Blow-up and global sign-changing solutions of the nonlinear heat equation

T. Cazenave[1] F. Dickstein[2] F. B. Weissler[3]

[1]Univ. Paris 6 & CNRS

[2]Univ. Fed. Rio de Janeiro

[3]Univ. Paris 13

BenasqueIII, August 2009

The nonlinear heat equation

$$\begin{cases} u_t - \Delta u = |u|^\alpha u & \text{in } (0,T)\times\Omega, \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \qquad \text{(NLH)}$$

Ω is a bounded domain or the whole $\mathbb{R}^N$,

$$\alpha > 0,$$

$$u_0 \in C_0(\Omega).$$

Some known results

$\mathcal{G} = \{u_0, u(t) \text{ is global}\}$,
$\mathcal{G}_0 = \{u_0 \in \mathcal{G}, u(t) \to 0, t \to +\infty\}$,
$\mathcal{B} = \{u_0, u(t) \text{ blows up}\}$.

- There exists $0 \neq u_0 \in \mathcal{G}_0$.
- Given $\varphi \neq 0$, $\lambda\varphi \in \mathcal{B}$ if λ is large.
- $\varphi \geq 0$, $\varphi \in \mathcal{G} \Longrightarrow \lambda\varphi \in G$ if $0 < \lambda < 1$.

  ($\mathcal{G}^+$ is starshaped with respect to 0.)

Some Questions:

- Is $\mathcal{G}$, $\mathcal{G}_0$ convex?
- Is $\mathcal{G}$, $\mathcal{G}_0$ starshaped with respect to 0?

Some known results

$\mathcal{G} = \{u_0, u(t) \text{ is global}\}$,
$\mathcal{G}_0 = \{u_0 \in \mathcal{G}, u(t) \longrightarrow 0, t \longrightarrow +\infty\}$,
$\mathcal{B} = \{u_0, u(t) \text{ blows up}\}$.

- There exists $0 \neq u_0 \in \mathcal{G}_0$.
- Given $\varphi \neq 0$, $\lambda\varphi \in \mathcal{B}$ if λ is large.
- $\varphi \geq 0$, $\varphi \in \mathcal{G} \Longrightarrow \lambda\varphi \in G$ if $0 < \lambda < 1$.

  ($\mathcal{G}^+$ is starshaped with respect to 0.)

Some Questions:

- Is $\mathcal{G}$, $\mathcal{G}_0$ convex?
- Is $\mathcal{G}$, $\mathcal{G}_0$ starshaped with respect to 0?

Some known results

$\mathcal{G} = \{u_0, u(t) \text{ is global}\}$,
$\mathcal{G}_0 = \{u_0 \in \mathcal{G}, u(t) \to 0, t \to +\infty\}$,
$\mathcal{B} = \{u_0, u(t) \text{ blows up}\}$.

- There exists $0 \neq u_0 \in \mathcal{G}_0$.
- Given $\varphi \neq 0$, $\lambda\varphi \in \mathcal{B}$ if λ is large.
- $\varphi \geq 0$, $\varphi \in \mathcal{G} \Longrightarrow \lambda\varphi \in G$ if $0 < \lambda < 1$.

  ($\mathcal{G}^+$ is starshaped with respect to 0.)

Some Questions:

- Is $\mathcal{G}$, $\mathcal{G}_0$ convex?
- Is $\mathcal{G}$, $\mathcal{G}_0$ starshaped with respect to 0?

Some known results

$\mathcal{G} = \{u_0, u(t) \text{ is global}\}$,
$\mathcal{G}_0 = \{u_0 \in \mathcal{G}, u(t) \to 0, t \to +\infty\}$,
$\mathcal{B} = \{u_0, u(t) \text{ blows up}\}$.

- There exists $0 \neq u_0 \in \mathcal{G}_0$.
- Given $\varphi \neq 0$, $\lambda\varphi \in \mathcal{B}$ if λ is large.
- $\varphi \geq 0$, $\varphi \in \mathcal{G} \Longrightarrow \lambda\varphi \in G$ if $0 < \lambda < 1$.

  ($\mathcal{G}^+$ is starshaped with respect to 0.)

Some Questions:

- Is $\mathcal{G}$, $\mathcal{G}_0$ convex?
- Is $\mathcal{G}$, $\mathcal{G}_0$ starshaped with respect to 0?

Some known results

$\mathcal{G} = \{u_0, u(t) \text{ is global}\}$,
$\mathcal{G}_0 = \{u_0 \in \mathcal{G}, u(t) \to 0, t \to +\infty\}$,
$\mathcal{B} = \{u_0, u(t) \text{ blows up}\}$.

- There exists $0 \neq u_0 \in \mathcal{G}_0$.
- Given $\varphi \neq 0$, $\lambda\varphi \in \mathcal{B}$ if λ is large.
- $\varphi \geq 0$, $\varphi \in \mathcal{G} \Longrightarrow \lambda\varphi \in G$ if $0 < \lambda < 1$.

  ($\mathcal{G}^+$ is starshaped with respect to 0.)

Some Questions:

- Is $\mathcal{G}$, $\mathcal{G}_0$ convex?
- Is $\mathcal{G}$, $\mathcal{G}_0$ starshaped with respect to 0?

Some known results

$\mathcal{G} = \{u_0, u(t) \text{ is global}\}$,
$\mathcal{G}_0 = \{u_0 \in \mathcal{G}, u(t) \to 0, t \to +\infty\}$,
$\mathcal{B} = \{u_0, u(t) \text{ blows up}\}$.

- There exists $0 \neq u_0 \in \mathcal{G}_0$.
- Given $\varphi \neq 0$, $\lambda\varphi \in \mathcal{B}$ if λ is large.
- $\varphi \geq 0$, $\varphi \in \mathcal{G} \Longrightarrow \lambda\varphi \in G$ if $0 < \lambda < 1$.

  ($\mathcal{G}^+$ is starshaped with respect to 0.)

Some Questions:

- Is $\mathcal{G}$, $\mathcal{G}_0$ convex?
- Is $\mathcal{G}$, $\mathcal{G}_0$ starshaped with respect to 0?

Some known results

$\mathcal{G} = \{u_0, u(t) \text{ is global}\}$,
$\mathcal{G}_0 = \{u_0 \in \mathcal{G}, u(t) \to 0, t \to +\infty\}$,
$\mathcal{B} = \{u_0, u(t) \text{ blows up}\}$.

- There exists $0 \neq u_0 \in \mathcal{G}_0$.
- Given $\varphi \neq 0$, $\lambda\varphi \in \mathcal{B}$ if λ is large.
- $\varphi \geq 0$, $\varphi \in \mathcal{G} \Longrightarrow \lambda\varphi \in G$ if $0 < \lambda < 1$.

  ($\mathcal{G}^+$ is starshaped with respect to 0.)

Some Questions:

- Is $\mathcal{G}$, $\mathcal{G}_0$ convex?
- Is $\mathcal{G}$, $\mathcal{G}_0$ starshaped with respect to 0?

The Neumann Problem

Consider

$$\begin{cases} u_t - \Delta u = |u|^\alpha u \quad \text{in } (0,T)\times\Omega, \\ u(0) = u_0 \quad \text{in } \Omega, \\ \partial_\eta u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{NP}$$

- $\{u_0 \geq 0, u_0 \neq 0\} \subset \mathcal{B}$.
- How about sign-changing solutions?

The Neumann Problem

Consider

$$\begin{cases} u_t - \Delta u = |u|^\alpha u & \text{in } (0,T)\times\Omega, \\ u(0) = u_0 & \text{in } \Omega, \\ \partial_\eta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{NP}$$

- $\{u_0 \geq 0, u_0 \neq 0\} \subset \mathcal{B}$.
- How about sign-changing solutions?

The Neumann Problem

Consider

$$\begin{cases} u_t - \Delta u = |u|^\alpha u & \text{in } (0,T)\times\Omega, \\ u(0) = u_0 & \text{in } \Omega, \\ \partial_\eta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{NP}$$

- $\{u_0 \geq 0, u_0 \neq 0\} \subset \mathcal{B}$.
- How about sign-changing solutions?

The Neumann Problem

Theorem - Suppose $\int \varphi \neq 0$. Then $\lambda\varphi \in \mathcal{B}$ if $\lambda > 0$ is small.

Proof - Assume $\int \varphi > 0$ and let $v(t,x) = u(t,x)/\lambda$. Then,
$v_t - \Delta v = \lambda^{\alpha}|v|^{\alpha}v$, $v(0) = \varphi$.
Consider, $z = e^{t\Delta}\varphi$. Since $\int \varphi > 0$, $z(t) > 0$ for t large.
Thus, $v(t) > 0$ if λ is small. □

Remark - Let u be a stationary solution in $\mathbb{R}$:
$-u'' = |u|^{\alpha}u$, $u'(-1) = u'(1) = 0$.
Then $\lambda u \in \mathcal{G}_0$ for $|\lambda| < 1$.

The Neumann Problem

Theorem - Suppose $\displaystyle\int \varphi \neq 0$. Then $\lambda\varphi \in \mathcal{B}$ if $\lambda > 0$ is small.

Proof - Assume $\int \varphi > 0$ and let $v(t,x) = u(t,x)/\lambda$. Then,
$v_t - \Delta v = \lambda^\alpha |v|^\alpha v$, $v(0) = \varphi$.
Consider, $z = e^{t\Delta}\varphi$. Since $\int \varphi > 0$, $z(t) > 0$ for t large.
Thus, $v(t) > 0$ if λ is small. □

Remark - Let u be a stationary solution in $\mathbb{R}$:
$-u'' = |u|^\alpha u$, $u'(-1) = u'(1) = 0$.
Then $\lambda u \in \mathcal{G}_0$ for $|\lambda| < 1$.

The Neumann Problem

Theorem - Suppose $\int \varphi \neq 0$. Then $\lambda\varphi \in \mathcal{B}$ if $\lambda > 0$ is small.

Proof - Assume $\int \varphi > 0$ and let $v(t,x) = u(t,x)/\lambda$. Then,
$v_t - \Delta v = \lambda^\alpha |v|^\alpha v$, $v(0) = \varphi$.
Consider, $z = e^{t\Delta}\varphi$. Since $\int \varphi > 0$, $z(t) > 0$ for t large.
Thus, $v(t) > 0$ if λ is small. □

Remark - Let u be a stationary solution in $\mathbb{R}$:
$-u'' = |u|^\alpha u$, $u'(-1) = u'(1) = 0$.
Then $\lambda u \in \mathcal{G}_0$ for $|\lambda| < 1$.

The Neumann Problem

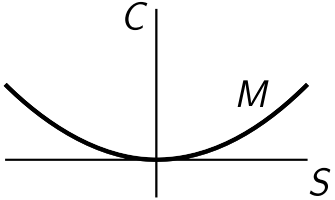
Theorem - There exists $u_0 \in \mathcal{G}_0$ such that $\displaystyle\int u_0 \neq 0$.

Proof - The linearization of (NP) around 0 is the heat equation. Let M be the nonlinear stable manifold near 0, $S = [1]^{\perp}$ the linear stable manifold, $C = [1]$.

The Neumann Problem

Theorem - There exists $u_0 \in \mathcal{G}_0$ such that $\int u_0 \neq 0$.

Proof - The linearization of (NP) around 0 is the heat equation. Let M be the nonlinear stable manifold near 0, $S = [1]^\perp$ the linear stable manifold, $C = [1]$.



The Neumann Problem

For $\varphi_s \in S$ we may choose $u_0 = \varepsilon\varphi_s + c \in M$. Suppose $c \equiv 0$. Define $I(t) = \int u(t)$. Then

$$I'(t) = \int |u(t)|^\alpha u(t) \Longrightarrow I'(0) = \int |\varphi_s|^\alpha \varphi_s.$$

We pick φ_s such that $I'(0) \neq 0$.
Then $I(t) \neq 0$ for $t \approx 0$. □

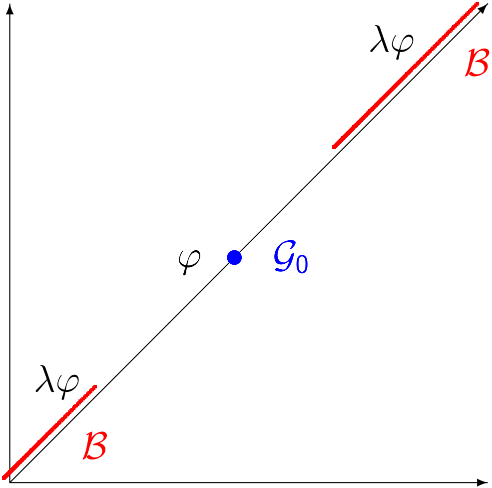
The Neumann Problem

For $\varphi_s \in S$ we may choose $u_0 = \varepsilon\varphi_s + c \in M$. Suppose $c \equiv 0$. Define $I(t) = \int u(t)$. Then

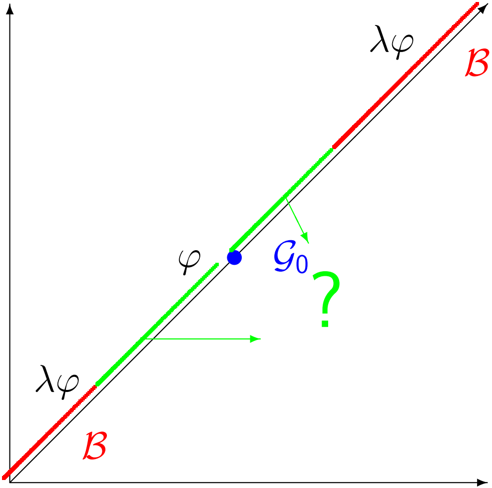
$$I'(t) = \int |u(t)|^{\alpha} u(t) \Longrightarrow I'(0) = \int |\varphi_s|^{\alpha}\varphi_s.$$

We pick φ_s such that $I'(0) \neq 0$.
Then $I(t) \neq 0$ for $t \approx 0$. $\square$

The Neumann case



The Neumann case



The Cauchy Problem

Consider

$$\begin{cases} u_t - \Delta u = |u|^\alpha u, & \text{in } (0,T) \times \mathbb{R}^N \\ u(0) = u_0, & \text{in } \mathbb{R}^N, \end{cases} \tag{CP}$$

(CP) for $\alpha < 2/N$ corresponds to the Neumann problem in the following sense.

The Cauchy Problem

Consider

$$\begin{cases} u_t - \Delta u = |u|^\alpha u, & \text{in } (0,T)\times \mathbb{R}^N \\ u(0) = u_0, & \text{in } \mathbb{R}^N, \end{cases} \tag{CP}$$

(CP) for $\alpha < 2/N$ corresponds to the Neumann problem in the following sense.

The Cauchy Problem

- $u_0 \geq 0$, $u_0 \neq 0$, then $u_0 \in \mathcal{B}$ (Fujita, '66).
- If $\varphi \in L^1$, $\int \varphi \neq 0$ then $\lambda\varphi \in \mathcal{B}$ if λ is small.
- There exists $\varphi \in L^1 \cap \mathcal{G}$, $\int \varphi \neq 0$.
- 0 is stable in L^∞.
- There exists $\varphi \in L^1 \cap \mathcal{G}_0$, $\int \varphi = 0$. $\lambda\varphi \in \mathcal{G}_0$ for all $|\lambda| < 1$ and $\lambda\varphi \in \mathcal{B}$ if λ is large.

The Cauchy Problem

- $u_0 \geq 0$, $u_0 \neq 0$, then $u_0 \in \mathcal{B}$ (Fujita, '66).
- If $\varphi \in L^1$, $\int \varphi \neq 0$ then $\lambda\varphi \in \mathcal{B}$ if λ is small.
- There exists $\varphi \in L^1 \cap \mathcal{G}$, $\int \varphi \neq 0$.
- 0 is stable in L^∞.
- There exists $\varphi \in L^1 \cap \mathcal{G}_0$, $\int \varphi = 0$. $\lambda\varphi \in \mathcal{G}_0$ for all $|\lambda| < 1$ and $\lambda\varphi \in \mathcal{B}$ if λ is large.

The Cauchy Problem

- $u_0 \geq 0$, $u_0 \neq 0$, then $u_0 \in \mathcal{B}$ (Fujita, '66).
- If $\varphi \in L^1$, $\int \varphi \neq 0$ then $\lambda\varphi \in \mathcal{B}$ if λ is small.
- There exists $\varphi \in L^1 \cap \mathcal{G}$, $\int \varphi \neq 0$.
- 0 is stable in L^∞.
- There exists $\varphi \in L^1 \cap \mathcal{G}_0$, $\int \varphi = 0$. $\lambda\varphi \in \mathcal{G}_0$ for all $|\lambda| < 1$ and $\lambda\varphi \in \mathcal{B}$ if λ is large.

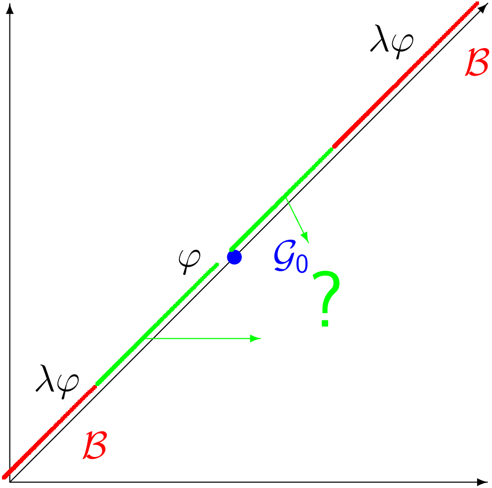
The Cauchy Problem

- $u_0 \geq 0$, $u_0 \neq 0$, then $u_0 \in \mathcal{B}$ (Fujita, '66).
- If $\varphi \in L^1$, $\int \varphi \neq 0$ then $\lambda\varphi \in \mathcal{B}$ if λ is small.
- There exists $\varphi \in L^1 \cap \mathcal{G}$, $\int \varphi \neq 0$.
- 0 is stable in L^∞.
- There exists $\varphi \in L^1 \cap \mathcal{G}_0$, $\int \varphi = 0$. $\lambda\varphi \in \mathcal{G}_0$ for all $|\lambda| < 1$ and $\lambda\varphi \in \mathcal{B}$ if λ is large.

The Cauchy Problem

- $u_0 \geq 0$, $u_0 \neq 0$, then $u_0 \in \mathcal{B}$ (Fujita, '66).
- If $\varphi \in L^1$, $\int \varphi \neq 0$ then $\lambda\varphi \in \mathcal{B}$ if λ is small.
- There exists $\varphi \in L^1 \cap \mathcal{G}$, $\int \varphi \neq 0$.
- 0 is stable in L^∞.
- There exists $\varphi \in L^1 \cap \mathcal{G}_0$, $\int \varphi = 0$. $\lambda\varphi \in \mathcal{G}_0$ for all $|\lambda| < 1$ and $\lambda\varphi \in \mathcal{B}$ if λ is large.

The Cauchy Problem



The Dirichlet Problem

$$\begin{cases} u_t - \Delta u = |u|^\alpha u, & \text{in } (0,T)\times\Omega \\ u(0) = u_0, & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega \end{cases} \tag{DP}$$

- If $\varphi \geq 0$, $\varphi \neq 0$ then there exists $\lambda^* > 0$ such that $\lambda\varphi \in \mathcal{G}_0$ if $\lambda < \lambda^*$, $\lambda\varphi \in \mathcal{B}$ if $\lambda > \lambda^*$.
- $\mathcal{G}^+$, $\mathcal{G}_0{}^+$ are convex.
- If $\alpha < \alpha_s = 4/(N-2)$ then (DP) admits stationary sign-changing solutions Ψ.

The Dirichlet Problem

$$\begin{cases} u_t - \Delta u = |u|^\alpha u, & \text{in } (0,T)\times\Omega \\ u(0) = u_0, & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega \end{cases} \tag{DP}$$

- If $\varphi \geq 0$, $\varphi \neq 0$ then there exists $\lambda^* > 0$ such that $\lambda\varphi \in \mathcal{G}_0$ if $\lambda < \lambda^*$, $\lambda\varphi \in \mathcal{B}$ if $\lambda > \lambda^*$.
- $\mathcal{G}^+$, $\mathcal{G}_0{}^+$ are convex.
- If $\alpha < \alpha_s = 4/(N-2)$ then (DP) admits stationary sign-changing solutions Ψ.

The Dirichlet Problem

$$\begin{cases} u_t - \Delta u = |u|^\alpha u, & \text{in } (0,T)\times\Omega \\ u(0) = u_0, & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega \end{cases} \qquad \text{(DP)}$$

- If $\varphi \geq 0$, $\varphi \neq 0$ then there exists $\lambda^* > 0$ such that $\lambda\varphi \in \mathcal{G}_0$ if $\lambda < \lambda^*$, $\lambda\varphi \in \mathcal{B}$ if $\lambda > \lambda^*$.
- $\mathcal{G}^+$, ${\mathcal{G}_0}^+$ are convex.
- If $\alpha < \alpha_s = 4/(N-2)$ then (DP) admits stationary sign-changing solutions Ψ.

The Dirichlet Problem

$$\begin{cases} u_t - \Delta u = |u|^\alpha u, & \text{in } (0,T)\times\Omega \\ u(0) = u_0, & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega \end{cases} \tag{DP}$$

- If $\varphi \geq 0$, $\varphi \neq 0$ then there exists $\lambda^* > 0$ such that $\lambda\varphi \in \mathcal{G}_0$ if $\lambda < \lambda^*$, $\lambda\varphi \in \mathcal{B}$ if $\lambda > \lambda^*$.
- $\mathcal{G}^+$, ${\mathcal{G}_0}^+$ are convex.
- If $\alpha < \alpha_s = 4/(N-2)$ then (DP) admits stationary sign-changing solutions Ψ.

The Dirichlet Problem

Proposition - Take Ψ a stationary sign-changing solution. Let φ_1 be a first eigenvector of $-\Delta - (\alpha + 1)|\Psi|^\alpha$. If $I = \int \Psi\varphi_1 \neq 0$ then $\lambda\Psi \in \mathcal{B}$ if $\lambda \approx 1$, $\lambda \neq 1$. In particular, $\mathcal{G}$ is not star-shaped around 0.

Proof - Assume $I > 0$, call u_λ the solution starting at $\lambda\Psi$ and set $z_\lambda = \dfrac{u_\lambda - \Psi}{\lambda - 1}$. Then

$$\begin{cases} \partial_t z_\lambda - \Delta z_\lambda = \dfrac{|u|^\alpha u - |\Psi|^\alpha \Psi}{\lambda - 1} \approx (\alpha + 1)|\Psi|^\alpha z_\lambda, \\ z_\lambda(0) = \Psi \end{cases}$$

The Dirichlet Problem

Proposition - Take Ψ a stationary sign-changing solution. Let φ_1 be a first eigenvector of $-\Delta - (\alpha+1)|\Psi|^\alpha$. If $I = \int \Psi\varphi_1 \neq 0$ then $\lambda\Psi \in \mathcal{B}$ if $\lambda \approx 1$, $\lambda \neq 1$. In particular, $\mathcal{G}$ is not star-shaped around 0.

Proof - Assume $I > 0$, call u_λ the solution starting at $\lambda\Psi$ and set $z_\lambda = \dfrac{u_\lambda - \Psi}{\lambda - 1}$. Then

$$\begin{cases} \partial_t z_\lambda - \Delta z_\lambda = \dfrac{|u|^\alpha u - |\Psi|^\alpha \Psi}{\lambda - 1} \approx (\alpha+1)|\Psi|^\alpha z_\lambda, \\ z_\lambda(0) = \Psi \end{cases}$$

$z_\lambda(t) \to z(t)$ in C^1, where

$$\begin{cases} \partial_t z - \Delta z - (\alpha+1)|\Psi|^\alpha z = 0, \\ z(0) = \Psi \end{cases}$$

Since $\int \Psi \varphi_1 > 0$, $z(t)$ becomes positive and so does $z_\lambda(t)$. This means $u_\lambda > \Psi$ for $\lambda > 1$ and $u_\lambda < \Psi$ for $\lambda < 1$.

But we know that u blows up if $u_0 > \Psi$ or $u_0 < \Psi$.

$z_\lambda(t) \to z(t)$ in C^1, where

$$\begin{cases} \partial_t z - \Delta z - (\alpha+1)|\Psi|^\alpha z = 0, \\ z(0) = \Psi \end{cases}$$

Since $\int \Psi \varphi_1 > 0$, $z(t)$ becomes positive and so does $z_\lambda(t)$. This means $u_\lambda > \Psi$ for $\lambda > 1$ and $u_\lambda < \Psi$ for $\lambda < 1$.

But we know that u blows up if $u_0 > \Psi$ or $u_0 < \Psi$.

Theorem - If $\alpha < \alpha_s$ then $\mathcal{G}$ is not convex.

Proof - We may assume that $I = 0$. For $u_0 \in \mathcal{G}$ let $u_{0,\lambda} = \lambda u_0 + (1-\lambda)\Psi$ and let $z_\lambda = (u_\lambda - \Psi)/\lambda$. Then,

$$\begin{cases} \partial_t z_\lambda - \Delta z_\lambda = \dfrac{|u_\lambda|^\alpha u_\lambda - |\Psi|^\alpha \Psi}{\lambda} \approx (\alpha+1)|\Psi|^\alpha z_\lambda, \\ z_\lambda(0) = u_0 - \Psi \end{cases}$$

Theorem - If $\alpha < \alpha_s$ then $\mathcal{G}$ is not convex.

Proof - We may assume that $I = 0$. For $u_0 \in \mathcal{G}$ let $u_{0,\lambda} = \lambda u_0 + (1-\lambda)\Psi$ and let $z_\lambda = (u_\lambda - \Psi)/\lambda$. Then,

$$\begin{cases} \partial_t z_\lambda - \Delta z_\lambda = \dfrac{|u_\lambda|^\alpha u_\lambda - |\Psi|^\alpha \Psi}{\lambda} \approx (\alpha+1)|\Psi|^\alpha z_\lambda, \\ z_\lambda(0) = u_0 - \Psi \end{cases}$$

$z_\lambda(t) \to z(t)$ in C^1, where

$$\begin{cases} \partial_t z - \Delta z - (\alpha+1)|\Psi|^\alpha z = 0, \\ z(0) = u_0 - \Psi \end{cases}$$

$\int z(0)\varphi_1 = \int u_0\varphi_1$. If $\int u_0\varphi_1 > 0$ then z becomes positive and so does z_λ for $\lambda \approx 0$. Thus $u_\lambda < \Psi \implies u_{0,\lambda} \in \mathcal{B}$.

To show the existence of $u_0 \in \mathcal{G}$ such that $\int u_0\varphi_1 > 0$ we use a dynamical system argument.

$z_\lambda(t) \to z(t)$ in C^1, where

$$\begin{cases} \partial_t z - \Delta z - (\alpha+1)|\Psi|^\alpha z = 0, \\ z(0) = u_0 - \Psi \end{cases}$$

$\int z(0)\varphi_1 = \int u_0\varphi_1$. If $\int u_0\varphi_1 > 0$ then z becomes positive and so does z_λ for $\lambda \approx 0$. Thus $u_\lambda < \Psi \implies u_{0,\lambda} \in \mathcal{B}$.

To show the existence of $u_0 \in \mathcal{G}$ such that $\int u_0\varphi_1 > 0$ we use a dynamical system argument.

$z_\lambda(t) \to z(t)$ in C^1, where

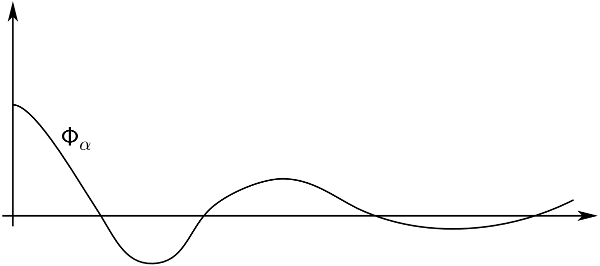
$$\begin{cases} \partial_t z - \Delta z - (\alpha+1)|\Psi|^\alpha z = 0, \\ z(0) = u_0 - \Psi \end{cases}$$

$\int z(0)\varphi_1 = \int u_0\varphi_1$. If $\int u_0\varphi_1 > 0$ then z becomes positive and so does z_λ for $\lambda \approx 0$. Thus $u_\lambda < \Psi \implies u_{0,\lambda} \in \mathcal{B}$.

To show the existence of $u_0 \in \mathcal{G}$ such that $\int u_0\varphi_1 > 0$ we use a dynamical system argument.

The case of a ball

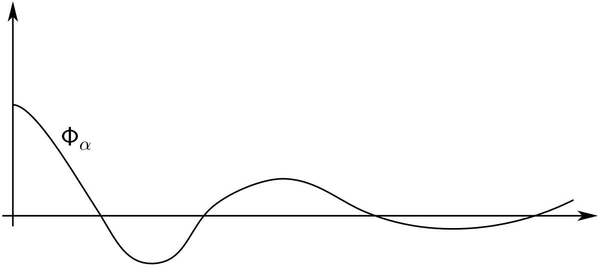
Consider $\Omega = B_1$. If $\alpha < \alpha_s = 4/(N-2)$ there are infinitely many stationnary (radial) solutions.



If $N = 1$, Ψ is symmetric, $I(\alpha) = 0$ for all α.
$\Psi_s(r) = (1 + (N(N-2))^{-1} r^2)^{-(N-2)/2}$ is a stationary global solution for $\alpha = \alpha_s$.

The case of a ball

Consider $\Omega = B_1$. If $\alpha < \alpha_s = 4/(N-2)$ there are infinitely many stationnary (radial) solutions.

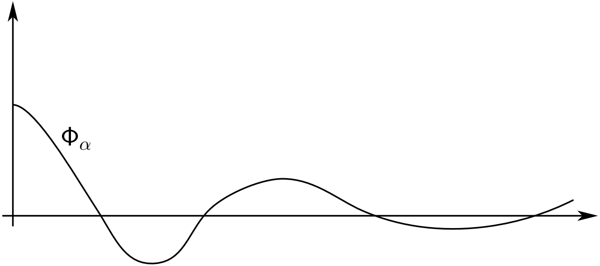


If $N = 1$, Ψ is symmetric, $I(\alpha) = 0$ for all α.

$\Psi_s(r) = (1 + (N(N-2))^{-1} r^2)^{-(N-2)/2}$ is a stationary global solution for $\alpha = \alpha_s$.

The case of a ball

Consider $\Omega = B_1$. If $\alpha < \alpha_s = 4/(N-2)$ there are infinitely many stationnary (radial) solutions.



If $N = 1$, Ψ is symmetric, $I(\alpha) = 0$ for all α.
$\Psi_s(r) = (1 + (N(N-2))^{-1} r^2)^{-(N-2)/2}$ is a stationary global solution for $\alpha = \alpha_s$.

Theorem 1 - Let $N \geq 3$, $\alpha < \alpha_s = 4/(N-2)$. Let Ψ be a stationary sign-changing solution of (DP). Then there exists $\alpha^* < \alpha_s$ such that $I(\alpha) > 0$ for $\alpha > \alpha^*$.

Proof -
Consider $\alpha = \alpha_s$ and $\Psi_s(r)$.
There exists φ_1 the first eigenvector of $-\Delta - (\alpha_s + 1)\Psi_s^{\alpha_s}$ in $\mathbb{R}^N$. Clearly $\int_{\mathbb{R}^N} \Psi_s \varphi_1 > 0$.
A limiting argument shows that $I(\alpha) > 0$ for $\alpha \approx \alpha_s$. □

Theorem 1 - Let $N \geq 3$, $\alpha < \alpha_s = 4/(N-2)$. Let Ψ be a stationary sign-changing solution of (DP). Then there exists $\alpha^* < \alpha_s$ such that $I(\alpha) > 0$ for $\alpha > \alpha^*$.

Proof -
Consider $\alpha = \alpha_s$ and $\Psi_s(r)$.
There exists φ_1 the first eigenvector of $-\Delta - (\alpha_s + 1)\Psi_s^{\alpha_s}$ in $\mathbb{R}^N$. Clearly $\int_{\mathbb{R}^N} \Psi_s \varphi_1 > 0$.
A limiting argument shows that $I(\alpha) > 0$ for $\alpha \approx \alpha_s$. □

Theorem 1 - Let $N \geq 3$, $\alpha < \alpha_s = 4/(N-2)$. Let Ψ be a stationary sign-changing solution of (DP). Then there exists $\alpha^* < \alpha_s$ such that $I(\alpha) > 0$ for $\alpha > \alpha^*$.

Proof -
Consider $\alpha = \alpha_s$ and $\Psi_s(r)$.
There exists φ_1 the first eigenvector of $-\Delta - (\alpha_s + 1)\Psi_s^{\alpha_s}$ in $\mathbb{R}^N$. Clearly $\int_{\mathbb{R}^N} \Psi_s \varphi_1 > 0$.
A limiting argument shows that $I(\alpha) > 0$ for $\alpha \approx \alpha_s$. □

Theorem 1 - Let $N \geq 3$, $\alpha < \alpha_s = 4/(N-2)$. Let Ψ be a stationary sign-changing solution of (DP). Then there exists $\alpha^* < \alpha_s$ such that $I(\alpha) > 0$ for $\alpha > \alpha^*$.

Proof -
Consider $\alpha = \alpha_s$ and $\Psi_s(r)$.
There exists φ_1 the first eigenvector of $-\Delta - (\alpha_s + 1)\Psi_s^{\alpha_s}$ in $\mathbb{R}^N$. Clearly $\int_{\mathbb{R}^N} \Psi_s \varphi_1 > 0$.
A limiting argument shows that $I(\alpha) > 0$ for $\alpha \approx \alpha_s$. □

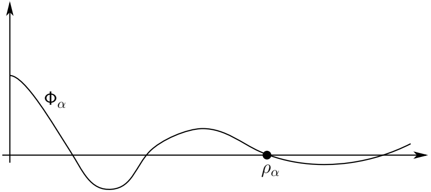
Theorem 2 - If $N = 3$, then $I(\alpha) > 0$ for α small.

Proof - Call ρ_α the second zero of ϕ_α and consider $\Omega = B(\rho_0)$.

$\Psi_\alpha(r) = \gamma^{2/\alpha}\Phi_\alpha(\gamma r)$, where $\gamma = \rho_\alpha/\rho_0$.
If η_1 is the first eigenvalue associated to Φ_α
then $\varphi_1(r) = \eta_1(\gamma r)$ and
$I(\alpha) = \int_{B_0} \Psi\varphi_1 = \gamma^{2/\alpha - N} \int_{B_\alpha} \Phi\eta_1 = \gamma^{2/\alpha - N} J(\alpha)$.
So it suffices to study the sign of J.

Theorem 2 - If $N = 3$, then $I(\alpha) > 0$ for α small.

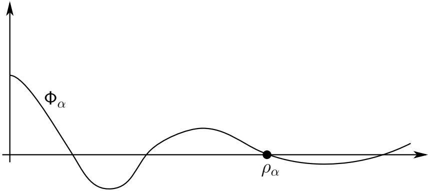
Proof - Call ρ_α the second zero of ϕ_α and consider $\Omega = B(\rho_0)$.



$\Psi_\alpha(r) = \gamma^{2/\alpha}\Phi_\alpha(\gamma r)$, where $\gamma = \rho_\alpha/\rho_0$.
If η_1 is the first eigenvalue associated to Φ_α
then $\varphi_1(r) = \eta_1(\gamma r)$ and
$I(\alpha) = \int_{B_0} \Psi\varphi_1 = \gamma^{2/\alpha - N} \int_{B_\alpha} \Phi\eta_1 = \gamma^{2/\alpha-N} J(\alpha)$.
So it suffices to study the sign of J.

Theorem 2 - If $N = 3$, then $I(\alpha) > 0$ for α small.

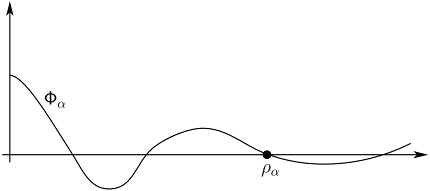
Proof - Call ρ_α the second zero of ϕ_α and consider $\Omega = B(\rho_0)$.



$\Psi_\alpha(r) = \gamma^{2/\alpha}\Phi_\alpha(\gamma r)$, where $\gamma = \rho_\alpha/\rho_0$.
If η_1 is the first eigenvalue associated to Φ_α
then $\varphi_1(r) = \eta_1(\gamma r)$ and
$I(\alpha) = \int_{B_0} \Psi\varphi_1 = \gamma^{2/\alpha - N} \int_{B_\alpha} \Phi\eta_1 = \gamma^{2/\alpha - N} J(\alpha)$.
So it suffices to study the sign of J.

Theorem 2 - If $N = 3$, then $I(\alpha) > 0$ for α small.

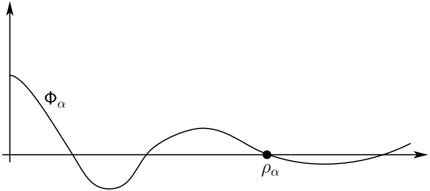
Proof - Call ρ_α the second zero of ϕ_α and consider $\Omega = B(\rho_0)$.



$\Psi_\alpha(r) = \gamma^{2/\alpha}\Phi_\alpha(\gamma r)$, where $\gamma = \rho_\alpha/\rho_0$.
If η_1 is the first eigenvalue associated to Φ_α
then $\varphi_1(r) = \eta_1(\gamma r)$ and
$I(\alpha) = \int_{B_0} \Psi\varphi_1 = \gamma^{2/\alpha - N} \int_{B_\alpha} \Phi\eta_1 = \gamma^{2/\alpha - N} J(\alpha)$.
So it suffices to study the sign of J.

Theorem 2 - If $N = 3$, then $I(\alpha) > 0$ for α small.

Proof - Call ρ_α the second zero of ϕ_α and consider $\Omega = B(\rho_0)$.



$\Psi_\alpha(r) = \gamma^{2/\alpha}\Phi_\alpha(\gamma r)$, where $\gamma = \rho_\alpha/\rho_0$.
If η_1 is the first eigenvalue associated to Φ_α
then $\varphi_1(r) = \eta_1(\gamma r)$ and
$I(\alpha) = \int_{B_0} \Psi\varphi_1 = \gamma^{2/\alpha - N} \int_{B_\alpha} \Phi\eta_1 = \gamma^{2/\alpha - N} J(\alpha)$.
So it suffices to study the sign of J.

$$-\Delta\Phi = |\Phi|^\alpha \Phi$$
$$-\Delta\eta_1 = (\alpha+1)|\Phi|^\alpha \eta_1 + \lambda_1\eta_1.$$

Note that $\lambda_1(\alpha) < 0$ even for $\alpha = 0$.

$$\lambda_1 J(\alpha) = \alpha \int_{B_\alpha} |\Phi|^\alpha \Phi\eta_1 \Rightarrow J'(\alpha) = J(0)/\lambda_1(0) = 0.$$

But

$$J(\alpha)/\alpha^2 \to (\lambda_1(0))^{-1} \int_{B_0} \log|\Phi_0|\Phi_0\eta_{1,0}.$$

$$-\Delta\Phi = |\Phi|^\alpha \Phi$$
$$-\Delta\eta_1 = (\alpha+1)|\Phi|^\alpha \eta_1 + \lambda_1 \eta_1.$$

Note that $\lambda_1(\alpha) < 0$ even for $\alpha = 0$.

$$\lambda_1 J(\alpha) = \alpha \int_{B_\alpha} |\Phi|^\alpha \Phi \eta_1 \Rightarrow J'(\alpha) = J(0)/\lambda_1(0) = 0.$$

But

$$J(\alpha)/\alpha^2 \to (\lambda_1(0))^{-1} \int_{B_0} \log|\Phi_0| \Phi_0 \eta_{1,0}.$$

$$-\Delta\Phi = |\Phi|^{\alpha}\Phi$$
$$-\Delta\eta_1 = (\alpha+1)|\Phi|^{\alpha}\eta_1 + \lambda_1\eta_1.$$

Note that $\lambda_1(\alpha) < 0$ even for $\alpha = 0$.

$$\lambda_1 J(\alpha) = \alpha \int_{B_\alpha} |\Phi|^{\alpha}\Phi\eta_1 \Rightarrow J'(\alpha) = J(0)/\lambda_1(0) = 0.$$

But

$$J(\alpha)/\alpha^2 \to (\lambda_1(0))^{-1} \int_{B_0} \log|\Phi_0|\Phi_0\eta_{1,0}.$$

$$-\Delta\Phi = |\Phi|^\alpha \Phi$$
$$-\Delta\eta_1 = (\alpha+1)|\Phi|^\alpha \eta_1 + \lambda_1 \eta_1.$$

Note that $\lambda_1(\alpha) < 0$ even for $\alpha = 0$.

$$\lambda_1 J(\alpha) = \alpha \int_{B_\alpha} |\Phi|^\alpha \Phi \eta_1 \Rightarrow J'(\alpha) = J(0)/\lambda_1(0) = 0.$$

But

$$J(\alpha)/\alpha^2 \to (\lambda_1(0))^{-1} \int_{B_0} \log|\Phi_0| \Phi_0 \eta_{1,0}.$$

When $N = 3$, $\eta_j = \sin jr/r$. For $j > 1$,

$$\int_0^{\pi} \log|\sin jr|/r \, \sin jr \sin r = -\int_0^{\pi} \log r \, \sin jr \sin r < 0.$$

This shows that $J(\alpha) > 0$ for α small.

When $N = 3$, $\eta_j = \sin jr/r$. For $j > 1$,

$$\int_0^\pi \log|\sin jr|/r \; \sin jr \sin r = -\int_0^\pi \log r \; \sin jr \sin r < 0.$$

This shows that $J(\alpha) > 0$ for α small.

Nehari Functional

Consider

$$E(u) = \frac{1}{2}\int |\nabla u|^2 - \frac{1}{\alpha+2}\int |u|^{\alpha+2},$$

$$N(u) = \int |\nabla u|^2 - \int |u|^{\alpha+2},$$

$$e_* = \inf\{E(u), N(u) = 0\}.$$

Known facts:

- $e_* > 0$.
- If u is a stationary solution, $N(u) = 0$.
- $E(u(t))$ is nonincreasing.
- $E^- \subset \mathcal{B}$.
- $W = \{E \leq e_*\} \cap \{N > 0\} \subset \mathcal{G}_0$.
- $Z = \{E \leq e_*\} \cap \{N < 0\} \subset \mathcal{B}$.
- $u_0 \in Gz \Longrightarrow u(t) \in W$ for t large.
- $u_0 \in B \Longrightarrow u(t) \in Z$ for $t \approx T$.
- $d/dt\|u\|^2 = -2N(u(t))$.

Known facts:

- $e_* > 0$.
- If u is a stationary solution, $N(u) = 0$.
- $E(u(t))$ is nonincreasing.
- $E^- \subset \mathcal{B}$.
- $W = \{E \leq e_*\} \cap \{N > 0\} \subset \mathcal{G}_0$.
- $Z = \{E \leq e_*\} \cap \{N < 0\} \subset \mathcal{B}$.
- $u_0 \in Gz \implies u(t) \in W$ for t large.
- $u_0 \in B \implies u(t) \in Z$ for $t \approx T$.
- $d/dt\|u\|^2 = -2N(u(t))$.

Known facts:

- $e_* > 0$.
- If u is a stationary solution, $N(u) = 0$.
- $E(u(t))$ is nonincreasing.
- $E^- \subset \mathcal{B}$.
- $W = \{E \leq e_*\} \cap \{N > 0\} \subset \mathcal{G}_0$.
- $Z = \{E \leq e_*\} \cap \{N < 0\} \subset \mathcal{B}$.
- $u_0 \in Gz \implies u(t) \in W$ for t large.
- $u_0 \in B \implies u(t) \in Z$ for $t \approx T$.
- $d/dt\|u\|^2 = -2N(u(t))$.

Known facts:

- $e_* > 0$.
- If u is a stationary solution, $N(u) = 0$.
- $E(u(t))$ is nonincreasing.
- $E^- \subset \mathcal{B}$.
- $W = \{E \leq e_*\} \cap \{N > 0\} \subset \mathcal{G}_0$.
- $Z = \{E \leq e_*\} \cap \{N < 0\} \subset \mathcal{B}$.
- $u_0 \in Gz \Longrightarrow u(t) \in W$ for t large.
- $u_0 \in B \Longrightarrow u(t) \in Z$ for $t \approx T$.
- $d/dt\|u\|^2 = -2N(u(t))$.

Known facts:

- $e_* > 0$.
- If u is a stationary solution, $N(u) = 0$.
- $E(u(t))$ is nonincreasing.
- $E^- \subset \mathcal{B}$.
- $W = \{E \leq e_*\} \cap \{N > 0\} \subset \mathcal{G}_0$.
- $Z = \{E \leq e_*\} \cap \{N < 0\} \subset \mathcal{B}$.
- $u_0 \in Gz \implies u(t) \in W$ for t large.
- $u_0 \in B \implies u(t) \in Z$ for $t \approx T$.
- $d/dt\|u\|^2 = -2N(u(t))$.

Known facts:

- $e_* > 0$.
- If u is a stationary solution, $N(u) = 0$.
- $E(u(t))$ is nonincreasing.
- $E^- \subset \mathcal{B}$.
- $W = \{E \leq e_*\} \cap \{N > 0\} \subset \mathcal{G}_0$.
- $Z = \{E \leq e_*\} \cap \{N < 0\} \subset \mathcal{B}$.
- $u_0 \in Gz \Longrightarrow u(t) \in W$ for t large.
- $u_0 \in B \Longrightarrow u(t) \in Z$ for $t \approx T$.
- $d/dt\|u\|^2 = -2N(u(t))$.

Known facts:

- $e_* > 0$.
- If u is a stationary solution, $N(u) = 0$.
- $E(u(t))$ is nonincreasing.
- $E^- \subset \mathcal{B}$.
- $W = \{E \leq e_*\} \cap \{N > 0\} \subset \mathcal{G}_0$.
- $Z = \{E \leq e_*\} \cap \{N < 0\} \subset \mathcal{B}$.
- $u_0 \in Gz \Longrightarrow u(t) \in W$ for t large.
- $u_0 \in B \Longrightarrow u(t) \in Z$ for $t \approx T$.
- $d/dt\|u\|^2 = -2N(u(t))$.

Known facts:

- $e_* > 0$.
- If u is a stationary solution, $N(u) = 0$.
- $E(u(t))$ is nonincreasing.
- $E^- \subset \mathcal{B}$.
- $W = \{E \leq e_*\} \cap \{N > 0\} \subset \mathcal{G}_0$.
- $Z = \{E \leq e_*\} \cap \{N < 0\} \subset \mathcal{B}$.
- $u_0 \in Gz \Longrightarrow u(t) \in W$ for t large.
- $u_0 \in B \Longrightarrow u(t) \in Z$ for $t \approx T$.
- $d/dt\|u\|^2 = -2N(u(t))$.

Known facts:

- $e_* > 0$.
- If u is a stationary solution, $N(u) = 0$.
- $E(u(t))$ is nonincreasing.
- $E^- \subset \mathcal{B}$.
- $W = \{E \le e_*\} \cap \{N > 0\} \subset \mathcal{G}_0$.
- $Z = \{E \le e_*\} \cap \{N < 0\} \subset \mathcal{B}$.
- $u_0 \in Gz \Longrightarrow u(t) \in W$ for t large.
- $u_0 \in B \Longrightarrow u(t) \in Z$ for $t \approx T$.
- $d/dt\|u\|^2 = -2N(u(t))$.

Known facts:

- $e_* > 0$.
- If u is a stationary solution, $N(u) = 0$.
- $E(u(t))$ is nonincreasing.
- $E^- \subset \mathcal{B}$.
- $W = \{E \leq e_*\} \cap \{N > 0\} \subset \mathcal{G}_0$.
- $Z = \{E \leq e_*\} \cap \{N < 0\} \subset \mathcal{B}$.
- $u_0 \in Gz \Longrightarrow u(t) \in W$ for t large.
- $u_0 \in B \Longrightarrow u(t) \in Z$ for $t \approx T$.
- $d/dt\|u\|^2 = -2N(u(t))$.

Question: $N^- \subset \mathcal{B}$?

The answer is no. Let Ψ be a stationary sign-changing solution of (DP) and let $u_0 = \Psi + \varepsilon\varphi_s + o(\varepsilon)\varphi_u \in M$. Since $N(\Psi) = 0$,

$$N(u_0) = \varepsilon N'(\Psi)\varphi_s + o(\varepsilon).$$

$$N'(\Psi)\varphi_s = 0 \Longrightarrow |\Psi|^\alpha\Psi \in [\varphi_1, \varphi_2, \ldots, \varphi_k] \subset C^{[\alpha]+2}.$$

This is not possible if α is not an even integer.

Question: $N^{-} \subset \mathcal{B}$?

The answer is no. Let Ψ be a stationary sign-changing solution of (DP) and let
$u_0 = \Psi + \varepsilon\varphi_s + o(\varepsilon)\varphi_u \in M$. Since $N(\Psi) = 0$,

$$N(u_0) = \varepsilon N'(\Psi)\varphi_s + o(\varepsilon).$$

$$N'(\Psi)\varphi_s = 0 \Longrightarrow |\Psi|^{\alpha}\Psi \in [\varphi_1, \varphi_2, \ldots, \varphi_k] \subset C^{[\alpha]+2}.$$

This is not possible if α is not an even integer.