Minimizing the area and maximizing the perimeter for Zindler sets

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Setting of the problems

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- Definition of bisecting chords

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- Definition of Zindler sets

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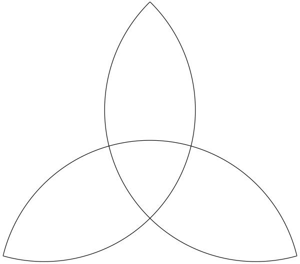

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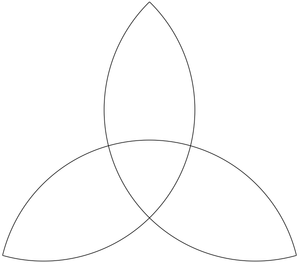

Answer to second question= semi-proof

What can we say of a convex Zindler set?

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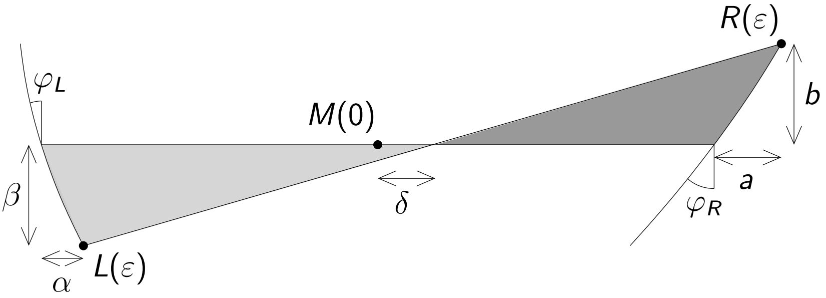

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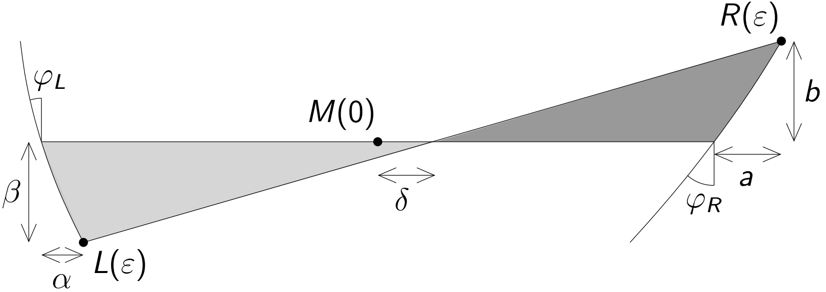

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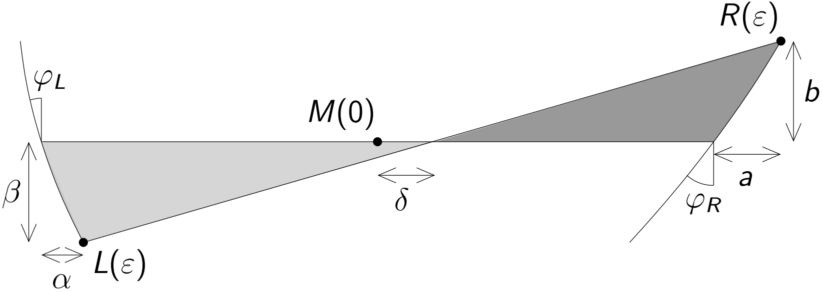

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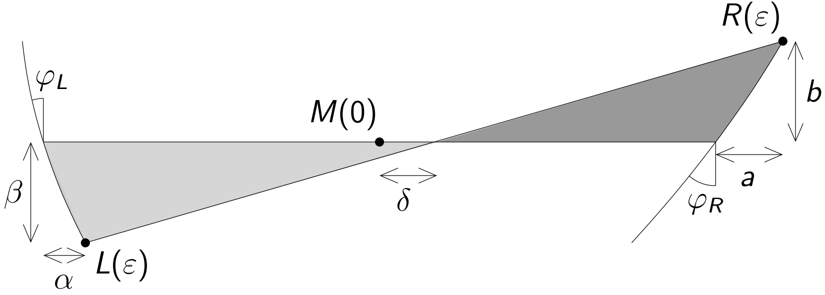

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Definition of $c : \mathbb{S}^1 \to \mathbb{R}$.

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but $$Area(c) = \pi - \sum_{\substack{n \text{ odd} \\ n\geq 3}} \frac{A_n^2+B_n^2}{n^2-1}$$

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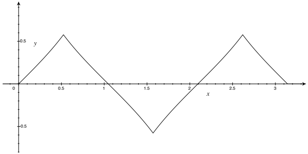

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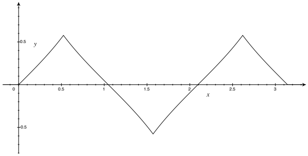

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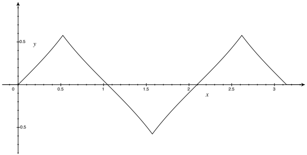

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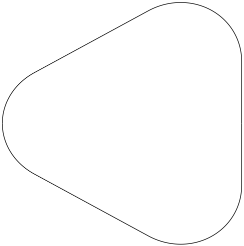

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Step I: An L^∞ estimate

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Step II: Three “magic” angles

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Step I: An L^∞ estimate

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Step III: Conclusion

Sketch of proof of Step II: Case A

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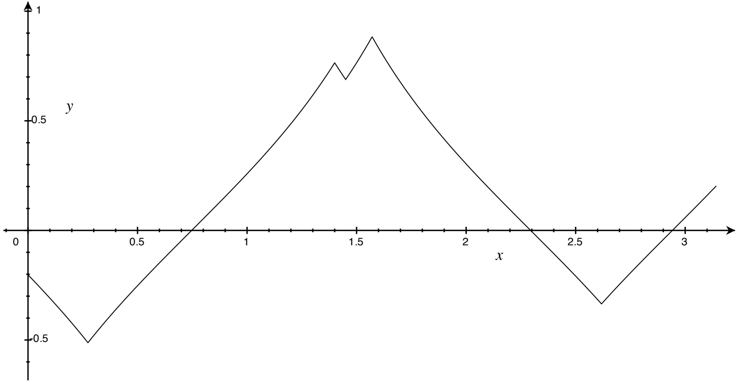

Sketch of proof of Step II: Case A

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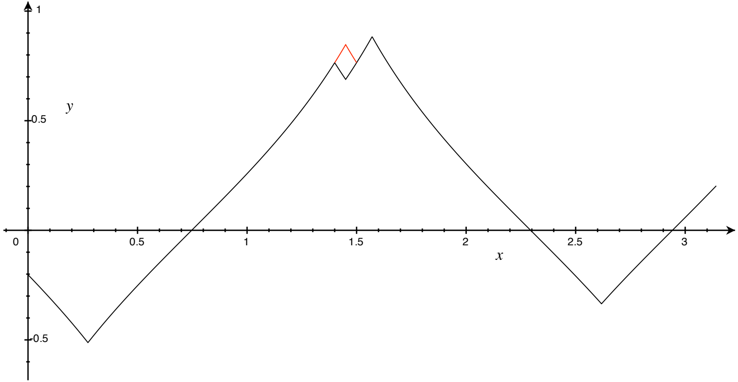

Sketch of proof of Step II: Case B

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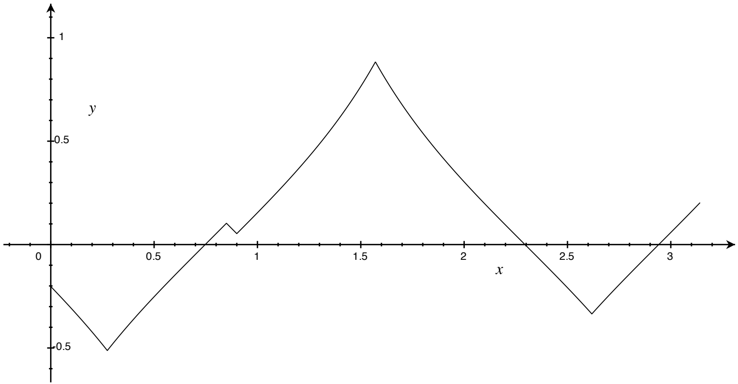

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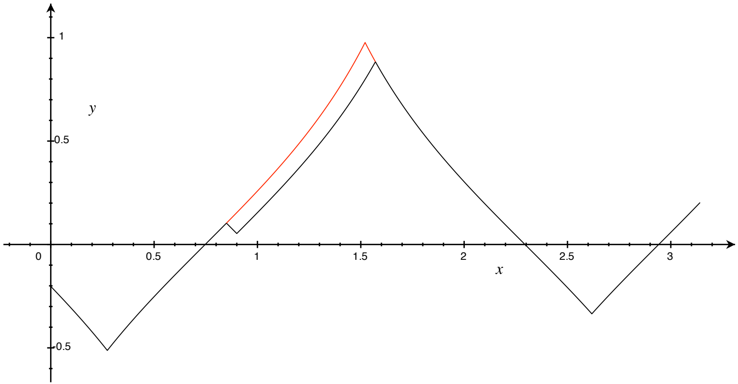

How do we prove the perimeter Theorem?

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Step I: There is no interval larger than $\pi/3$ where $c > 0$

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Step II: Small intervals are a really bad idea

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Problem III: Among all the Zindler sets? Among all general sets?