Non-diffusive large time behaviour for a Hamilton-Jacobi equation with degenerate diffusion

Philippe Laurençot

CNRS, Institut de Mathématiques de Toulouse

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Joint works with:

Jean-Philippe Bartier (Paris Dauphine)

Juan Luis Vázquez (UAM, Madrid)

Outline

1. A semilinear diffusive Hamilton-Jacobi equation
2. A degenerate parabolic equation with gradient absorption
   - Speed of propagation
3. Large time behaviour: $q \in (1, p-1)$
   - Convergence to self-similarity
   - Waiting time

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Motivation

Consider a parabolic PDE in $\mathbb{R}^N$ featuring a diffusion term and a reaction term.
During the time evolution, there is a competition between diffusion and reaction: which effect governs the large time dynamics and how?

Question thoroughly investigated for the semilinear heat equation

$$\partial_t u - \Delta u + u^q = 0 \ , \quad (t,x) \in (0,\infty) \times \mathbb{R}^N$$

and its quasilinear counterparts with Δu^m or $\Delta_p u$ instead of Δu.

A semilinear diffusive Hamilton-Jacobi equation

Less was known for

$$\begin{aligned} \partial_t u - \Delta u + |\nabla u|^q &= 0 \ , \quad (t,x) \in (0,\infty) \times \mathbb{R}^N \ , \\ u(0) &= u_0 \ , \quad x \in \mathbb{R}^N \ , \end{aligned}$$

where

- $q > 0$,
- u_0 is non-negative, bounded, continuous, and compactly supported ($u_0 \not\equiv 0$).

Diffusion-dominated large time behaviour

Assume that

$$q > q_\star := \frac{N+2}{N+1} \,.$$

Then

$$I_1(\infty) := \lim_{t\to\infty} \|u(t)\|_1 \in (0, \|u_0\|_1) \,,$$

and, for every $p \in [1, \infty]$,

$$\lim_{t\to\infty} t^{N(p-1)/2p} \|u(t) - I_1(\infty)\, g(t)\|_p = 0 \,,$$

with

$$g(t,x) := \frac{1}{t^{N/2}}\, G\left(\frac{x}{t^{1/2}}\right) \quad \text{and} \quad G(x) := \frac{1}{(4\pi)^{N/2}} \exp\left(-\frac{|x|^2}{4}\right) .$$

Balance in the large time behaviour

Assume that

$$q_1 := 1 < q < q_\star := \frac{N+2}{N+1}\,.$$

Then, for every $p \in [1,\infty]$,

$$\lim_{t\to\infty} t^{[(N+1)(q_\star - q)/(2(q-1))]+[N(p-1)/2p]}\,\|u(t) - W(t)\|_p = 0\,.$$

The limit W is the *very singular solution*:

$$\lim_{t\to 0}\int_{B(0,r)} W(t,x)\,dx = \infty \quad \text{and} \quad \lim_{t\to 0}\int_{\{|x|\geq r\}} W(t,x)\,dx = 0$$

and is self-similar: $W(t,x) = t^{-(2-q)/(2(q-1))}\, W\left(1, xt^{-1/2}\right)$.

Absorption-dominated large time behaviour

Assume that

$$0 < q < q_1 := 1\,.$$

Then there is $T_\star > 0$ such that

$$u(t,x) = 0 \quad \text{for} \quad (t,x) \in [T_\star, \infty) \times \mathbb{R}^N\,.$$

In addition,

$$\|u(t)\|_\infty \geq \kappa \ (T_\star - t)^{1/(1-q)}\ , \quad t \in [0, T_\star]\ ,$$

Large time behaviour: critical exponents

- $q = q_\star$: Diffusion-dominated behaviour but the dynamics already feels the influence of the absorption term $\longrightarrow$ extra scaling logarithmic factors. Gallay & L. (2007)
- $q = q_1$: Same homogeneity as a linear equation: not completely clear. Benachour, Roynette & Vallois (1996,1997)

Large time behaviour: summary

Competition between diffusion and absorption: two critical exponents

$$q_1 := 1 \quad \text{and} \quad q_\star := \frac{N+2}{N+1}$$

- $q_\star < q$: Diffusion-dominated behaviour. Benachour, Karch & L. (2004)
- $q_1 < q < q_\star$: Balance between the diffusion and the absorption $\longrightarrow$ Very Singular Solutions. Benachour & L. (2001), Qi & Wang (2001), Benachour, Koch & L. (2004), Benachour, Karch & L. (2004), Fang & Kwak (2007)
- $0 < q < q_1$: Absorption-dominated behaviour $\longrightarrow$ Extinction in finite time. Benachour, L., Schmitt & Souplet (2002), Gilding (2005)

Absorption-dominated large time behaviour: $q \in (0, 1)$

The exponent $q_1 = 1$ is somehow “doubly” critical:

- the nonlinearity governs the dynamics,
- the nonlinearity is no longer Lipschitz continuous.

The latter property gives rise to singular phenomena such as extinction in finite time.

Aim: elucidate the true role of the absorption term when it governs the dynamics without singular phenomena.

Outline

A degenerate parabolic equation with gradient absorption

$$\begin{aligned} \partial_t u - \Delta_p u + |\nabla u|^q &= 0 \,, \quad (t,x) \in (0,\infty) \times \mathbb{R}^N \,, && (1) \\ u(0) &= u_0 \,, \quad x \in \mathbb{R}^N \,, && (2) \end{aligned}$$

where

- $\Delta_p u := \text{ div } \left(|\nabla u|^{p-2} \, \nabla u \right)$ with $p > 2$,
- $q > 1$,
- u_0 is non-negative, bounded, continuous, and compactly supported ($u_0 \not\equiv 0$).

$p > 2$: finite speed of propagation

Large time behaviour

Competition between diffusion and absorption: two critical exponents

$$q_1 := p - 1 > 1 \quad \text{and} \quad q_\star := p - \frac{N}{N+1}$$

The following behaviour is expected:

- $q_\star < q$: Diffusion-dominated behaviour
- $q_1 < q < q_\star$: Balance between diffusion and absorption
- $1 < q < q_1$: Absorption-dominated behaviour

Propagation of the support

For $t \geq 0$ we put

$$\varrho(t) := \inf\left\{R > 0 \quad \text{such that} \quad u(t,x) = 0 \quad \text{for} \quad |x| > R\right\}.$$

Since u is a subsolution to the p-Laplacian equation, then

$$0 \leq \varrho(t) \leq C\,(1+t)^{1/(N(p-2)+p)} < \infty \quad \text{for all} \quad t \geq 0\,.$$

Question : does the absorption term slow down the expansion of the support?

Propagation of the support: slow expansion

- If $q \in (p-1, q_\star)$ then

$$\varrho(t) \leq C\ (1+t)^{(q-p+1)/(2q-p)} \quad \text{for} \quad t \geq 0\,.$$

- If $q = p-1$ then

$$\varrho(t) \leq C\ \ln(2+t) \quad \text{for} \quad t \geq 0\,.$$

Andreucci, Tedeev & Ughi (2004), Bartier & L. (2008)

Propagation of the support: localization

- If $q \in (1, p-1)$ then

$$\limsup_{t\to\infty} \varrho(t) < \infty\,.$$

Bartier & L. (2008)

Remark: if h solves $\partial_t h + |\nabla h|^q = 0$ in $(0,\infty) \times \mathbb{R}^N$ with $h(0,x) = u_0(x)$, then

$$\{x \in \mathbb{R}^N : \ \ h(t,x) > 0\} = \mathcal{P}_0 := \{x \in \mathbb{R}^N : \ \ u_0(x) > 0\}\,.$$

Large time behaviour for the HJ equation

More precisely, if h is the *viscosity solution* to

$$\begin{aligned} \partial_t h + |\nabla h|^q &= 0\,, \quad (t,x) \in (0,\infty) \times \mathbb{R}^N\,, \\ h(0,x) &= u_0(x)\,, \quad x \in \mathbb{R}^N\,, \end{aligned}$$

then

$$\{x \in \mathbb{R}^N : \ \ h(t,x) > 0\} = \mathcal{P}_0 := \{x \in \mathbb{R}^N : \ \ u_0(x) > 0\}\,.$$

and

$$\lim_{t\to\infty} \left\| t^{1/(q-1)}\, h(t) - h_\infty \right\|_\infty = 0$$

with

$$h_\infty(x) = \frac{(q-1)}{q^{q/(q-1)}}\, d\left(x, \mathbb{R}^N \setminus \mathcal{P}_0\right)^{q/(q-1)}\,, \qquad x \in \mathbb{R}^N\,,$$

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The positivity set

For $t \geq 0$ we put $\mathcal{P}(t) := \{x \in \mathbb{R}^N : \ u(t,x) > 0\}$.

Lemma

For $t_1 \in [0, \infty)$ and $t_2 \in (t_1, \infty)$ we have

$$\mathcal{P}(t_1) \subseteq \mathcal{P}(t_2)$$

and

$$\mathcal{P}_\infty := \bigcup_{t \geq 0} \mathcal{P}(t) \quad \textit{is a bounded open subset of } \ \mathbb{R}^N .$$

Comparison with a suitable subsolution.

L. & Vázquez (2007)

Convergence to self-similarity

Theorem

$$\lim_{t\to\infty} \left\| t^{1/(q-1)}\, u(t) - V_\infty \right\|_\infty = 0\,,$$

where

$$V_\infty(x) := \frac{q-1}{q^{q/(q-1)}}\, d\left(x, \mathbb{R}^N \setminus \mathcal{P}_\infty\right)^{q/(q-1)} \quad \textit{for} \quad x \in \mathbb{R}^N\,.$$

L. & Vázquez (2007)

Remark: $h_\infty(t,x) := t^{-1/(q-1)}\, V_\infty(x)$ is a self-similar (viscosity) solution to $\partial_t h_\infty + |\nabla h_\infty|^q = 0$ such that $h_\infty(t,x) > 0$ for $x \in \mathcal{P}_\infty$ and $h_\infty(t,x) = 0$ for $x \notin \mathcal{P}_\infty$.

Proof - Decay estimates

For each $t \geq 0$

$$\|u(t)\|_1 + \|u(t)\|_\infty + \|\nabla u(t)\|_\infty \leq C_1 \ t^{-1/(q-1)} .$$

For each $x \in \mathcal{P}_\infty$, there are $T_x \geq 0$ and $\varepsilon_x > 0$ such that

$$u(t,x) \geq \varepsilon_x \ (1+t)^{-1/(q-1)} \quad \text{for} \quad t \geq T_x .$$

Proof - Self-similar variables

We introduce the *self-similar* variables $\tau := \ln\left(1+(q-1)t\right)/(q-1)$ and x and put

$$u(t,x) =: (1+(q-1)t)^{-1/(q-1)}\ v\left(\frac{\ln(1+(q-1)t)}{(q-1)}, x\right)$$

for $(t,x) \in [0,\infty) \times \mathbb{R}^N$. Then v solves

$$\begin{aligned} \partial_\tau v + |\nabla v|^q - v &= e^{-(p-1-q)\tau}\, \Delta_p v\ , \quad (\tau,x) \in (0,\infty) \times \mathbb{R}^N\,, \\ v(0) &= u_0\,, \quad x \in \mathbb{R}^N\,. \end{aligned}$$

Proof - Half-relaxed limits

For $(\tau, x) \in [0, \infty) \times \mathbb{R}^N$, we introduce

$$\begin{aligned} v_*(x) &:= \liminf_{(\sigma,y,s)\to(\tau,x,\infty)} v(\sigma+s, y)\,, \\ v^*(x) &:= \limsup_{(\sigma,y,s)\to(\tau,x,\infty)} v(\sigma+s, y)\,. \end{aligned}$$

- v_* and v^* do not depend on $\tau \geq 0$ and $v_* \leq v^*$,
- v_* is a viscosity supersolution to $|\nabla v_*|^q - v_* = 0$ in $\mathbb{R}^N$,
- v^* is a viscosity subsolution to $|\nabla v^*|^q - v^* = 0$ in $\mathbb{R}^N$,
- $v^*(x) = v_*(x) = 0$ for $x \notin \mathcal{P}_\infty$,
- $0 < v_*(x) \leq v^*(x)$ for $x \in \mathcal{P}_\infty$.

Proof - Identification of the limit

$$v_* = v^* \quad \text{and} \quad \lim_{\tau \to \infty} \|v(\tau) - v_*\|_\infty = 0\,.$$

Setting

$$V_*(x) = \frac{q}{q-1}\, v_*(x)^{(q-1)/q}\,, \quad x \in \mathbb{R}^N\,,$$

the function V_* solves the *eikonal* equation $|\nabla V_*| - 1 = 0$ in $\mathcal{P}_\infty$ with $V_* = 0$ on $\partial\mathcal{P}_\infty$, hence

$$V_*(x) = \mathrm{dist}\left(x, \mathbb{R}^N \setminus \mathcal{P}_\infty\right)\,, \quad x \in \mathcal{P}_\infty\,.$$

The limit positivity set $\mathcal{P}_\infty$

Unknown in the asymptotics: the limit positivity set

$$\mathcal{P}_\infty = \bigcup_{t \geq 0} \{x \in \mathbb{R}^N : \quad u(t,x) > 0\}\,.$$

Additional information?

Evolution of the positivity set

- Infinite waiting time: if u_0 is sufficiently flat at the edges of its positivity set then

$$\mathcal{P}(t) = \mathcal{P}(0) \quad \text{for every} \quad t \geq 0\,,$$

and thus $\mathcal{P}_\infty = \mathcal{P}(0)$.
Comparison with a supersolution Knerr (1977).

- Instantaneous expansion: if u_0 vanishes sufficiently slowly in the neighbourhood of a point $x_0 \in \partial\mathcal{P}(0)$ then

$$u(t, x_0) > 0 \quad \text{for all} \quad t > 0\,.$$

Local integral estimates Aronson & Caffarelli (1983).