THE MONGE PROBLEM IN $\mathbb{R}^d$

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Abstract. We consider the Monge problem in a convex bounded subset of $\mathbb{R}^d$. The cost is given by a general norm, and we prove the existence of an optimal transport map under the classical assumption that the first marginal is absolutely continuous with respect to the Lebesgue measure. The approach we propose to solve this problem does not use the disintegration of measures.

1. Introduction

The Monge problem has origin in the *Mémoire sur la théorie des déblais et remblais* written by G. Monge [23], and may be stated as follows:

$$\inf\left\{\int_\Omega |x-T(x)|d\mu(x)\ :\ T\in\mathcal{T}(\mu,\nu)\right\},\tag{1.1}$$

where Ω is the closure of a convex open subset of $\mathbb{R}^d$, $|\cdot|$ denotes the usual Euclidean norm of $\mathbb{R}^d$, μ,ν are Borel probability measures on Ω and $\mathcal{T}(\mu,\nu)$ denotes the set of transport maps from μ to ν, i.e. the class of Borel maps T such that $T_\sharp\mu=\nu$ (where $T_\sharp\mu(B):=\mu(T^{-1}(B))$ for each Borel set B).

In this paper we prove the following existence result for a generalization of the problem, where the Euclidean norm $|\cdot|$ is replaced by a general norm on $\mathbb{R}^d$.

Theorem 1.1. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^d$ and assume that μ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^d$, then the problem*

$$\min\left\{\int_\Omega \|x-T(x)\|d\mu(x)\ :\ T\in\mathcal{T}(\mu,\nu)\right\}\tag{1.2}$$

has at least one solution.

We emphasize the fact that we make no regularity assumption on the norm $\|\cdot\|$. On the other hand, the assumption that the first marginal μ should be absolutely continuous with respect to the Lebesgue measure is classical and may be justified by Theorem 8.3 in Ambrosio *et al.* [4], which states that for any $s<d$ there exists a measure $\mu<<\mathcal{H}^s$ for which (1.2) does not have any solution.

The main difficulties in (1.2) are due to the facts that the objective functional is non-linear in T and the set $\mathcal{T}(\mu,\nu)$ does not possess the right compactness properties to apply the direct methods of the Calculus of Variations. A suitable relaxation was

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introduced by Kantorovich [21, 22] and it proved to be a decisive tool to deal with this problem. Define the set of transport plans from μ to ν as

$$\Pi(\mu,\nu) := \{\gamma \in \mathcal{P}(\Omega\times\Omega) \mid \pi^1_\sharp\gamma = \mu,\ \pi^2_\sharp\gamma = \nu\},$$

where $\mathcal{P}(\Omega\times\Omega)$ denotes the set of Borel probability measures on $\Omega\times\Omega$ and π^i denotes the standard projection in the Cartesian product. The set $\Pi(\mu,\nu)$ is always non-empty as it contains at least $\mu\otimes\nu$. Then Kantorovich proposed to study the problem

$$\min\left\{\int_{\Omega\times\Omega}\|x-y\|d\gamma(x,y)\ :\ \gamma\in\Pi(\mu,\nu)\right\}. \tag{1.3}$$

Problem (1.3) is convex and linear in γ, then the existence of a minimizer may be obtained by the direct method of the Calculus of Variations. To obtain the existence of a minimizer for (1.2) it is then sufficient to prove that some solution $\gamma\in\Pi(\mu,\nu)$ of (1.3) is in fact induced by a transport $T\in\mathcal{T}(\mu,\nu)$, i.e. may be written as $\gamma = (id\times T)_\sharp\mu$.

Before describing the present work, let us review briefly other existence results for (1.2). Sudakov [31] first proposed an efficient strategy to solve (1.2) for a general norm $\|\cdot\|$ on $\mathbb{R}^d$. However this method involved a crucial step on the disintegration of an optimal measure γ for (1.3) which was not completed correctly at that time, and has recently been justified in the case of a strictly convex norm by Caravenna [11]. Meanwhile, the problem (1.1) has been solved by Evans *et al.* [19] with the additional regularity assumption that μ and ν have Lipschitz-continuous densities with respect to $\mathcal{L}^d$, and then by Ambrosio [1] and Trudinger *et al.* [32] for μ and ν with integrable density. For C^2 uniformly convex norms the problem (1.2) has been solved by Caffarelli *et al.* [10] and Ambrosio *et al.* [4], and finally for crystalline norms in $\mathbb{R}^d$ and general norms in $\mathbb{R}^2$ by Ambrosio *et al.* [3]. The original proof of Sudakov was based on the reduction of the transport problems to affine regions of smaller dimension, and all the proof we listed above are based on the reduction of the problem to a 1-dimensional problem via a change of variable or area-formula. In [12], we designed a different method which does not require the reduction to 1-dimensional settings. However, we were able to carry on one of the steps of our proof only in the case of strictly convex norms.

In this paper, we prove the existence of a solution to (1.2) for a general norm $\|\cdot\|$ on $\mathbb{R}^d$. The originality of our method for the proof of Theorem 1.1 above is that it does not require disintegration of measures and relies on a simple but powerful regularity result (see Lemma 3.3 below), which is inspired by a previous regularity result obtained in the study of an optimal transportation problem with cost functional in non-integral form in [13]. In section §2, we introduce a variational approximation to select solutions of (1.3) that have a particular monotonicity property. Section §3 is devoted to the notion of density-regular points of a transport γ and in particular to Lemma 3.3, which states that a transport map $\gamma\in\Pi(\mu,\nu)$ is concentrated on such points. In the following section §4, we infer from the preceding some technical regularity result for the particular solutions of (1.3) previously selected. The proof of our main result Theorem 1.1 is finally derived in §5, while some final comments are collected in §6.

2. Variational approximation to select monotone transport plans

Following the line of [3, 10, 29], we introduce a variational approximation to select optimal transport plans for (1.3) which have some additional properties, and in the next sections we shall prove that these particular optimal transport plans are induced by transport maps. This procedure of choosing particular minimizers is the root of the idea of asymptotic development by Γ-convergence (see [5] and [6]) .

We denote by $\mathcal{O}_1(\mu,\nu)$ the set of optimal transport plans for (1.3), and consider the auxiliary problem:

$$\min\left\{\int_{\Omega\times\Omega}|y-x|^2 d\gamma(x,y)\ :\ \gamma\in\mathcal{O}_1(\mu,\nu)\right\}, \tag{2.1}$$

where we remark the fact that the cost in consideration involves the euclidean norm $|\cdot|$ of $\mathbb{R}^d$. Following §3.1 in [29], we introduce an approximating procedure for some particular solutions of (2.1) (see Lemma 2.3 below). Given two Borel probability measures α and β on Ω, we denote by

$$\mathcal{W}_1(\alpha,\beta) := \min\left\{\int_{\Omega\times\Omega}\|x-y\|d\gamma\ :\ \gamma\in\Pi(\alpha,\beta)\right\}$$

the usual 1−Wasserstein distance associated to the norm $\|\cdot\|$. Notice that problem (1.3) then corresponds to $\mathcal{W}_1(\mu,\nu)$. For $\varepsilon>0$, we also set

$$C_\varepsilon(\gamma;\nu) := \frac{1}{\varepsilon}\mathcal{W}_1(\pi^2_\sharp\gamma,\nu) + \int_{\Omega\times\Omega}\|x-y\|d\gamma + \varepsilon\int_{\Omega\times\Omega}|x-y|^2 d\gamma + \varepsilon^{3d+2}\mathrm{Card}(\pi^2_\sharp\gamma)$$

for any $\gamma\in\mathcal{P}(\Omega\times\Omega)$, where Card$(\cdot)$ denotes the cardinality of the support of the measure. We emphasize the fact that the norm $\|\cdot\|$ appears in the two first terms of C_ε while the Euclidean norm $|\cdot|$ appears only in the third term. We then consider the following family of minimization problems $(D_\varepsilon)_{\varepsilon>0}$ associated to (1.3) and (2.1):

$$(D_\varepsilon)\qquad \min\{C_\varepsilon(\gamma;\nu)\ :\gamma\in\mathcal{P}(\Omega\times\Omega),\ \pi^1_\sharp\gamma=\mu\}.$$

For any $\varepsilon>0$ the problem (D_ε) admits at least one solution γ_ε, with discrete second marginal $\pi^2_\sharp\gamma_\varepsilon$.

We finally introduce the standard family of interpolated projections.

Definition 2.1. For $t\in[0,1]$ we will denote by P^t the map

$$\begin{array}{rccl} P^t: & \Omega\times\Omega & \to & \Omega \\ & (x,y) & \mapsto & (1-t)x+ty. \end{array}$$

The following Proposition collects some properties of the minimizers of (D_ε) for later use, mainly inspired from [29].

Proposition 2.2. *Let B be a Borel subset of $\Omega\times\Omega$. Let $\varepsilon>0$ and γ_ε be a solution for (D_ε), and set $\mu_{\varepsilon,B} := \pi^1_\sharp\gamma_\varepsilon\lfloor B$ and $\nu_{\varepsilon,B} := \pi^2_\sharp\gamma_\varepsilon\lfloor B$. Then it holds*

(1) *the measure $\gamma_\varepsilon\lfloor B$ is a solution of the problem*

$$(D_{\varepsilon,B})\qquad \min\left\{\int_{\Omega\times\Omega}(\|x-y\|+\varepsilon|x-y|^2)d\gamma\ :\gamma\in\Pi(\mu_{\varepsilon,B},\nu_{\varepsilon,B})\right\}$$

where $\Pi(\mu_{\varepsilon,B},\nu_{\varepsilon,B})$ *denotes the set of non-negative Borel measures with marginals* $\mu_{\varepsilon,B}$ *and* $\nu_{\varepsilon,B}$*;*

(2) *if* $\mu_{\varepsilon,B}\in L^\infty(\Omega)$ *then for any* $t\in(0,1)$ *it holds*

$$\|{P^t}_\sharp(\gamma_\varepsilon\lfloor B)\|_{L^\infty}\leq(1-t)^{-d}\|\mu_{\varepsilon,B}\|_{L^\infty}\,.$$

Proof. Since γ_ε is a solution of (D_ε), it is a solution of

$$\min\left\{\int_{\Omega\times\Omega}(\|x-y\|+\varepsilon|x-y|^2)d\gamma\ :\gamma\in\Pi(\mu,\pi^2_\sharp\gamma_\varepsilon)\right\}. \tag{2.2}$$

The claim (1) then follows from the linearity of the functional in problem (2.2) (e.g. see proof of Lemma 4.2 in [3]).

The claim (2) is a direct application of Lemma 2 in §3.2 of [29], since by (1) the measure $\gamma_\varepsilon\lfloor B$ is an optimal transport plan between $\mu_{\varepsilon,B}$, which is absolutely continuous with respect to $\mathcal{L}^d$, and the discrete measure $\nu_{\varepsilon,B}$ for the strictly convex cost $(x,y)\mapsto\|x-y\|+\varepsilon|x-y|^2$ (see also the Appendix below). □

The link between the family of problems (D_ε) and (2.1) is given in the following Lemma, whose proof coincides with that of Lemma 1 in §3.1 of [29] and will be given in the appendix for sake of completeness.

Lemma 2.3. *For any* $\varepsilon>0$ *let* γ_ε *be a solution of* (D_ε)*, then the sequence* $(\pi^2_\sharp\gamma_\varepsilon)$ w^**-converges to* ν *as* $\varepsilon\to0$*. Moreover, any* w^**-limit as* $\varepsilon_k\to0$ *of a subsequence of solutions* $(\gamma_{\varepsilon_k})_{k\in\mathbb{N}}$ *is a solution of (2.1).*

The above Lemma suggests to introduce the following set of optimal transport plans for (1.3).

Definition 2.4. We shall denote by $\mathcal{O}_2(\mu,\nu)$ the minimizers for (2.1) which are w^*-limits as $\varepsilon_k\to0$ of a subsequence $(\gamma_{\varepsilon_k})_{k\in\mathbb{N}}$ of minimizers of (D_{ε_k}).

We observe that, by definition, the minimizers γ_ε of problem (D_ε) are all probability measures on $\Omega\times\Omega$, and since their marginals converge as $\varepsilon\to0$ to μ and ν, we infer that $\mathcal{O}_2(\mu,\nu)$ is not empty.

It is an important fact in the following that the local properties stated in Proposition 2.2 pass to the limit and are still valid for the elements of $\mathcal{O}_2(\mu,\nu)$. Notice that, in general, the restrictions of a sequence of weakly converging measures does not converge without additional assumptions. The following lemma states that this is the case when considering a sequence of transport plans with the same first marginals.

Lemma 2.5. *Let* $(\gamma_\varepsilon)_\varepsilon$ *a sequence in* $\mathcal{P}(\Omega\times\Omega)$ *with* w^*-*limit* $\gamma\in\mathcal{P}(\Omega\times\Omega)$ *as* $\varepsilon\to0$*, and such that* $\pi^1_\sharp\gamma_\varepsilon=\pi^1_\sharp\gamma=\mu$ *for any* $\varepsilon>0$*, with* $\mu<<\mathcal{L}^d$*. Then for any Borel set* $G\subset\Omega$ *it holds* $\gamma_\varepsilon\lfloor G\times\Omega\overset{*}{\rightharpoonup}\gamma\lfloor G\times\Omega$*.*

Proof. We have to prove that $\forall\ \varphi\in\mathcal{C}_b(\Omega\times\Omega)$

$$\int_{\Omega\times\Omega}\chi_G(x)\varphi(x,y)d\gamma_\varepsilon(x,y)\to\int_{\Omega\times\Omega}\chi_G(x)\varphi(x,y)d\gamma(x,y)\qquad as\ \varepsilon\to0. \tag{2.3}$$

Since $\mu << \mathcal{L}^d$, it follows from Lusin's Theorem that for all $\alpha > 0$ there exists a closed set F_α such that

$$\chi_{G|F_\alpha} \text{ is continuous and } \mu(\Omega \setminus F_\alpha) \leq \alpha.$$

As a consequence for every $\alpha > 0$ one has

$$\text{the restriction of } (x,y) \mapsto \chi_G(x)\varphi(x,y) \ \text{ to } F_\alpha \times \Omega \text{ is continuous}$$

and

$$\limsup_{\varepsilon \to 0} \gamma_\varepsilon((\Omega \setminus F_\alpha) \times \Omega) \leq \mu(\Omega \setminus F_\alpha) \leq \alpha.$$

Then since $(x,y) \mapsto \chi_G(x)\varphi(x,y)$ is bounded and then equiintegrable with respect to $(\gamma_\varepsilon)_{\varepsilon>0}$, (2.3) follows from Proposition 5.1.10 of [2]. □

Finally, since an element of $\mathcal{O}_2(\mu,\nu)$ is a solution of (2.1), it enjoys a cyclical-monotonicity property inherited from the cost $(x,y) \mapsto |y-x|^2$ (see remark 2.7 below), stated in the following Proposition, whose proof may be derived from that of Lemma 4.1 in [3] and is given in [12] (see Proposition 3.2 therein).

Proposition 2.6. *Let γ be a solution of (2.1), then γ is concentrated on a σ-compact set Γ with the following property:*

$$\forall (x,y),(x',y') \in \Gamma, \qquad x \in [x',y'] \ \Rightarrow \ (x-x')\cdot(y-y') \geq 0, \tag{2.4}$$

where $\cdot$ denotes the usual scalar product on $\mathbb{R}^d$.

Remark 2.7. A solution γ of the classical transport problem associated to $|\cdot|^2$:

$$\min\left\{\int_{\Omega\times\Omega} |y-x|^2 d\lambda(x,y) \ : \ \lambda \in \Pi(\mu,\nu)\right\},$$

is known to be concentrated on a $|\cdot|^2$-cyclically monotone set Γ, that is:

$$\forall (x,y),(x',y') \in \Gamma, \qquad (x-x')\cdot(y-y') \geq 0.$$

In (2.4), the restriction that x should be in $[x',y']$ to get the inequality has origin in the fact that the constraint in (2.1) is $\mathcal{O}_1(\mu,\nu)$ in place of $\Pi(\mu,\nu)$.

Remark 2.8. The reason to deal with σ-compact sets Γ, in the above proposition as well as in the following, is that the projection $\pi^1(\Gamma)$ is also σ-compact, and in particular is a Borel set.

3. A property of transport plans

We begin by considering some general properties of transport plans. This section is independent of the transport problem (1.3), and some of the techniques detailed below are refinements of similar ones which were first applied in [13] in the framework of non-classical transportation problems involving cost functionals not in integral form.

Definition 3.1. Let $\gamma \in \Pi(\mu,\nu)$ be a transport plan and Γ a σ-compact set on which it is concentrated. For $y \in \Omega$ and $r > 0$ we define

$$\Gamma^{-1}(\overline{B(y,r)}) := \pi^1(\Gamma \cap (\Omega \times \overline{B(y,r)})).$$

In other words, when given a σ-compact set Γ on which γ is concentrated, the set $\Gamma^{-1}(\overline{B(y,r)})$ is the set of those points whose mass (with respect to μ) is partially or completely transported to $\overline{B(y,r)}$ by the restriction of γ to Γ. We may justify this slight abuse of notations by the fact that γ should be thought of as a device that transports mass. Notice also that $\Gamma^{-1}(\overline{B(y,r)})$ is a σ-compact set.

Since this notion is important in the sequel, we recall that when a function f is *locally integrable* for the Lebesgue measure $\mathcal{L}^d$, one has

$$\lim_{r\to 0} \frac{1}{\mathcal{L}^d(B(x,r))} \int_{B(x,r)} |f(z) - f(x)| dz \;=\; 0$$

for almost every x in Ω. These points x are usually called Lebesgue points of f. When A is an $\mathcal{L}^d$-measurable subset of Ω, we shall call Lebesgue point of A any element $x \in A$ which is a Lebesgue point of the characteristic function $f = \chi_A$ of A, and then satisfies

$$\lim_{r\to 0} \frac{\mathcal{L}^d(A \cap B(x,r))}{\mathcal{L}^d(B(x,r))} \;=\; 1.$$

In the following, we shall denote by $\mathrm{Leb}(f)$ (resp. $\mathrm{Leb}(A)$) the set of points $x \in \Omega$ (resp. $x \in A$) which are Lebesgue points of f (resp. A). Moreover we will denote by *support*(f) the set of points $x \in \Omega$ such that $\int_{B(x,r)} f(z)dz > 0$ for any $r > 0$.

Definition 3.2. We will call density of an absolutely continuous measure λ the function

$$g(x) = \limsup_{r\to 0} \frac{\lambda(B(x,r))}{\mathcal{L}^d(B(x,r))}.$$

Then the Lebesgue points of the density of λ are uniquely determined as well as the value of g at those points.

The following Lemma is an essential step in the proof of Proposition 4.2 and Theorem 5.1 below. This result is a refinement of Lemma 5.2 from [13] and Lemma 4.3 in [12], and its proof follows the line of those Lemmas. It in fact encompasses those results, as Remark 3.5 below shows.

Lemma 3.3. *Assume that $\mu << \mathcal{L}^d$ with density denoted by f. Let $\gamma \in \Pi(\mu,\nu)$, and Γ a set on which γ is concentrated. Then there exists a σ-compact subset $D(\Gamma)$ of $\Gamma \cap$ support(γ) on which γ is concentrated, and such that for any $(x,y) \in D(\Gamma)$ and $r > 0$, there exist $\tilde{y} \in \Omega$ and $\tilde{r} > 0$ such that*

$$y \in B(\tilde{y},\tilde{r}) \subset B(y,r), \quad x \in \mathrm{Leb}(f) \cap \mathrm{Leb}(\tilde{f}), \quad f(x) < +\infty \;\; \textit{and} \;\; \tilde{f}(x) > 0 \qquad (3.1)$$

where $\tilde{f}$ is the density of $\pi^1_\sharp \gamma \lfloor \Omega \times B(\tilde{y},\tilde{r})$ with respect to $\mathcal{L}^d$.

Proof. Let $(y_n)_n$ be a dense sequence in Ω. For each $(n,k) \in \mathbb{N}^2$ we set $\gamma_{n,k} := \gamma\lfloor \Omega \times B(y_n, \frac{1}{k+1})$ and define $f_{n,k}$ to be the density of $\pi^1_\sharp \gamma_{n,k}$ with respect to $\mathcal{L}^d$. We notice that for any $(x,y) \in \Omega\times\Omega$ and $r > 0$ there exists $n,k \in \mathbb{N}$ such that $y \in B(y_n, \frac{1}{k+1}) \subset B(y,r)$, and that if (x,y) is in the support of γ then it is in the support of $\gamma_{n,k}$ and x is in the support of $f_{n,k}$. Let now

$$A_{n,k} := [\Omega \setminus (\mathrm{Leb}(f) \cap \mathrm{Leb}(f_{n,k}) \cap \{f < +\infty\} \cap \{f_{n,k} > 0\})] \times B(y_n, \frac{1}{k+1}).$$

for all $n, k \in \mathbb{N}$.

We claim that $\gamma(\cup_{n,k} A_{n,k}) = 0$. Indeed for fixed $n, k \in \mathbb{N}$, the set $\Omega \setminus (\mathrm{Leb}(f) \cap \mathrm{Leb}(f_{n,k}) \cap \{f < +\infty\})$ has $\mathcal{L}^d$ measure 0, so that it also has μ-measure 0 and then $\pi^1_\sharp \gamma_{n,k}$ measure 0. The set $\Omega \setminus \{f_{n,k} > 0\}$ also has $\pi^1_\sharp \gamma_{n,k}$ measure 0, so that $\gamma(A_{n,k}) = \gamma_{n,k}(A_{n,k}) = 0$. This proves the claim and we conclude that γ is concentrated on the set $(support(\gamma) \cap \Gamma) \setminus \cup_{n,k} A_{n,k}$, which has all the desired properties but the σ-compactness. This last property is achieved thanks to the inner regularity of γ. □

The above discussion and Lemma yield us to introduce the following notions:

Definition 3.4. The couple $(x, y) \in \Gamma$ is a Γ-*regular point* if x is a Lebesgue point of $\Gamma^{-1}(\overline{B(y, r)})$ for any positive r; it is a Γ-*density-regular point* if for any $r > 0$ there exists $(\tilde{y}, \tilde{r})$ such that (3.1) holds.

Remark 3.5. By definition any element $(x, y) \in D(\Gamma)$ (with the notations of Lemma 3.3) is a Γ-density-regular point, we notice that it is also a Γ-regular point. Indeed, for $r > 0$ there exists $(\tilde{y}, \tilde{r})$ such that (3.1) holds, then since $\tilde{f}(x) > 0$ and $x \in \mathrm{Leb}(\tilde{f})$ it follows that x belongs to $\mathrm{Leb}(\{\tilde{f} > 0\})$. By the definition of $\tilde{f}$ it comes

$$\int_{\{\tilde{f}>0\} \setminus \Gamma^{-1}(\overline{B(y,r)})} \tilde{f} d\mathcal{L}^d = \gamma(\Omega \times B(\tilde{y}, \tilde{r}) \setminus \Gamma) = 0$$

so that $\mathcal{L}^d(\{\tilde{f} > 0\} \setminus \Gamma^{-1}(\overline{B(y, r)})) = 0$. As a consequence, x belongs to $\mathrm{Leb}(\Gamma^{-1}(\overline{B(y, r)}))$.

Lemma 3.3 above therefore states that any transport plan γ is concentrated on a Borel set consisting of regular as well as density-regular points.

4. A property of the selected optimal transport plans

In this section, we obtain a regularity result (Proposition 4.2 below) for the transport plans which belong to $\mathcal{O}_2(\mu, \nu)$ (see Definition 2.4). Following the formalism of [4] we introduce the notion of transport set related to a subset Γ of $\mathbb{R}^d \times \mathbb{R}^d$.

Definition 4.1. Let Γ be a subset of $\mathbb{R}^d \times \mathbb{R}^d$, the transport set $T(\Gamma)$ of Γ is

$$T(\Gamma) := \{(1 - t)x + ty \mid (x, y) \in \Gamma,\ t \in (0, 1)\}.$$

Notice that if Γ is σ-compact then $T(\Gamma)$ is also σ-compact.

Proposition 4.2. *Assume that* $\mu << \mathcal{L}^d$ *and let* $\gamma \in \mathcal{O}_2(\mu, \nu)$ *be concentrated on a* σ*-compact set* Γ*. Then for any* $(x, y) \in D(\Gamma)$ *(obtained by Lemma 3.3) with* $x \neq y$ *and for* $r > 0$ *small enough it holds*

$$\liminf_{\delta \to 0^+} \frac{\mathcal{L}^d \left[T\left(\Gamma \cap \left[B(x, \frac{\delta}{2}) \times B(y, r)\right]\right) \cap B(x, \delta)\right]}{\mathcal{L}^d(B(x, \delta))} \ > \ 0. \tag{4.1}$$

Proof. We denote by f the density of μ. Consider $(x, y) \in D(\Gamma)$ with $x \neq y$ and $0 < r << |x - y|$. Let $\tilde{y}$ and $\tilde{r}$ be as in Lemma 3.3; we recall that $\pi^1_\sharp \gamma \lfloor \Omega \times B(\tilde{y}, \tilde{r})$ is absolutely continuous with respect to $\mathcal{L}^d$, with density denoted by $\tilde{f}$, that $\tilde{f}(x) > 0$ and

$$\lim_{s \to 0} \frac{1}{\mathcal{L}^d(B(x, s))} \int_{B(x,s)} |\tilde{f}(z) - \tilde{f}(x)| = \lim_{s \to 0} \frac{1}{\mathcal{L}^d(B(x, s))} \int_{B(x,s)} |f(z) - f(x)| = 0.$$

Let $G := \{z \in \Omega \mid \frac{1}{2}\tilde{f}(x) \leq \tilde{f}(z) \text{ and } f(z) \leq f(x)+1\}$. Possibly subtracting a set of $\mathcal{L}^d$-measure 0 we may consider G a Borel set and

$$\lim_{s\to 0} \frac{\mathcal{L}^d(G\cap B(x,s))}{\mathcal{L}^d(B(x,s))} = 1.$$

Fix $\delta > 0$ so that

$$\frac{\delta}{|x-y|+r} < 1 \quad \text{and} \quad \forall s \in (0,\delta),\ \mathcal{L}^d(G\cap B(x,s)) \geq \frac{1}{2}\mathcal{L}(B(x,s)) \tag{4.2}$$

and fix $t \in (0, \frac{\delta}{2(|x-y|+r)})$. Then for every $z \in B(x,\frac{\delta}{2})$ and every $w \in B(y,r)$ it holds

$$(1-t)z + tw \in B(x,\delta). \tag{4.3}$$

For such a choice of δ define the subset $G_\delta := \overline{B(x,\frac{\delta}{2})} \cap G$ of G and notice that

$$\mathcal{L}^d(G_\delta \cap B(x,\frac{\delta}{2})) \geq \frac{1}{2}\mathcal{L}^d(B(x,\frac{\delta}{2})). \tag{4.4}$$

Let $A_\delta := G_\delta \times B(\tilde{y},\tilde{r})$ and consider the measure $\gamma_{A_\delta} := \gamma\lfloor A_\delta$. We observe that $\pi^1_\sharp\gamma_{A_\delta}$ is absolutely continuous with respect to $\mathcal{L}^d$ and we denote by f_{A_δ} its density. Then one has

$$\frac{1}{2}\tilde{f}(x) \ \leq\ f_{A_\delta} \ \leq\ f \ \leq\ f(x)+1 \quad \text{on } G_\delta. \tag{4.5}$$

It then follows from (4.3), (4.4) and (4.5) that

$$\frac{\tilde{f}(x)}{4}\mathcal{L}^d(B(x,\frac{\delta}{2})) \leq \pi^1_\sharp\gamma_{A_\delta}(B(x,\frac{\delta}{2})) \leq P^t_\sharp\gamma_{A_\delta}(B(x,\delta)). \tag{4.6}$$

Since γ belongs to $\mathcal{O}_2(\mu,\nu)$, it is a w^*-limit of a subsequence $(\gamma_{\varepsilon_k})_k$ of minimizers of (D_{ε_k}). We notice that claim (2) of Proposition 2.2 holds for $\gamma_{\varepsilon_k}\lfloor G_\delta\times\Omega$, so that:

$$\|P^t_\sharp\gamma_{\varepsilon_k}\lfloor G_\delta\times\Omega\|_{L^\infty} \leq (1-t)^{-d}\|\pi^1_\sharp\gamma_{\varepsilon_k}\lfloor G_\delta\times\Omega\|_\infty = (1-t)^{-d}\|f\lfloor G_\delta\|_\infty.$$

By Lemma 2.5 it follows that $\gamma\lfloor G_\delta\times\Omega$ is the w^*-limit of the subsequence $(\gamma_{\varepsilon_k}\lfloor G_\delta\times\Omega)_k$. The sequence $(P^t_\sharp\gamma_{\varepsilon_k}\lfloor G_\delta\times\Omega)_k$ then converges weakly in $L^\infty(\Omega)$ to $P^t_\sharp\gamma\lfloor G_\delta\times\Omega$, and in particular letting $k\to+\infty$ in the above estimate yields

$$\|P^t_\sharp\gamma_{A_\delta}\|_{L^\infty} \leq \|P^t_\sharp\gamma\lfloor G_\delta\times\Omega\|_{L^\infty} \leq 2^d(f(x)+1). \tag{4.7}$$

On the other hand we claim that whenever a measure $\lambda\in\mathcal{M}(\Omega\times\Omega)$ is supported on a Borel set Λ the measure $P^t_\sharp\lambda$ is supported on $T(\Lambda)$. Indeed

$$P^t_\sharp\lambda(\Omega\setminus T(\Lambda)) = \lambda((P^t)^{-1}(\Omega\setminus T(\Lambda))) \leq \lambda(\Omega\times\Omega\setminus\Lambda) = 0.$$

As a consequence $P^t_\sharp\gamma_{A_\delta}$ is supported on $T(\Gamma\cap\big[B(x,\frac{\delta}{2})\times B(y,r)\big])$.

Then again the choice of t and (4.7) imply that

$$\begin{aligned} P^t_\sharp\gamma_{A_\delta}(B(x,\delta)) &= P^t_\sharp\gamma_{A_\delta}\left(T(\Gamma\cap[B(x,\frac{\delta}{2})\times B(y,r)])\cap B(x,\delta)\right) \\ &\leq 2^d(f(x)+1)\mathcal{L}^d\left(T(\Gamma\cap[B(x,\frac{\delta}{2})\times B(y,r)])\cap B(x,\delta)\right). \end{aligned} \tag{4.8}$$

The proof is now complete since (4.6) and (4.8) yield

$$\mathcal{L}^d(T(\Gamma\cap[B(x,\frac{\delta}{2})\times B(y,r)])\cap B(x,\delta))\geq\frac{\tilde{f}(x)}{2^{d+2}(f(x)+1)}\mathcal{L}^d(B(x,\frac{\delta}{2}))$$

for any $\delta>0$ small enough for (4.2) to hold. □

5. Proof of the main theorem

We now conclude with the proof of Theorem 1.1, which is a consequence of the following result.

Theorem 5.1. *Assume that $\mu << \mathcal{L}^d$. Then every element γ of $\mathcal{O}_2(\mu,\nu)$ is induced by a transport map $T_\gamma\in\mathcal{T}(\mu,\nu)$, i.e. $\gamma=(id\times T_\gamma)_\sharp\mu$.*

Proof. Let $\gamma\in\mathcal{O}_2(\mu,\nu)$, we prove that it is induced by a transport map $T_\gamma\in\mathcal{T}(\mu,\nu)$. By Proposition 2.1 in [1], it is sufficient to prove that γ is concentrated on a Borel graph.

It follows from Proposition 2.6 that γ is concentrated on a σ-compact set Γ satisfying (2.4). We then apply Proposition 4.2 to get that γ is concentrated on a σ-compact subset $D(\Gamma)$ of $\Gamma\cap supp(\gamma)$ and on which (4.1) is satisfied.

We claim that $D(\Gamma)$ is contained in a graph. To prove this, we show that if (x_0,y_0) and (x_0,y_1) both belong to $D(\Gamma)$ then $y_0=y_1$. We argue by contradiction, and assume that $y_1\neq y_0$. As a consequence, one either has $(y_1-y_0)\cdot(y_0-x_0)<0$ or $(y_0-y_1)\cdot(y_1-x_0)<0$. Without loss of generality, we assume that

$$(y_1-y_0)\cdot(y_0-x_0)\ <\ 0.$$

We fix $r>0$ small enough so that

$$\forall x\in B(x_0,r),\ \forall y'\in\overline{B(y_0,r)},\ \forall y\in\overline{B(y_1,r)},\qquad (y-y')\cdot(y'-x)\ <\ 0. \tag{5.1}$$

Since $(x_0,y_1)\in D(\Gamma)$, we infer that x_0 is a Lebesgue point for $\Gamma^{-1}(\overline{B(y_1,r)})$. Moreover, we also get from $(x_0,y_0)\in D(\Gamma)$ and (4.1) that

$$\liminf_{\delta\to0^+}\frac{\mathcal{L}^d\left(T\left(\Gamma\cap\left[B(x_0,\frac{\delta}{2})\times B(y_0,r)\right]\right)\cap B(x_0,\delta)\right)}{\mathcal{L}^d(B(x_0,\delta))}\ >\ 0.$$

As a consequence, for δ small enough there exists (x',y') and (x,y) in Γ such that

$$x'\in B(x_0,\frac{\delta}{2}),\quad y'\in\overline{B(y_0,r)},\quad x\in[x',y']\cap B(x_0,\delta)\quad \textit{and}\quad y\in\overline{B(y_1,r)}.$$

It follows from (2.4) applied to (x',y') and (x,y) that

$$(y-y')\cdot(x-x')\geq0$$

but since $x\in[x',y']$ one also has $x-x'=\frac{|x-x'|}{|y'-x|}(y'-x)$ which contradicts (5.1). □

Remark 5.2. It seems natural to expect that the set $\mathcal{O}_2(\mu,\nu)$ has a unique element, using the same type of uniqueness argument as in the Step 5 of the proof of Theorem B in [3]. However the set $\mathcal{O}_2(\mu,\nu)$, as defined in Definition 2.4, is not necessarily convex, and this argument does not apply here.

6. Comments

The strategy for proving Theorem 5.1 above relies on two fundamental ingredients: the cyclical-monotonicity for particular solutions of (1.3) obtained in Proposition 2.6, and the density result for the set of transport rays obtained in Proposition 4.2. This strategy was already that developed in [12] for the special case of a strictly convex norm.

The originality in the use of Proposition 2.6 is that, since the norm $\|\cdot\|$ is not assumed to be strictly convex, it may happen that the points x, x', y, y' in consideration are not aligned. In the strictly convex case this property of alignment is fundamental since it basically allows to reduce the problem (1.3) to a family of one-dimensional problems, on which one can use the property of monotonicity of the selected optimal transport plan (solution of (2.1)). In the general - not necessarily strictly convex - case, we need to use the full information that the selected particular solution is concentrated on a set which is cyclically monotone in the classical sense of convex analysis.

The fact that the result stated in Proposition 4.2, although quite natural, happens somewhat difficult to obtain (and in particular was not derived in its full generality in Proposition 5.2 of [12]), may be illustrated by the following example. Let us first recall the following result in [3]:

Theorem 6.1 (Theorem A of [3]). *There exist a Borel set $M \subset [-1,1]^3$ with $|M| = 8$ and two Borel maps $f_i : M \to [-2,2] \times [-2,2]$ for $i = 1,2$ such that the following holds. For $x \in M$ denote by l_x the segment connecting $(f_1(x), -2)$ to $(f_2(x), 2)$ then*

(1) *$\{x\} = l_x \cap M$ for all $x \in M$,*
(2) *$l_x \cap l_y = \emptyset$ for all $x, y \in M$ different.*

If one considers $\Gamma := \{(x, F(x)) : x \in M\}$ with $F(x) := (f_2(x), 2)$, then we observe that the open transport set $T(\Gamma)$ has density 0 at every point of $\pi^1(\Gamma) = M$ (although M has full measure in $[-1,1]^3$). We notice that Γ supports the transport plan $(id \times F)_\sharp(\mathcal{L}^3 \lfloor M)$ which is an optimal transport plan between its marginals for the cost $\|x\| := \max\{|x_1|, |x_2|, 3|x_3|\}$. The Lemma 3.3 (and the notion of Γ-density-regular points) as well as the approximating procedure provided in [29] (and recalled in §2) then appear as the necessary cornerstones to derive Proposition 4.2. In fact, it had been noticed in section §7 of [12], that using some estimate for the so called "transport density" may allow to obtain some technical result analogous to Proposition 4.2. Altough this is not exactly what we did in the present paper, the inequality (4.6) in the proof of Proposition 4.2 contains that type of estimate.

We now discuss further possible extensions of the methodology developed here to prove Theorem 5.1. The above example first indicates that for some very bad cases, the open transport $T(\Gamma)$ may have density 0 at any point of $\pi^1(\Gamma)$ when the norm is not strictly convex. This may be a limit of the definition of the open transport set that we use: a possible alternative would be to consider the set of all geodesics joining two points instead of considering only the segments. This would give a "fat" transport set. For the moment, our approach cannot be extended to this kind of transport sets without some substantial addition. We also notice that the construction we make in this paper does not make explicit use of the geometry of the segments, but it is based on some property of segments which may be enjoyed by more general family of curves. Then we believe that there is the possibility that the same approach could bring to the proof of existence of optimal transports also in other geometric settings where this result is currently out of sight.

We finally conclude by noticing that our strategy also provides a very efficient way to recover the existence result for an optimal transport map for the classical case of the Euclidean norm (or a C^2 strictly convex norm). Indeed, in that case the approximating procedure of §2 is useless and Proposition 4.2 holds for any solution γ of (1.3) because of the following property: if u is a potential for (1.3) (*i.e.* a solution of the classical dual problem for (1.3)), then there exists a countable union of sets $\cup_i T_i$ on which μ is concentrated and such that the gradient ∇u is Lipschitz-continuous on each T_i (for instance see [1, 10, 32]). This, together with the fact that the transport rays do not cross, allows to derive the desired density.

Appendix

For the sake of completeness, we give some details of the arguments of the proofs of Proposition 2.2 as well as Lemma 2.3. These proofs are adapted from that of Theorem 1 and Lemmas 1 and 2 of [29].

Proof of Proposition 2.2 (2). Fix $\varepsilon > 0$ and $t > 0$. Let $\{y_i\}_{i\in I}$ be the finite support of $\nu_{\varepsilon,B}$. For $i \in I$ we set $\Omega_i := support(\gamma_\varepsilon\lfloor\Omega\times\{y_i\})$ and $\Omega_i(t) := P_t(\Omega_i \times \{y_i\})$. Then if A is a Borel subset of Ω we have

$$\begin{aligned} P^t_\sharp(\gamma_\varepsilon\lfloor B)(A) &\le \sum_{i\in I}(\gamma_\varepsilon\lfloor B)((P^t)^{-1}(A\cap\Omega_i(t))) \\ &= \sum_{i\in I}\mu_{\varepsilon,B}\left(\frac{1}{1-t}(A\cap\Omega_i(t) - t\,y_i)\right) \\ &\le \sum_{i\in I}(1-t)^{-d}\|\mu_{\varepsilon,B}\|_{L^\infty}\mathcal{L}^d(A\cap\Omega_i(t)). \end{aligned}$$

The conclusion then follows whenever

$$\sum_{i\in I}\mathcal{L}^d(A\cap\Omega_i(t)) \;=\; \mathcal{L}^d\left(\bigcup_{i\in I}A\cap\Omega_i(t)\right) \quad (\le\; \mathcal{L}^d(A)).$$

This equality indeed follows from the fact that the sets $\Omega_i(t)$ and $\Omega_j(t)$ are disjoint when $i \ne j$. We prove this by contradiction, and assume that $(1-t)x_i+ty_i = (1-t)x_j+ty_j$ for some $x_i \in \Omega_i$, $x_j \in \Omega_j$ with $i \ne j$. Notice that since $y_i \ne y_j$, one also has $y_i-x_i \ne y_j-x_j$. The cost $c : (x,y) \mapsto \|x-y\|+\varepsilon|x-y|^2$ being continuous, the support of γ_ε is a c-cyclically monotone set, and thus one has

$$c(y_i-x_i)+c(y_j-x_j) \;\le\; c(y_j-x_i)+c(y_i-x_j).$$

Since $y_j - x_i = t(y_i - x_i) + (1-t)(y_j - x_j)$ and $y_i - x_j = (1-t)(y_i - x_i) + t(y_j - x_j)$, we conclude from the strict convexity of c that

$$c(y_j-x_i)+c(y_i-x_j) \;<\; c(y_i-x_i)+c(y_j-x_j)$$

which is a contradiction. ∎

Proof of Lemma 2.3. Assume that $\Omega \subset B(0,R)$. For $n \geq 1$ let p_n be a measurable map from Ω to a grid of at most $(2Rn)^d$ points with the property that $|p_n(x) - x| \leq \frac{1}{n}$ for any $x \in \Omega$. Let γ be a solution of (2.1), for every $n \geq 1$ we set $\gamma^n := (id \times p_n)_\sharp \gamma$.

We now write the optimality of γ_ε for (D_ε) so that for any $\varepsilon > 0$ and $n \geq 1$ it holds

$$\begin{aligned} C_\varepsilon(\gamma_\varepsilon;\nu) &= \frac{1}{\varepsilon}\mathcal{W}_1(\pi^2_\sharp\gamma_\varepsilon,\nu) + \int_{\Omega\times\Omega}\|x-y\|d\gamma_\varepsilon + \varepsilon\int_{\Omega\times\Omega}|x-y|^2 d\gamma_\varepsilon + \varepsilon^{3d+2}\mathrm{Card}(\pi^2_\sharp\gamma_\varepsilon) \\ &\leq C_\varepsilon(\gamma^n;\nu) \\ &= \frac{1}{\varepsilon}\mathcal{W}_1(p_{n\sharp}\nu,\nu) + \int_{\Omega\times\Omega}\|x-y\|d\gamma^n + \varepsilon\int_{\Omega\times\Omega}|x-y|^2 d\gamma^n + \varepsilon^{3d+2}\mathrm{Card}(p_{n\sharp}\nu) \\ &\leq \frac{1}{n\,\varepsilon} + \int_{\Omega\times\Omega}\|x-y\|d\gamma^n + \varepsilon\int_{\Omega\times\Omega}|x-y|^2 d\gamma^n + \varepsilon^{3d+2}(2Rn)^d. \end{aligned}$$

Keeping the first term in $C_\varepsilon(\gamma_\varepsilon;\nu)$, multiplying by ε and letting $\varepsilon \to 0$ then yields

$$\forall n \geq 1, \quad \limsup_{\varepsilon\to 0}\mathcal{W}_1(\pi^2_\sharp\gamma_\varepsilon,\nu) \leq \frac{1}{n}.$$

Letting $n \to +\infty$ we get the w^*-convergence of $\pi^2_\sharp\gamma_\varepsilon$ to ν. As a consequence, any w^*-cluster point of $(\gamma_\varepsilon)_\varepsilon$ as $\varepsilon \to 0$ belongs to $\Pi(\mu,\nu)$.

Keeping the second term in $C_\varepsilon(\gamma_\varepsilon,\nu)$ and taking $n(\varepsilon) \approx \varepsilon^{-2}$ yields

$$\int_{\Omega\times\Omega}\|x-y\|d\gamma_\varepsilon \leq \varepsilon + \int_{\Omega\times\Omega}\|x-y\|d\gamma^{n(\varepsilon)} + \varepsilon\int_{\Omega\times\Omega}|x-y|^2 d\gamma^{n(\varepsilon)} + \varepsilon^{d+2}(2R)^d.$$

We let $\varepsilon \to 0$ and notice that

$$\int_{\Omega\times\Omega}\|x-y\|d\gamma^{n(\varepsilon)} \to \int_{\Omega\times\Omega}\|x-y\|d\gamma = \mathcal{W}_1(\mu,\nu),$$

so that any w^*-cluster point of $(\gamma_\varepsilon)_\varepsilon$ is a solution of (1.3).

We now notice that

$$\int_{\Omega\times\Omega}\|x-y\|d\gamma_\varepsilon \geq \mathcal{W}_1(\mu,\pi^2_\sharp\gamma_\varepsilon) \geq \mathcal{W}_1(\mu,\nu) - \mathcal{W}_1(\nu,\pi^2_\sharp\gamma_\varepsilon)$$

and

$$\int_{\Omega\times\Omega}\|x-y\|d\gamma^n \leq \int_{\Omega\times\Omega}\|x-y\|d\gamma + \int_{\Omega\times\Omega}\|p_n(y)-y\|d\gamma^n \leq \mathcal{W}_1(\mu,\nu) + \frac{1}{n}$$

where we used the optimality of γ for (1.3). Keeping the three first terms in $C_\varepsilon(\gamma_\varepsilon,\nu)$, we then obtain that

$$(\frac{1}{\varepsilon}-1)\mathcal{W}_1(\nu,\pi^2_\sharp\gamma_\varepsilon) + \varepsilon\int_{\Omega\times\Omega}|x-y|^2 d\gamma_\varepsilon \leq \frac{1+\varepsilon}{n\,\varepsilon} + \varepsilon\int_{\Omega\times\Omega}|x-y|^2 d\gamma^n + \varepsilon^{3d+2}(2Rn)^d.$$

The first term on the right hand side is non-negative for ε small enough, then dividing by ε and taking $n(\varepsilon) \approx \varepsilon^{-3}$ yield

$$\int_{\Omega\times\Omega}|x-y|^2 d\gamma_\varepsilon \leq (1+\varepsilon)\varepsilon + \int_{\Omega\times\Omega}|x-y|^2 d\gamma^{n(\varepsilon)} + \varepsilon(2R)^d.$$

so that any w^*-cluster point of $(\gamma_\varepsilon)_\varepsilon$ is a solution of (2.1). □

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