# What is the optimal shape of a pipe?

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# Who is right ?

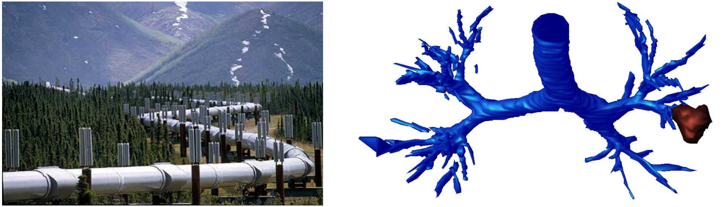


FIG.: The example of a pipeline and of the bronchial tree

# Outlines of the talk

# Principle of inverse modelling

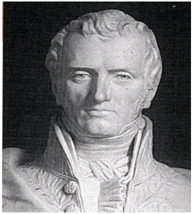
**Question**

Do the shapes in Nature try to optimize some criterion ?

- Let us consider an organ or a part of the human body.
- We write a mathematical model (e.g. a PDE) which describes the behaviour of this organ.
- We imagine a numerical criterion that Nature would like to optimize.
- We determine the optimal shape for this criterion and this model.
- We compare the theoretical shape with the real ones.

# Mathematical and Physical models (1)

- $\mathcal{U}$ = set of simply connected domains of $\mathbb{R}^3$ for which the inlet $E$ and the outlet $S$ are fixed.
- We assume that $\Omega \in \mathcal{U}$ is crossed by a newtonian viscous incompressible fluid, driven by the stationnary Navier-Stokes system.
- $\mathbf{u} = \mathbf{u}(x_1, x_2, x_3)$ = velocity of the fluid and $p = p(x_1, x_2, x_3)$ = pressure of the fluid.



# Mathematical and Physical models (2)

## The PDE

The fluid is driven by the Navier-Stokes PDE :

$$\begin{cases} -\mu\triangle\mathbf{u} + \nabla p + \boxed{\mathbf{u}\cdot\nabla\mathbf{u}} = 0 & \mathbf{x}\in\Omega \\ \text{div }\mathbf{u} = 0 & \mathbf{x}\in\Omega \end{cases}$$

**Boundary conditions**

1. Inlet $E$ : we assume that the velocity of the fluid is known (parabolic profile).
2. Lateral boundary $\Gamma$ : we impose a *no-slip* boundary condition (i.e. $\mathbf{u} = 0$ on $\Gamma$).
3. Outlet $S$ : we impose a condition of normal constraint.

# Mathematical and Physical models (3)

## The criterion

We define :

- The stretching tensor : $\varepsilon(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$.
- The strain tensor : $\sigma(\mathbf{u}, p) = -pI_3 + 2\mu\varepsilon(\mathbf{u})$.

A good criterion from a physical point of view is :

$$\boxed{J(\Omega) = 2\mu \int_\Omega |\varepsilon(\mathbf{u})|^2 dx}.$$

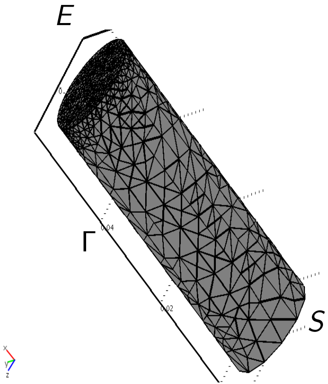
# Study of the shape optimization problem : 2 directions

1. **A theoretical approach. (joint work with Antoine Henrot, École des Mines de Nancy)**
   Is the cylinder an optimal shape to minimize the energy dissipated by the fluid ?

2. **A numerical approach. (joint work with Benjamin Mauroy, CNRS, Paris)**
   What is the shape of a bifurcation (for instance, the trachea and the daughter branches) minimizing the energy dissipated by a fluid ?

# The optimal shape of a pipe

- **Question :** do the cylinder minimize the energy dissipated by the fluid ?
- Let us consider a cylinder with length $L > 0$ and radius $R > 0$.



# Is the cylinder optimal ? (1)

## The PDE on the cylinder

$$\begin{cases} -\mu\triangle \mathbf{u} + \nabla p + \boxed{\mathbf{u}\cdot\nabla\mathbf{u}} = 0 & \mathbf{x}\in\Omega \\ \text{div } \mathbf{u} = 0 & \mathbf{x}\in\Omega \\ \mathbf{u} = \mathbf{u_0} & \mathbf{x}\in E \\ \mathbf{u} = \mathbf{0} & \mathbf{x}\in\Gamma \text{ (No-slip condition)} \\ \sigma(\mathbf{u},p)\cdot\mathbf{n} = 0 & \mathbf{x}\in S \text{ (normal flow)}, \end{cases}$$

with :

- $\mathbf{u_0}$ = parabolic velocity profile ;
- $\sigma(\mathbf{u},p) = -pI_3 + \mu(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$ = strain tensor.

# Is the cylinder optimal ? (2)

Let us remind that :

$$J(\Omega) = 2\mu \int_{\Omega} |\varepsilon(\mathbf{u})|^2 dx.$$

**Theorem. A non optimality result (A. Henrot, Y.P.)**

The cylinder is not solution of the following shape optimization problem :

$$\begin{cases} \min J(\Omega) \\ \text{Vol}(\Omega) \text{ is given.} \end{cases}$$

# Is the cylinder optimal ? (3)

## Outlines of the proof (1)

- **Step 1 : calculus of the shape derivative.**
  Let $f(t) := J((I + t\mathbf{V})\Omega)$, for $t$ small and $\mathbf{V}$, a smooth vector field.
  The shape derivative of the criterion $J$ is :

$$f'(0) = \mathrm{d}J(\Omega, \mathbf{V}) = 2\mu \int_\Gamma \left(\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) - |\varepsilon(\mathbf{u})|^2\right)(\mathbf{V} \cdot \mathbf{n})\mathrm{d}\sigma,$$

where $\mathbf{v}$ is an adjoint state ($\simeq$ linearized Navier-Stokes equation) :

$$(AS) \begin{cases} -\mu\triangle\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{v} + \nabla q = -2\mu\triangle\mathbf{u} & \mathbf{x} \in \Omega \\ \mathrm{div}\ \mathbf{v} = 0 & \mathbf{x} \in \Omega \\ \mathbf{v} = \mathbf{0} & \mathbf{x} \in E \cup \Gamma \\ \sigma(\mathbf{v}, q)\cdot\mathbf{n} + (\mathbf{u}\cdot\mathbf{n}) - 4\mu\varepsilon(\mathbf{u})\cdot\mathbf{n} = 0 & \mathbf{x} \in S. \end{cases}$$

# Is the cylinder ? (4)

## Outlines of the proof (2)

- **Step 2 : mathematical analysis of the adjoint state.**
  A symmetry result : there exists three functions $w$, $w_3$ and $\tilde{q}$ s.t. $\forall(x_1, x_2, x_3) \in \Omega$ :
  - $v_i(x_1, x_2, x_3) = x_i w(r, x_3)$, $i \in \{1, 2\}$.
  - $v_3(x_1, x_2, x_3) = w_3(r, x_3)$,
  - $q(x_1, x_2, x_3) = \tilde{q}(r, x_3)$.

  Moreover,

$$\boxed{(\mathbf{v}, q) \in C^1(\overline{\Omega}) \times C^0(\overline{\Omega}).}$$

# Is the cylinder optimal ? (5)

## Outlines of the proof (3)

- **Step 3 : a first order optimality condition.**
  Let us use the previous symmetry result.
  There exists $\lambda \in \mathbb{R}$ s.t. :

$$\mathrm{d}J(\Omega, \mathbf{V}) = \lambda \int_{\Gamma} (\mathbf{V} \cdot \mathbf{n}) ds,$$

which rewrites :

$$\boxed{\frac{\partial v_3}{\partial n} = 0 \text{ on } \Gamma}.$$

# Is the cylinder optimal ? (5)

## Outlines of the proof (3)

- **Step 4 : Conclusion.** Let us introduce the functions :

$$w_0(r, x_3) := \int_0^{x_3} w(r, z)\mathrm{d}z \text{ and } \psi(z) = \int_{\Gamma_z} (\tilde{q} - 2cr^2 w_0) r\mathrm{d}r\mathrm{d}\theta.$$

**Lemme**

The function $\psi$ is affine.

Idea of the proof. We apply the divergence operator to the PDE :

$$-\mu\triangle\mathbf{v} + \nabla q + \nabla\mathbf{u}\cdot\mathbf{v} - \nabla\mathbf{v}\cdot\mathbf{u} = -2\mu\triangle\mathbf{u},$$

then, we integrate this equation on a strip of the cylinder.

# Is the cylinder optoimal ? (6)

## Outlines of the proof (4)

Ingredients to conclude :

- $\longrightarrow$ The pair $(\mathbf{v}, q)$ belongs to $C^1 \times C^0$ in $\overline{\Omega}$.
- $\longrightarrow$ We integrate the PDE giving $v_3$ separately on $E$ and on $S$.
- $\longrightarrow$ We use the overdetermined condition $\frac{\partial v_3}{\partial n} = 0$ on $\Gamma$.

We obtain :

$$\boxed{\psi'(L) = -16\mu c\pi R^2} \text{ and } \boxed{\psi'(0) = -8\mu c\pi R^2}.$$

$\psi$ is affine, then it is absurd !

# Extension of the previous result

**Theorem. (A. Henrot, Y.P.)**

The cylinder is not optimal :

- in the case of a Navier-Stokes system, in 2D and 3D ;
- in the case of a Stokes system, in 2D and 3D.

$\longrightarrow$ **A natural question : has the solution of the optimization problem a cylindrical symmetry ?**

# Symmetry of the optimum (1)

Only in the case of a Stokes system, that is possible to state the :

**Theorem. (A. HENROT, Y.P.)**

There exists a domain $\Omega$ minimizing the dissipated energy under volume constraint which has a plane of symmetry containing the vertical axis (going from the center of $E$ to the center of $S$).

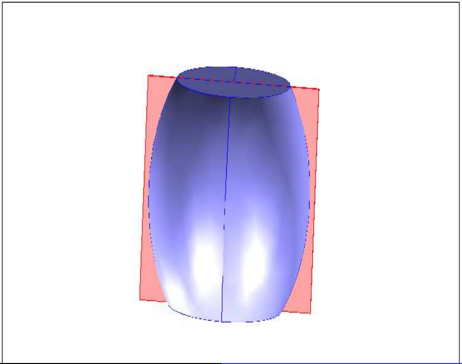
$\longrightarrow$ An element of answer : has the optimum a cylindrical symmetry ?

G. Arumugam and O. Pironneau showed that, in the case of a Poiseuille flow (when the flow is proportional to he drop pressure between the inlet and the outlet of the pipe), one improves the criterion $J$ by creating some vertical riblets.

# Symmetry of the optimum (2)

Let $\Omega$, a solution of he shape optimization problem.

- **Step 1 : Selection of a domain with measure $|\Omega|/2$.**
  There exists a plane containing the vertical axis, spliting $\Omega$ in two domains with the same measure.



# Symmetry of the optimum (3)

- **Step 2 : "Symmetrisation" of the domain $\Omega$.**

Let $\Omega_1$ and $\Omega_2$ be the two domains with same measure.

If $\int_{\Omega_1} |\varepsilon(\mathbf{u})|^2 dx \leq \int_{\Omega_2} |\varepsilon(\mathbf{u})|^2 dx$, let :

$$\widehat{\mathbf{u}}(\mathbf{x}) = \begin{cases} \mathbf{u}(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega_1 \\ \mathbf{u}(\sigma(\mathbf{x})) & \text{if } \mathbf{x} \in \sigma(\Omega_1) \end{cases} \quad \text{and } \widehat{p}(\mathbf{x}) = \begin{cases} p(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega_1 \\ p(\sigma(\mathbf{x})) & \text{if } \mathbf{x} \in \sigma(\Omega_1) \end{cases}$$

where $\sigma$ is the symmetry operator with respect to the plane spliting $\Omega$ in two domains with the same measure, and $\widehat{\Omega} = \Omega_1 \cup \sigma(\Omega_1)$.

# Symmetry de l'optimum (4)

- **Step 3 : Conclusion.**

$$\begin{aligned} J(\widehat{\Omega}) &= \min_{\mathbf{u} \mid \mathrm{div}\mathbf{u}=0} \left( 2\mu \int_{\widehat{\Omega}} |\varepsilon(\mathbf{u})|^2 \mathrm{d}x \right) \\ &\leq 2\mu \int_{\widehat{\Omega}} |\varepsilon(\widehat{\mathbf{u}})|^2 \mathrm{d}x \\ &\leq J(\Omega) \end{aligned}$$

$\longrightarrow$ $\widehat{\Omega}$ is admissible ($|\widehat{\Omega}| = |\Omega|$).
$\longrightarrow$ The previous inequalities are equalities.

$\widehat{\Omega}$ minimizes the criterion $J$ in the class of admissible shapes.

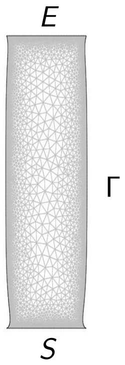
# Symmetry of the optimum (5)

$\longrightarrow$ Is this propertie true for every minimizer of this problem ?
Yes, is the minimizer $\Omega$ is $\mathcal{C}^2$, using the analyticity of the solutions of the Stokes problem.

## Open Problems

- Quid of the Navier-Stokes case ? (The "symmetrisation" technic cannot be used *a priori*.)
- Is it possible to prove stronger symmetry properties in the Stokes and Navier-Stokes cases ?

# On the optimal shape of a pipe
## Numerical confirmation of the non optimality result



# How dissipate a fluid through a bifurcation ? (1)

## What choice of modelling ?

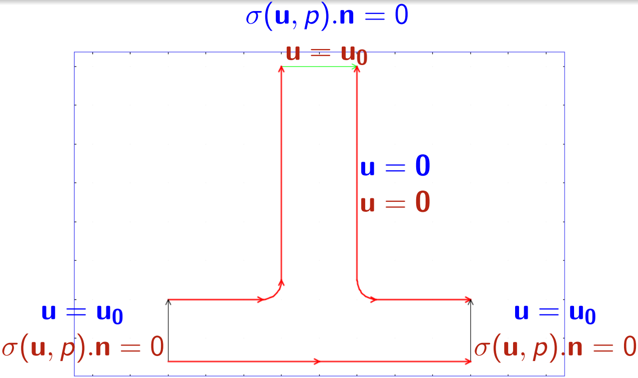



FIG.: The different boundary conditions

# How dissipate a fluid through a bifurcation ? (2)

## An Augmented Lagrangien like algorithm

Let $\tau > 0$ and $\varepsilon_{stop} > 0$.
We define the augmented Lagrangian of the problem :

$$\mathcal{L}_b(\Omega, \mu) = J(\Omega) + \mu\,(\text{mes }(\Omega) - V_0) + \frac{b}{2}(\text{mes }(\Omega) - V_0)^2.$$

**Description of the algorithm**

$\longrightarrow$ **Initialisation.** Let $\Omega_0$ be fixed (initial shape of the tree) and $\mu_0 \in \mathbb{R}$.

$\longrightarrow$ **Iteration $m$ : a gradient method**

$\longrightarrow$ **Calculus of the descent direction : $-\nabla\mathcal{L}_b(\Omega_m, \mu_m)$.**

- Resolution of the Navier-Stokes problem (solution $\mathbf{u_m}$).
- Resolution of the adjoint state (solution $\mathbf{v_m}$).

# How dissipate a fluid through a bifurcation ? (3)

$\longrightarrow$ Determination of the displacement $\mathbf{d_m}$. We choose $\mathbf{d_m}$ solution of :

$$\langle \mathbf{d_m}, \mathbf{w} \rangle_{H^1(\Omega_m)} = -\int_{\Gamma_m} \nabla \mathcal{L}_b(\Omega_m, \mu_m) \cdot \mathbf{w} ds, \ \forall \mathbf{w} \in B(\Omega_m),$$

with $B(\Omega_m) \overset{\text{déf}}{=} \left\{ \mathbf{w} \in H^1(\Omega_m) \mid \mathbf{w}_{|_{E \cup S}} = 0 \right\}$.
Then :

$$\begin{aligned} 0 \leq \langle \mathbf{d_m}, \mathbf{d_m} \rangle_{H^1(\Omega_m)} &= -\int_{\Gamma_m} \nabla \mathcal{L}_b(\Omega_m, \mu_m) \cdot \mathbf{d_m} ds \\ &= -\langle d\mathcal{L}_b(\Omega_m, \mu_m), \mathbf{d_m} \rangle. \end{aligned}$$

# How dissipate a fluid through a bifurcation ? (4)

$\longrightarrow$ Determination of $\Omega_{m+1}$ : $\Omega_{m+1} = (I + \varepsilon_m \mathbf{d}_m)(\Omega_m)$.

$\longrightarrow$ Reinitializing of the Lagrange multiplier :

$$\mu_{m+1} = \mu_m + \tau\left(\text{mes}\left(\Omega_{m+1}\right) - V_0\right)$$

$\longrightarrow$ Stopping criterion.

# How dissipate a fluid through a bifurcation ? (5)

## Some numerical results

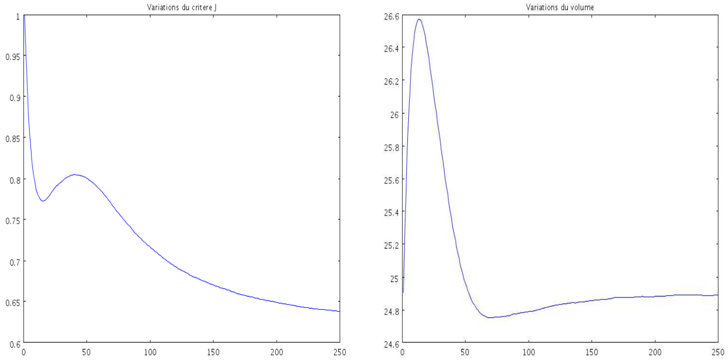



FIG.: On the right, volume as a function of the iteration and n the left, the criterion as a function of the iterations

# How dissipate a fluid through a whole tree ? (1)

- Case of a tree driven by a Poiseuille fluid (with Xavier Dubois de la Sablonière, Supélec) : theoretical study of the shape optimization problem. (existence of a minimizing sequence closing all the branches of the tree except one)
- Case of a tree driven by a Navier-Stokes fluid (with Benjamin Mauroy, CNRS) : numerical study. (confirmation of the result obtained in the Poiseuille case)

# How dissipate a fluid through a whole tree ? (2)

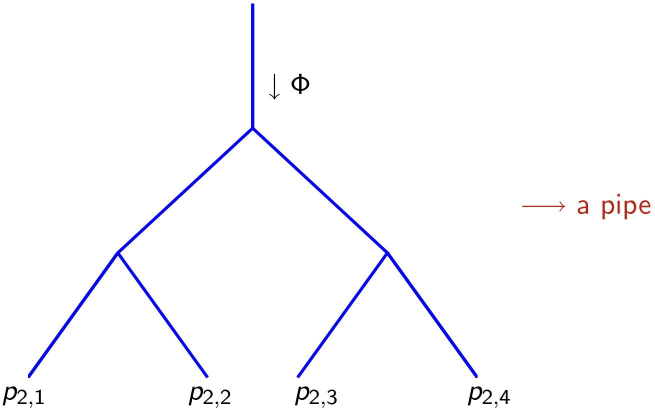



FIG.: An example of dichotomous tree of three generations

# Prospects (1)

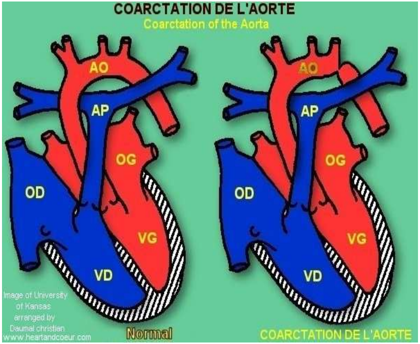
- Would the cylindrical pipe be optimal for other reasonnable data at the inlet and the outlet ?
- Existence en characterization of an optimum in a class of simply connected domains having a cylindrical symmetry (joint work with Maitine Bergounioux, MAPMO, Orléans)
- The study of an other criterion may be interesting :

$$J_1(\Omega) := \int_S p(s)\mathrm{d}s - \int_E p(s)\mathrm{d}s \text{ (drop pressure)}$$

Give the minimization of such a criterion the shape of the bronchial tree ?

# Prospects (2)

- $J_2(\Omega) := \int_{\partial\Omega} |\sigma(\mathbf{u}, p)|^2 \mathrm{d}x$. (constraints)
  Application to the aorta coarctation problem.
  $\Longrightarrow$ Joint work with Benjamin MAUROY.



# Thank you for your attention !