (L_1 v, D_{x_n} v)_0, \quad (L_2 v, v)_0, \quad \text{on both sides}$$

  L_j tangential operators of order j.
  $(\cdot,\cdot)_0$: scalar product at the interface.

- We can use the trans. conditions to analyse $\mathcal{B}(v)$
  In general $\mathcal{B}(v) \not\geq 0$.
  In the case of **Doubova – Osses – Puel (02)**: $\mathcal{B}(v) \geq 0$

Boundary problem

$P_\varphi v = g$ in $x_n > 0$

$$\text{We write } \underline{v} = \begin{cases} v \text{ si } x_n > 0 \\ 0 \text{ si } x_n < 0 \end{cases}$$

$$P_\varphi \underline{v} = \underline{g} + \gamma_0 \delta'_{x_n=0} + \gamma_1 \delta_{x_n=0}$$

where γ_j is a function of $v_{|x_n=0}$ and $D_{x_n} v_{|x_n=0}$.

Let Q be a parametrix of P_φ (when possible)

$$\underline{v} = Q\underline{g} + C_0\gamma_0 + C_1\gamma_1 + h^N R\underline{v}$$

Trace on $x_n = 0^+$:

$$v_{|x_n=0} = Q\underline{g}_{|x_n=0} + C'_0\gamma_0 + C'_1\gamma_1 + h^N R\underline{v}_{|x_n=0}$$

Does this give a relation between $v_{|x_n=0}$ and $D_{x_n} v_{|x_n=0}$?

Calderón projectors 1/2

C_0, C_1 have the following form (principal part)

$$C_j \gamma_j = \iint e^{ix'\xi'} \gamma_j(x') \Big(\int e^{ix_n \xi_n} \frac{\xi_n^j}{p_\varphi(x,\xi)} \, d\xi_n \Big) \, d\xi' \, dx'$$

$p_\varphi(x, \xi', \xi_n)$ principal symbol of P_φ.

Key point $\longrightarrow$ root of $p_\varphi(x, \xi', \xi_n)$ in ξ_n

Why ? $\longrightarrow$ residue formula

Projecteurs de Calderón 2/2

We have $p_\varphi(x,\xi) = (\xi_n - r_1(x,\xi'))(\xi_n - r_2(x,\xi')) \rightarrow$ 3 cases

- if $\mathrm{Im} r_1(x,\xi') < 0$ and $\mathrm{Im} r_2(x,\xi') < 0$ then $\int e^{ix_n\xi_n} \frac{\xi_n^j}{p(x,\xi)} d\xi_n = 0$
$\rightarrow f$ determine both traces of u

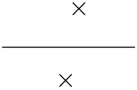
- si $\mathrm{Im} r_1(x,\xi') > 0$ and $\mathrm{Im} r_2(x,\xi') < 0$ then

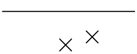
$$\int e^{ix_n\xi_n} \frac{\xi_n^j}{p(x,\xi)} d\xi_n = 2i\pi e^{ix_n r_1(x,\xi')} \frac{r_1^j(x,\xi')}{r_1(x,\xi') - r_2(x,\xi')}$$

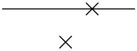
$\rightarrow$ We are back to the model problem $\rightarrow$ one relation between the two traces.

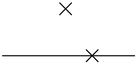
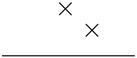
- If $\mathrm{Im} r_1(x,\xi') > 0$ and $\mathrm{Im} r_2(x,\xi') > 0$
$\rightarrow$ No relation between the traces

Root localisation

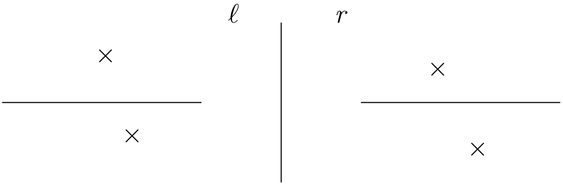
 one relation between the traces, $u_{|x_n=0}$ and $D_{x_n} u_{|x_n=0}$.

 fully determined traces

 Carleman method $\rightarrow$ one relation between the traces

 or  no relation between the traces

Back to the coupled problem: a "good" case

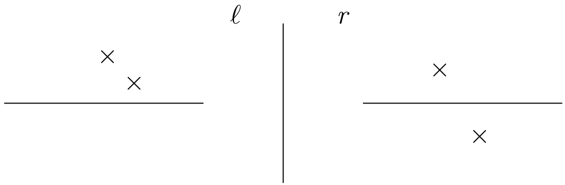


This happens when $|\xi'|$ is large.

- one relation between the traces on the l.h.s.
- one relation between the traces on the r.h.s.
- two transmission conditions

4 equations for 4 unknowns $\rightarrow$ we can solve "algebraically"

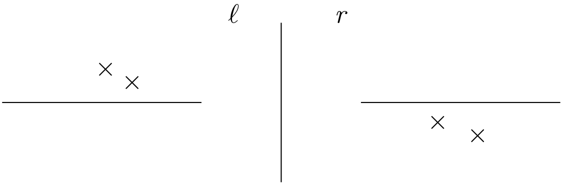
A "bad" case



- no condition on the traces on the l.h.s.
- one relation between the traces on the r.h.s.
- two transmission conditions

3 equations for 4 unknowns $\rightarrow$ impossible to solve

Another "good" case



- no condition on the traces on the l.h.s.
- fully determined traces on the r.h.s. $\rightarrow$ 2 equations
- two transmission conditions

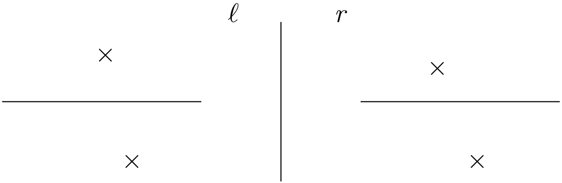
4 equations for 4 unknowns $\rightarrow$ we can solve "algebraically"

Strategy

Conjugated operator:

$$P_\varphi = h^2 e^{\varphi/h} P e^{-\varphi/h}$$

The weight function φ is chosen so as to have a succession of "good" cases for all the frequency regimes in ξ'



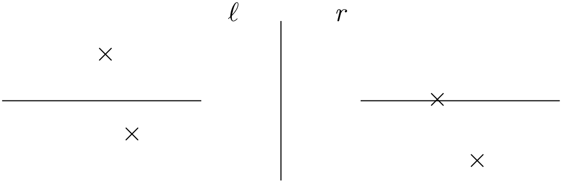
high frequences

Strategy

Conjugated operator:

$$P_\varphi = h^2 e^{\varphi/h} P e^{-\varphi/h}$$

The weight function φ is chosen so as to have a succession of "good" cases for all the frequency regimes in ξ'



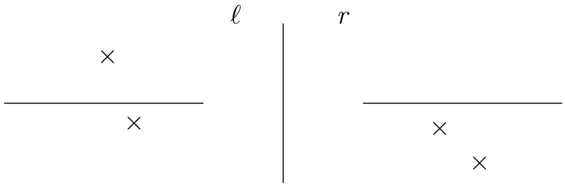
medium frequencies

Strategy

Conjugated operator:

$$P_\varphi = h^2 e^{\varphi/h} P e^{-\varphi/h}$$

The weight function φ is chosen so as to have a succession of "good" cases for all the frequency regimes in ξ'



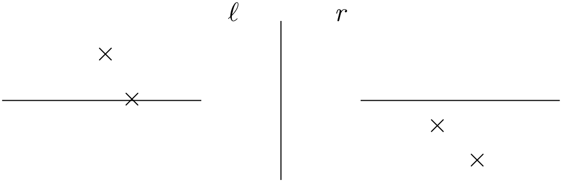
low frequencies

Strategy

Conjugated operator:

$$P_\varphi = h^2 e^{\varphi/h} P e^{-\varphi/h}$$

The weight function φ is chosen so as to have a succession of "good" cases for all the frequency regimes in ξ'



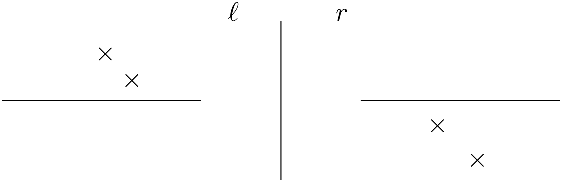
low frequencies

Strategy

Conjugated operator:

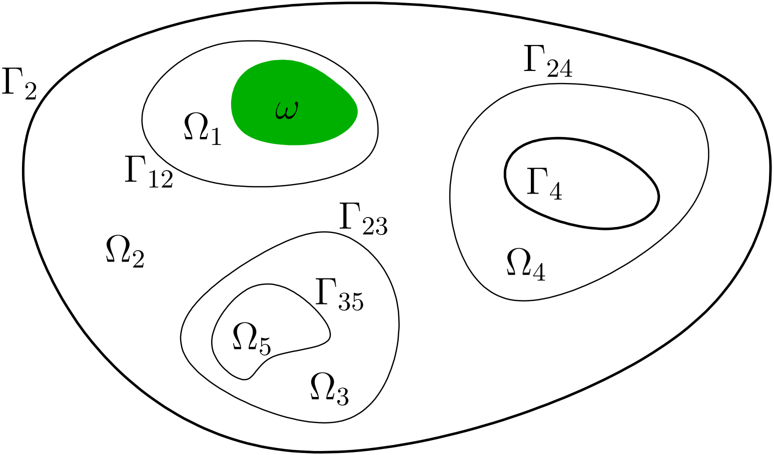
$$P_\varphi = h^2 e^{\varphi/h} P e^{-\varphi/h}$$

The weight function φ is chosen so as to have a succession of "good" cases for all the frequency regimes in ξ'

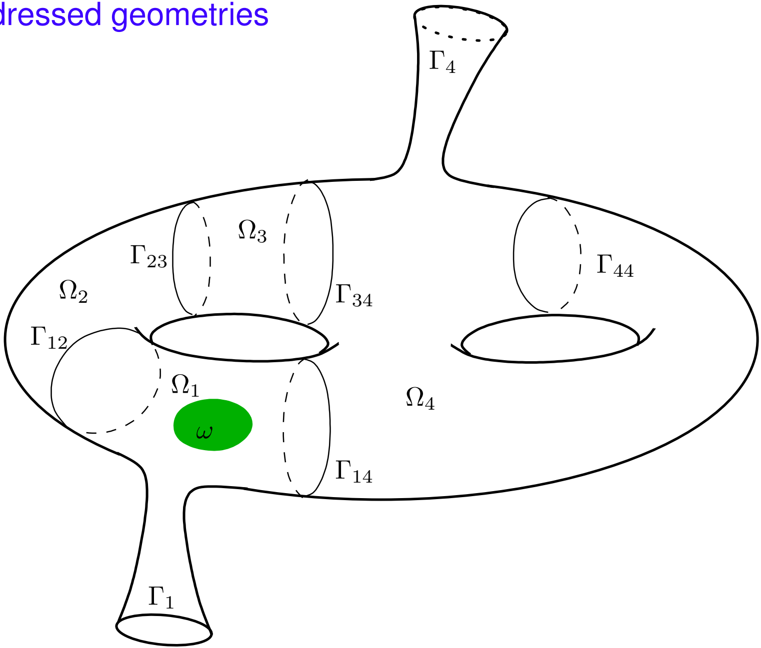


low frequencies

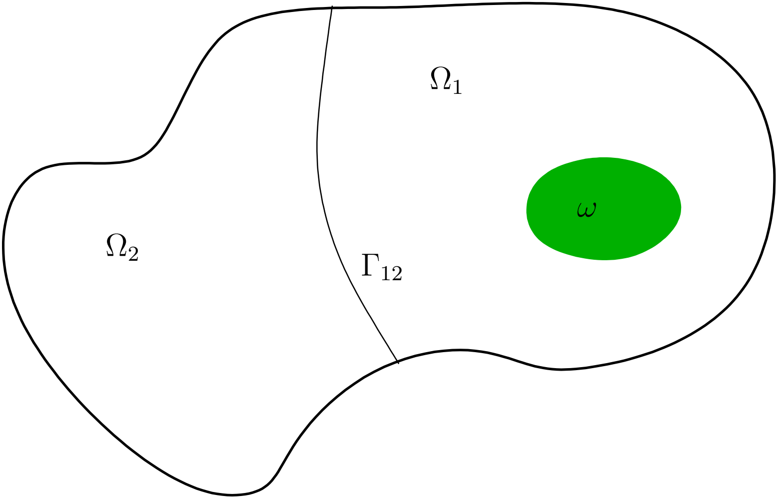
Addressed geometries



Addressed geometries



Some open problems



Some open problems

- discontinuous diffusion matrix (important for applications)
- crossing interfaces