Blow-up and global sign-changing solutions of the nonlinear heat equation

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TC, FD, FBW ()

sign-changing solutions

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The nonlinear heat equation

$$\begin{cases} u_t - \Delta u = |u|^{\alpha} u & \text{in } (0, T) \times \Omega, \\ u(0) = u_0 & \text{in } \Omega, \end{cases}$$
(NLH)

 Ω is a bounded domain or the whole \mathbb{R}^N , lpha > 0, $u_0 \in C_0(\Omega).$

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Image: Image:

$$\begin{split} \mathcal{G} &= \{u_0, u(t) \text{ is global}\}, \\ \mathcal{G}_0 &= \{u_0 \in \mathcal{G}, u(t) \to 0, t \to +\infty\}, \\ \mathcal{B} &= \{u_0, u(t) \text{ blows up}\}. \end{split}$$

• There exists $0 \neq u_0 \in \mathcal{G}_0$.

• Given $\varphi \neq 0$, $\lambda \varphi \in \mathcal{B}$ if λ is large.

• $\varphi \geq 0, \ \varphi \in \mathcal{G} \Longrightarrow \lambda \varphi \in \mathcal{G} \text{ if } 0 < \lambda < 1.$

 $(\mathcal{G}^+ \text{ is starshaped with respect to 0.})$

Some Questions:

• Is \mathcal{G} , \mathcal{G}_0 convex?

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Theorem - Suppose $\int \varphi \neq 0$. Then $\lambda \varphi \in \mathcal{B}$ if $\lambda > 0$ is small.

Proof - Assume $\int \varphi > 0$ and let $v(t, x) = u(t, x)/\lambda$. Then, $v = \Delta v = \lambda^{\alpha} |v|^{\alpha} v |v(0) = \omega$

Consider, $z = e^{t\Delta}\varphi$. Since $\int \varphi > 0$, z(t) > 0 for t large. Thus, v(t) > 0 if λ is small.

Remark - Let u be a stationary solution in \mathbb{R} : $-u^{''} = |u|^{\alpha}u, u'(-1) = u'(1) = 0.$ Then $\lambda u \in \mathcal{G}_0$ for $|\lambda| < 1$.

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Proof - The linearization of (NP) around 0 is the heat equation. Let M be the nonlinear stable manifold near 0, $S = [1]^{\perp}$ the linear stable manifold, C = [1].



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The Neumann Problem For $\varphi_s \in S$ we may choose $u_0 = \varepsilon \varphi_s + c \in M$. Suppose $c \equiv 0$. Define $I(t) = \int u(t)$. Then

$$I'(t) = \int |u(t)|^{lpha} u(t) \Longrightarrow I'(0) = \int |arphi_s|^{lpha} arphi_s.$$

We pick φ_s such that $I'(0) \neq 0$. Then $I(t) \neq 0$ for $t \approx 0$.

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The Neumann case



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(CP) for $\alpha < 2/N$ corresponds to the Neumann problem in the following sense.

- $u_0 \geq 0$, $u_0 \neq 0$, then $u_0 \in \mathcal{B}$ (Fujita, '66).
- If $\varphi \in L^1$, $\int \varphi \neq 0$ then $\lambda \varphi \in \mathcal{B}$ if λ is small.
- There exists $\varphi \in L^1 \cap \mathcal{G}$, $\int \varphi \neq 0$.
- 0 is stable in L^{∞} .
- There exists $\varphi \in L^1 \cap \mathcal{G}_0$, $\int \varphi = 0$. $\lambda \varphi \in \mathcal{G}_0$ for all $|\lambda| < 1$ and $\lambda \varphi \in \mathcal{B}$ if λ is large.

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(DP)

- If $\varphi \ge 0$, $\varphi \ne 0$ then there exists $\lambda^* > 0$ such that $\lambda \varphi \in \mathcal{G}_0$ if $\lambda < \lambda^*$, $\lambda \varphi \in \mathcal{B}$ if $\lambda > \lambda^*$.
- \mathcal{G}^+ , \mathcal{G}_0^+ are convex.
- If α < α_s = 4/(N 2) then (DP) admits stationary sign-changing solutions Ψ.

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Proposition - Take Ψ a stationary sign-changing solution. Let φ_1 be a first eigenvector of $-\Delta - (\alpha + 1)|\Psi|^{\alpha}$. If $I = \int \Psi \varphi_1 \neq 0$ then $\lambda \Psi \in \mathcal{B}$ if $\lambda \approx 1$, $\lambda \neq 1$. In particular, \mathcal{G} is not star-shaped around 0.

Proof - Assume l > 0, call u_{λ} the solution starting at $\lambda \Psi$ and set $z_{\lambda} = \frac{u_{\lambda} - \Psi}{\lambda - 1}$. Then

$$\begin{cases} \partial_t z_{\lambda} - \Delta z_{\lambda} = \frac{|u|^{\alpha} u - |\Psi|^{\alpha} \Psi}{\lambda - 1} \approx (\alpha + 1) |\Psi|^{\alpha} z_{\lambda}, \\ z_{\lambda}(0) = \Psi \end{cases}$$

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Since $\int \Psi \varphi_1 > 0$, z(t) becomes positive and so does $z_{\lambda}(t)$. This means $u_{\lambda} > \Psi$ for $\lambda > 1$ and $u_{\lambda} < \Psi$ for $\lambda < 1$.

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Theorem - If $\alpha < \alpha_s$ then \mathcal{G} is not convex.

Proof - We may assume that I = 0. For $u_0 \in \mathcal{G}$ let $u_{0,\lambda} = \lambda u_0 + (1 - \lambda)\Psi$ and let $z_\lambda = (u_\lambda - \Psi)/\lambda$. Then,

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 $\int z(0)\varphi_1 = \int u_0\varphi_1$. If $\int u_0\varphi_1 > 0$ then z becomes positive and so does z_λ for $\lambda \approx 0$. Thus $u_\lambda < \Psi \Longrightarrow u_{0,\lambda} \in \mathcal{B}$.

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The case of a ball

Consider $\Omega = B_1$. If $\alpha < \alpha_s = 4/(N-2)$ there are infinitely many stationnary (radial) solutions.



If N = 1, Ψ is symmetric, $I(\alpha) = 0$ for all α . $\Psi_s(r) = (1 + (N(N-2))^{-1}r^2)^{-(N-2)/2}$ is a stationary global solution for $\alpha = \alpha_s$.

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17 / 24

Proof -Consider $\alpha = \alpha_s$ and $\Psi_s(r)$. There exists φ_1 the first eigenvector of $-\Delta - (\alpha_s + 1)\Psi_s^{\alpha_s}$ in \mathbb{R}^N . Clearly $\int_{\mathbb{R}^N} \Psi_s \varphi_1 > 0$. A limiting argument shows that $I(\alpha) > 0$ for $\alpha \approx \alpha_s$.

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Proof - Call ρ_{α} the second zero of ϕ_{α} and consider $\Omega = B(\rho_0)$.



 $\Psi_{\alpha}(r) = \gamma^{2/\alpha} \Phi_{\alpha}(\gamma r)$, where $\gamma = \rho_{\alpha}/\rho_0$. If η_1 is the first eigenvalue associated to Φ_{α} then $\varphi_1(r) = \eta_1(\gamma r)$ and $I(\alpha) = \int_{B_0} \Psi \varphi_1 = \gamma^{2/\alpha - N} \int_{B_\alpha} \Phi \eta_1 = \gamma^{2/\alpha - N} J(\alpha)$. So it suffices to study the sign of J.

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This shows that J(lpha)> 0 for lpha small.

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Benasquelli, August 2009 21 / 24

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Nehari Functional Consider

$$E(u) = \frac{1}{2} \int |\nabla u|^2 - \frac{1}{\alpha + 2} \int |u|^{\alpha + 2},$$
$$N(u) = \int |\nabla u|^2 - \int |u|^{\alpha + 2},$$
$$e_* = \inf\{E(u), N(u) = 0\}.$$

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- $e_* > 0$.
- If u is a stationary solution, N(u) = 0.
- E(u(t)) is nonincreasing.
- $E^- \subset \mathcal{B}$.
- $W = \{E \le e_*\} \cap \{N > 0\} \subset \mathcal{G}_0.$
- $Z = \{E \leq e_*\} \cap \{N < 0\} \subset \mathcal{B}.$
- $u_0 \in Gz \Longrightarrow u(t) \in W$ for t large.
- $u_0 \in B \Longrightarrow u(t) \in Z$ for $t \approx T$.
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Question: $N^- \subset \mathcal{B}$?

The answer is no. Let Ψ be a stationary sign-changing solution of (DP) and let $u_0 = \Psi + \varepsilon \varphi_s + o(\varepsilon) \varphi_u \in M$. Since $N(\Psi) = 0$,

$$N(u_0) = \varepsilon N'(\Psi)\varphi_s + o(\varepsilon).$$

 $\mathcal{N}'(\Psi)\varphi_s = \mathbf{0} \Longrightarrow |\Psi|^{lpha}\Psi \in [\varphi_1, \varphi_2, \dots, \varphi_k] \subset \mathcal{C}^{[lpha]+2}.$

This is not possible if α is not an even integer.

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