Partial Differential Equations, Optimal Design and Numerics

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Controllability and stabilization of fluid-solid structure models

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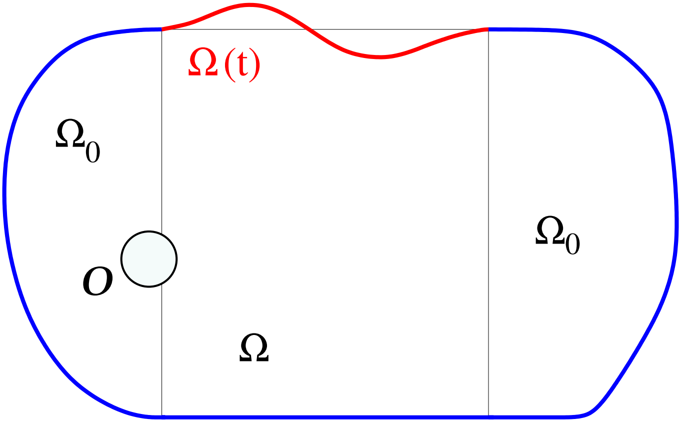
A Fluid–Structure system with a control in the fluid equation



Problem 1.

(Locally) Stabilize (with an arbitrary decay rate) the coupled system (NSE + Damped Beam Eq.) with a control acting only in the fluid equation.

Problem 2.

Control to zero a system coupling the LNSE + Finite Dim. Beam Eq. (Finite Dimensional approximation of the Beam Equation)

Motivation. Better understanding for the control of fluid–structure systems.

Known results

• Null controllabilty of a system coupling NSE + Rigid Body (2007, Imanuvilov & Takahashi, Boulakia & Osses).

• Null controllability of Stokes Eq. + Helmholtz Eq. (2009, R. & Vanninathan) $\longrightarrow$ Problem 2.

• Stabilization of NSE + Damped Beam Eq. with a force acting only in the beam equation for rectangular type domain (2008, R.) $\longrightarrow$ Problem 1.

Notation

Domain occupied by the fluid at time t

$$\Omega_{\eta(t)} = \Big\{(x,y) \mid x \in (0,L),\ 0 < y < 1 + \eta(x,t)\Big\} \cup \Omega_0.$$

Boundary corresponding to the structure at time t

$$\Gamma_{s,\eta(t)} = \Big\{(x,y) \mid x \in (0,L),\ y = 1 + \eta(x,t)\Big\}.$$

The space-time domain and boundary

$$\widetilde{Q}_T = \bigcup_{t\in(0,T)} \Omega_{\eta(t)}, \quad \widetilde{\Sigma}_s^T = \bigcup_{t\in(0,T)} \Gamma_{s,\eta(t)}, \quad \Sigma_0^T = \Gamma_0 \times (0,T).$$

We have

$$0 = \int_{\Omega_{\eta(t)}} \mathrm{div}\, u = \int_{\Gamma_{s,\eta(t)}} u(t) \cdot n(t) = \int_0^L \eta_t = \int_{\Gamma_s} \eta_t,$$

because

$$n(t) = \left(\frac{-\eta_x}{\sqrt{1+\eta_x^2}}, \frac{1}{\sqrt{1+\eta_x^2}} \right)^T.$$

We choose $\eta_{1,0}$ and $\eta_{2,0}$ the In. Cond. in

$$L_0^2(\Gamma_s) = \left\{ \eta \mid \int_{\Gamma_s} \eta = 0 \right\},$$

and we denote by M the orthogonal projection on $L_0^2(\Gamma_s)$.

The fluid equation

$$u_t + (u \cdot \nabla)u - \operatorname{div}\sigma(u,p) = f\,\chi_{\mathcal{O}}, \quad \operatorname{div} u = 0 \quad \text{in } \widetilde{Q}_\infty,$$

$$u = \eta_t \vec{e}_2 \ \text{ on } \widetilde{\Sigma}^\infty, \quad u = 0 \ \text{ on } \Sigma_0^\infty, \quad u(0) = u_0 \text{ in } \Omega_0,$$

$$\sigma(u,p) = \nu(\nabla u + \nabla u^T) - p\,I.$$

The structure equation

$$\eta_{tt} - \beta\eta_{xx} - \delta\eta_{txx} + \alpha M\eta_{xxxx} = M(p + H(u,\eta)) \ \text{ on } \Sigma_s^\infty,$$

$$\eta = 0 \quad \text{and} \quad \eta_x = 0 \ \text{ on } \{0, L\} \times (0,\infty)$$

$$\eta(0) = \eta_{1,0} \quad \text{and} \quad \eta_t(0) = \eta_{2,0} \ \text{ in } \Gamma_s,$$

and

$$H(u,\eta) = -\nu(\nabla u + \nabla u^T)(-\eta_x \vec{e}_1 + \vec{e}_2) \cdot \vec{e}_2.$$

The nonlinear and linearized models

• First change of variable $\longrightarrow$ Fixed domain Ω.

• Second change of variable to take into account the exponential decay rate $-\omega$

• Writing of the linearized model

Well posedness of the nonlinear system

- H. Beirao da Veiga (2004), 2D, Existence and uniqueness of local strong solution with small data.

- A. Chambolle, B. Desjardins, M. J. Esteban, C. Grandmont (2005), 3D, Existence of weak solutions.

- M. Guidorzi, M. Padula, P. I. Plotnikov (2008), 2D, Existence of weak solutions.

- J. Lequeurre (2009), 2D, Existence and uniqueness of local strong solution without smallness condition on the data.

We make the change of variable

$$(x, y) \longmapsto (x, z) = \left(x, \frac{y}{1 + \eta(x, t)} \right),$$

transforms $\Omega_{\eta(t)}$ onto $\Omega = (0, L) \times (0, 1) \cup \Omega_0$. Setting

$$\hat{u}(x, z, t) = u(x, y, t), \quad \hat{p}(x, z, t) = p(x, y, t),$$

the nonlinear system is rewritten in the form

$$\hat{u}_t + (\hat{u} \cdot \nabla)\hat{u} - \nu \Delta \hat{u} - \nabla \hat{p} = \hat{F}(\eta, \hat{u}, \nabla \hat{p}), \quad \mathrm{div}\, \hat{u} = \hat{G}(\eta, \hat{u})$$

$$\hat{u} = \eta_t \vec{e}_2 \text{ on } \Sigma_s^\infty, \quad \hat{u} = 0 \text{ on } \Sigma_0^\infty, \quad \hat{u}(0) = u_0 \text{ in } \Omega,$$

$$\eta_{tt} - \beta \eta_{xx} - \delta \eta_{txx} + \alpha \eta_{xxxx} = \hat{p} + \hat{H}(\hat{u}, \eta) \text{ on } \Sigma_s^\infty,$$

$$\eta(0) = \eta_{1,0} \quad \text{and} \quad \eta_t(0) = \eta_{2,0} \text{ in } \Gamma_s,$$

where

$$
\begin{aligned}
&\hat{F}(\eta, \hat{u}, \nabla \hat{p}) \\
&= -\eta \hat{u}_t + \left(z\eta_t + \nu z \left(\tfrac{\eta_x^2}{1+\eta} - \eta_{xx} \right) \right) \hat{u}_z \\
&\quad + \nu \left(-2z\eta_x \hat{u}_{xz} + \eta \hat{u}_{xx} + \left(\tfrac{z^2\eta_x^2 - \eta}{1+\eta} \right) \hat{u}_{zz} \right) \\
&\quad + z(\eta_x \hat{p}_z - \eta \hat{p}_x)\vec{e}_1 - (1+\eta)\hat{u}_1 \hat{u}_x + (z\eta_x \hat{u}_1 - \hat{u}_2)\hat{u}_z,
\end{aligned}
$$

$$
\hat{G}(\eta, \hat{u}) = -\eta \hat{u}_{1,x} + z\eta_x \hat{u}_{1,z} = \mathrm{div}\left(-\eta \hat{u}_1 \vec{e}_1 + z\eta_x \hat{u}_1 \vec{e}_2 \right),
$$

and

$$
\hat{H}(\hat{u}, \hat{p}, \eta) = \nu \left(\tfrac{\eta_x}{1+\eta} \hat{u}_{1,z} + \eta_x \hat{u}_{2,x} - \tfrac{2+\eta_x^2}{1+\eta} \hat{u}_{2,z} \right).
$$

For $-\omega < 0$, we make the following change of variables:

$$\tilde{u} = e^{\omega t}\hat{u}, \quad \tilde{p} = e^{\omega t}\hat{p}, \quad \tilde{\eta}_1 = e^{\omega t}\eta_1, \quad \tilde{\eta}_2 = e^{\omega t}\eta_2.$$

The system is transformed into

$$\tilde{u}_t + e^{-\omega t}(\tilde{u}\cdot\nabla)\tilde{u} - \nu\Delta\tilde{u} - \nabla\tilde{p} - \omega\tilde{u} = e^{-\omega t}\tilde{F}(\tilde{\eta}_1, \tilde{\eta}_2, \tilde{u}, \nabla\tilde{p}),$$

$$\operatorname{div}\tilde{u} = e^{-\omega t}\tilde{G}(\tilde{\eta}_1, \tilde{u}) \quad \text{in } Q_\infty,$$

$$\tilde{u} = \tilde{\eta}_2\vec{e}_2 \ \text{ on } \Sigma^s_\infty, \quad \tilde{u} = 0 \ \text{ on } \Sigma^0_\infty, \quad \tilde{u}(0) = u_0 \text{ in } \Omega,$$

$$\tilde{\eta}_{1,t} = \tilde{\eta}_2 + \omega\tilde{\eta}_1 \quad \text{on } \Sigma^s_\infty,$$

$$\tilde{\eta}_{2,t} - \omega\tilde{\eta}_2 - \beta\tilde{\eta}_{1,xx} - \delta\tilde{\eta}_{2,xx} + \alpha\tilde{\eta}_{1,xxxx}$$

$$= \tilde{p} - 2\nu\tilde{u}_{2,z} + e^{-\omega t}\tilde{H}(\tilde{u}, \tilde{\eta}_1) + \tilde{f} \ \text{ on } \Sigma^\infty_s,$$

$$\tilde{\eta}_1(0) = \eta_{1,0} \quad \text{and} \quad \tilde{\eta}_2(0) = \eta_{2,0} \ \text{ in } \Gamma_s.$$

The linearized system (around 0) is

$$v_t - \operatorname{div} \sigma(v, p) - \omega v = 0,$$

$$\operatorname{div} v = 0 \quad \text{in } Q_\infty,$$

$$v = \eta_2 \vec{e}_2 \text{ on } \Sigma_s^\infty, \quad v = 0 \text{ on } \Sigma_0^\infty, \quad v(0) = v_0 \text{ in } \Omega,$$

$$\eta_{1,t} = \eta_2 + \omega\eta_1 \quad \text{on } \Sigma_s^\infty,$$

$$\eta_{2,t} - \omega\eta_2 - \beta\eta_{1,xx} - \delta\eta_{2,xx} + \alpha\eta_{1,xxxx} = p + f \text{ on } \Sigma_s^\infty,$$

$$\eta_1(0) = \eta_{1,0} \quad \text{and} \quad \eta_2(0) = \eta_{2,0} \text{ in } \Gamma_s.$$

Observe that

$$v_{1,x} + v_{2,z} = 0 \quad \text{implies} \quad v_{2,z}|_{\Gamma_s} = 0.$$

Plan of the talk

1. Local stabilization

Stabilizability of the linearized model

- The semigroup is analytic.
- The resolvent is compact.
- Approximate controllability of the linearized system.

Local feedback stabilization of the nonlinear model

2. Null controllability of the Stokes eq. + Finite dimensional elastic structure

- Carleman estimates for the Stokes equations with NHBC

- Gradient estimate of the pressure

- Trace estimate of the pressure

- Elimination of the structure deformation (Compactness argument)

- Elimination of the local estimate of the pressure (Method by Gonzales-Burgos, Guerrero, Puel)

Part 1. Definition of the semigroup

$$\mathbf{V}^0_n(\Omega) = \Big\{ y \in \mathbf{L}^2(\Omega) \mid \mathrm{div}\, y = 0,\ \ y \cdot n = 0 \text{ on } \Gamma \Big\},$$

$$\mathbf{L}^2(\Omega) = \mathbf{V}^0_n(\Omega) \oplus \mathrm{grad}\, H^1(\Omega),$$

$$P \ : \ \mathbf{L}^2(\Omega) \longmapsto \mathbf{V}^0_n(\Omega).$$

We denote by A_0 either the Stokes operator in $\mathbf{V}^0_n(\Omega)$ with domain

$$\mathbf{V}^2(\Omega) \cap \mathbf{V}^1_0(\Omega) = \mathbf{H}^2(\Omega) \cap \mathbf{H}^1_0(\Omega) \cap \mathbf{V}^0_n(\Omega),$$

or its extension to $(\mathbf{V}^2(\Omega) \cap \mathbf{V}^1_0(\Omega))'$ (by extrapolation) as an unbounded operator with domain $\mathbf{V}^0_n(\Omega)$.

For $\omega = 0$, the equation satisfied by v can be split into two equations

$$Pv_t = A_0 Pv + (-A_0) PD(\eta_2 \vec{e}_2 \, \chi_{\Gamma_s}), \quad v(0) = v_0 \quad \text{in} \quad \Omega,$$

$$(I - P)v = (I - P)D(\eta_2 \vec{e}_2 \, \chi_{\Gamma_s}) = \nabla q,$$

$$(I - P)v_t(t) = (I - P)D(\eta_{2,t}(t) \vec{e}_2 \, \chi_{\Gamma_s}),$$

$$(I - P)v_0 = (I - P)v(0) \qquad \text{i.e.} \qquad v_0 \cdot \vec{e}_2 = \eta_2^0 \quad \text{on} \quad \Gamma_s,$$

and $D(\eta_2 \vec{e}_2 \, \chi_{\Gamma_s}) = w$ is the solution to

$$-\nu \Delta w + \nabla \rho = 0, \quad \operatorname{div} w = 0 \text{ in } \Omega, \quad w = (\eta_2 \vec{e}_2 \, \chi_{\Gamma_s}) \text{ on } \Gamma.$$

Idea: Eliminate $(I - P)v$

The pressure $p = \pi - q_t$ where

$$\Delta \pi = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial \pi}{\partial n} = \Delta P v \cdot n \quad \text{on} \quad \Gamma,$$

$$\Delta q(t) = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial q}{\partial n} = \eta_2 \quad \text{on} \quad \Gamma_s, \quad \frac{\partial q}{\partial n} = 0 \quad \text{on} \quad \Gamma_0.$$

Let $N \in \mathcal{L}(L_0^2(\Gamma_s), H^{3/2}(\Omega))$ and $N_0 \in \mathcal{L}(H^{-1/2}(\Gamma), H^1(\Omega))$ be

$$N_0(\Delta P v \cdot n) = \pi \quad \text{and} \quad N(\eta_2) = q,$$

and let $\gamma_s \in \mathcal{L}(H^1(\Omega), L_0^2(\Gamma_s))$ be

$$\gamma_s q = q|_{\Gamma_s} - \frac{1}{|\Gamma_s|} \int_{\Gamma_s} q.$$

For $\omega = 0$, we rewrite the system in the form

$$Pv_t = A_0 Pv + (-A_0)PD(\eta_2 \vec{e}_2\, \chi_{\Gamma_s}), \quad v(0) = v_0 \quad \text{in} \quad \Omega,$$

$$(I - P)v = (I - P)D(\eta_2 \vec{e}_2\, \chi_{\Gamma_s}) = \nabla q,$$

$$\eta_{1,t} = \eta_2,$$

$$\eta_{2,t} - \beta\eta_{1,xx} - \delta\eta_{2,xx} + \alpha\eta_{1,xxxx}$$
$$= -\gamma_s N(\eta_{2,t}) + \gamma_s N_0(\Delta Pv \cdot n) + f \quad \text{on } \Sigma^s_\infty,$$

$$\eta_1(0) = \eta_{1,0} \quad \text{and} \quad \eta_2(0) = \eta_{2,0} \quad \text{in } \Gamma_s.$$

The equation satisfied by η_2 is

$$(I + \gamma_s N)\eta_{2,t} - \beta\eta_{1,xx} - \delta\eta_{2,xx} + \alpha\eta_{1,xxxx}$$
$$= \gamma_s N_0(\Delta Pv \cdot n) + f \quad \text{on } \Sigma^s_\infty,$$

We can rewrite the system in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} Pv \\ \eta_1 \\ \eta_2 \end{pmatrix} = \mathcal{A}\begin{pmatrix} Pv \\ \eta_1 \\ \eta_2 \end{pmatrix},$$

with

$$\mathcal{A} =$$

$$\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & (I+\gamma_s N)^{-1} \end{pmatrix}\begin{pmatrix} A_0 & 0 & (-A_0)PD \\ 0 & 0 & I \\ \gamma_s N_0(\Delta(\cdot)\cdot n) & A_{\alpha,\beta} & \delta\Delta_s \end{pmatrix}$$

$$D(A_{\alpha,\beta}) = H^4(\Gamma_s)\cap H_0^2(\Gamma_s)\cap L_0^2(\Gamma_s), \quad A_{\alpha,\beta}\eta = \beta\eta_{xx} - \alpha\eta_{xxxx},$$

and

$$D(\mathcal{A}) =$$

$$\Big\{(Pv, \eta_1, \eta_2) \in \mathbf{V}^2_n(\Omega) \times (H^4 \cap H^2_0 \cap L^2_0)(\Gamma_s) \times (H^2_0 \cap L^2_0)(\Gamma_s) \;\Big|\; Pv|_\Gamma = -\nabla_\tau (I - P) D(\eta_2\, \chi_{\Gamma_s})\Big\}.$$

The compatibility condition

$$Pv|_\Gamma = -\nabla_\tau (I - P) D(\eta_2 \vec{e}_2\, \chi_{\Gamma_s})$$

is equivalent to

$$v = \eta_2\, \vec{e}_2 \chi_{\Gamma_s} \quad \text{on} \quad \Gamma.$$

Theorem. The operator $(\mathcal{A}, D(\mathcal{A}))$ is the infinitesimal generator of an analytic semigroup on $H = \mathbf{V}^0_n(\Omega) \times (H^2_0(\Gamma_s) \cap L^2_0(\Gamma_s)) \times L^2_0(\Gamma_s)$.

Proof. We have

$$\mathcal{A} = \begin{pmatrix} A_0 & 0 & (-A_0)PD \\ 0 & 0 & I \\ 0 & A_{\alpha,\beta} & \delta\Delta_s \end{pmatrix} + B = \mathcal{A}_1 + B,$$

with

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (I+\gamma_s N)^{-1}\gamma_s N_0(\Delta(\cdot)\cdot n)) & K_s\, A_{\alpha,\beta} & \delta K_s\, \Delta_s \end{pmatrix},$$

with $K_s = (I+\gamma_s N)^{-1} - I$.

We prove that the operator $(\mathcal{A}_1, D(\mathcal{A}))$ is the infinitesimal generator of an analytic semigroup on $H = \mathbf{V}_n^0(\Omega) \times (H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s)) \times L_0^2(\Gamma_s)$, and that $(B, D(B))$ is $\mathcal{A}_1$-bounded with relative bound zero:

$$D(B) \supset D((-\mathcal{A}_1)^\rho) \qquad \text{with } 0 < \rho < 1.$$

Theorem. The resolvent of $(\mathcal{A}, D(\mathcal{A}))$ is compact in $H = \mathbf{V}_n^0(\Omega) \times (H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s)) \times L_0^2(\Gamma_s)$.

Stabilizability/Approximate controllability

Theorem. The system SE + DBE is approximately controllable. It is also stabilizable with any decay rate.

Proof. The reachable set $R(T)$ at time T when f describes $L^2(0, T; L^2(\mathcal{O}))$ is dense.

If $(\phi_T, \zeta_{1,T}, \zeta_{2,T}) \in (R(T))^\perp$ then

$$\int_0^T \int_{\mathcal{O}} f\,\phi = 0 \quad \text{where } (\phi, \zeta) \quad \text{solves the adjoint syst.}$$

By the unique continuation result for the Stokes equation it yields

$$\phi = 0 \quad \text{in} \quad Q.$$

Next, coming back to the adjoint system we prove that $(\phi_T, \zeta_{1,T}, \zeta_{2,T}) = (0, 0, 0)$.

Feedback control of the linearized model

A feedback law can be obtained by solving an optimal control problem over the time interval $(0, \infty)$

$$(\mathcal{P}^{\infty}_{0,v_0,\eta_{1,0},\eta_{2,0}})$$
$$\inf \Big\{ I(v, \eta_1, \eta_2, f) \ \mid\ (v, \eta_1, \eta_2, f) \text{ satisfies } (LS),\ f \in L^2(\Sigma_s) \Big\},$$

where

$$I(v, \eta_1, \eta_2, f) = \frac{1}{2}\int_0^\infty \int_\Omega |v|^2\, dxdt + \frac{1}{2}\int_0^\infty \|\eta_1\|^2_{H^2_0(\Gamma_s)}\, dt + \frac{1}{2}\int_0^\infty \int_{\Gamma_s} |\eta_2|^2\, dxdt + \frac{1}{2}\int_0^\infty \|f(t)\|^2_{L^2(\Gamma_s)}\, dt.$$

Theorem. (Existence of a feedback operator)

For all $Pv_0 \in \mathbf{V}^0_n(\Omega)$, $\eta_{1,0} \in (H^2_0 \cap L^2_0)(\Gamma_s)$, and $\eta_{2,0} \in L^2_0(\Gamma_s)$,

Problem $(\mathcal{P}^\infty_{0,v_0,\eta_{1,0},\eta_{2,0}})$ admits a unique solution. There exists $\Pi \in \mathcal{L}(\mathbf{V}^0(\Omega) \times (H^2_0 \cap L^2_0)(\Gamma_s) \times L^2_0(\Gamma_s))$, obeying $\Pi = \Pi^*$, and the optimal control can be writen in feedback form

$$f(t) = -\Pi_1(v(t), \eta_1(t), \eta_2(t)).$$

Theorem. (Regularity of optimal solution)

If $Pv_0 \in \mathbf{V}_n^1(\Omega)$, $\eta_{1,0} \in (H^3 \cap H_0^2 \cap L_0^2)(\Gamma_s)$, $\eta_{2,0} \in (H_0^1 \cap L_0^2)(\Gamma_s)$, and if Pv_0 and $\eta_{2,0}$ satisfy the C.C.,

then the optimal solution to Problem $(\mathcal{P}^{\infty}_{0,v_0,\eta_{1,0},\eta_{2,0}})$ belongs to $\mathbf{H}^{2,1}(Q_\infty) \times H^{4,2}(\Sigma_s^\infty) \times H^{2,1}(\Sigma_s^\infty)$ and

$$\begin{aligned}&\|v\|_{\mathbf{H}^{2,1}(Q_\infty)} + \|\eta_1\|_{H^{4,2}(\Sigma_s)} + \|\eta_2\|_{H^{2,1}(\Sigma_s)}\\ &\le C\big(\|Pv_0\|_{\mathbf{V}_n^1(\Omega)} + \|\eta_{1,0}\|_{H^3(\Gamma_s)\cap H_0^2(\Gamma_s)} + \|\eta_{2,0}\|_{H_0^1(\Gamma_s)}\big).\end{aligned}$$

Local feedback stabilization of the nonlinear model

The nonhomogeneous closed loop linear system

$$v_t - \nu\Delta v - \nabla p - \omega v = -\Pi_1(v, \eta_1, \eta_2)\chi_{\mathcal{O}} + e^{-\omega t}\widetilde{F},$$

$$\mathrm{div}\, v = e^{-\omega t}\widetilde{G} = e^{-\omega t}\mathrm{div}\, \widetilde{w} \quad \text{in } Q_\infty,$$

$$v = \eta_2 \vec{e}_2 \text{ on } \Sigma_s^\infty, \quad v = 0 \text{ on } \Sigma_0^\infty, \quad v(0) = v_0 \text{ in } \Omega,$$

$$\eta_{1,t} = \eta_2 + \omega\eta_1 \quad \text{on } \Sigma_s^\infty,$$

$$\eta_{2,t} - \omega\eta_2 - \beta\eta_{1,xx} - \delta\eta_{2,xx} + \alpha\eta_{1,xxxx} = p - 2\nu v_{2,z} + e^{-\omega t}\widetilde{H} \text{ on } \Sigma_s^\infty$$

$$\eta_1(0) = \eta_{1,0} \quad \text{and} \quad \eta_2(0) = \eta_{2,0} \text{ in } \Gamma_s.$$

Theorem.

If $Pv_0 \in \mathbf{V}^1_n(\Omega)$, $\eta_{1,0} \in H^3(\Gamma_s) \cap H^2_0(\Gamma_s)$, $\eta_{2,0} \in H^1_0(\Gamma_s)$ + C.C., $\widetilde{F} \in L^2(Q_\infty)$, $\widetilde{w} \in \mathbf{H}^{2,1}(Q_\infty)$, $\widetilde{H} \in L^2(0,\infty; L^2(\Gamma_s))$,

then the nonhomogeneous closed loop linear system admits a unique solution belonging to $\mathbf{H}^{2,1}(Q_\infty) \times H^{4,2}(\Sigma_s) \times H^{2,1}(\Sigma_s)$, the pressure belongs to $L^2(0,\infty; H^1(\Omega))$, and

$$
\begin{aligned}
&\|v\|_{\mathbf{H}^{2,1}(Q_\infty)} + \|p\|_{L^2(0,\infty;H^1(\Omega))} + \|\eta_1\|_{H^{4,2}(\Sigma_s)} \\
&\leq C\big(\|Pv_0\|_{\mathbf{V}^1_n(\Omega)} + \|\eta_{1,0}\|_{H^3(\Gamma_s)\cap H^2_0(\Gamma_s)} + \|\eta_{2,0}\|_{H^1_0(\Gamma_s)} \\
&\qquad + \|e^{-\omega\cdot}\widetilde{F}\|_{L^2(Q_\infty)} + \|e^{-\omega\cdot}\widetilde{w}\|_{\mathbf{H}^{2,1}(Q_\infty)} \\
&\qquad + \|e^{-\omega\cdot}\widetilde{H}\|_{L^2(\Sigma_s)}\big).
\end{aligned}
$$

Theorem. There exist $\mu_0 > 0$ and an increasing function θ_0 from $\mathbb{R}^+$ into itself such that if $0 < \mu \le \mu_0$ and

$$\|v_0\|_{\mathbf{V}^1_n(\Omega)} + \|\eta_{1,0}\|_{(H^3\cap H^2_0)(\Gamma_s)} + \|\eta_{2,0}\|_{H^1_0(\Gamma_s)} \le \theta_0(\mu) \qquad + C.C.,$$

then the nonlinear closed loop system admits a unique solution in the set

$$\Big\{\|e^{-\omega(\cdot)}v\|_{\mathbf{H}^{2,1}(Q_\infty)}+\|e^{-\omega(\cdot)}\eta_1\|_{H^{4,2}(\Sigma_s^\infty)}+\|e^{-\omega(\cdot)}\eta_2\|_{H^{2,1}(\Sigma_s^\infty)} \le \mu\Big\},$$

and

$$\|v(t)\|_{\mathbf{H}^1(\Omega)} + \|\eta_1(t)\|_{(H^3\cap H^2_0)(\Gamma_s)} + \|\eta_2(t)\|_{H^1_0(\Gamma_s)}$$

$$\le Ce^{-\omega\, t}\big(\|v_0\|_{\mathbf{V}^1_n(\Omega)} + \|\eta_{1,0}\|_{(H^3\cap H^2_0)(\Gamma_s)} + \|\eta_{2,0}\|_{H^1_0(\Gamma_s)}\big).$$

Part 2. A model - Coupling S.E. + F.D. beam Eq.

$$
\begin{array}{lll}
v_t - \nu\Delta v + \nabla p = f\,\chi_{\mathcal{O}} \quad \text{and} \quad \operatorname{div} v = 0 & \text{in} & Q, \\
v = 0 & \text{on} & \Sigma_0, \\
v = \eta_t\,\vec{e}_2 & \text{on} & \Sigma_s, \\
v(0) = v_0 & \text{in} & \Omega, \\
\eta_{tt} - \beta\eta_{xx} - \delta\eta_{txx} + \alpha\eta_{xxxx} = Mp & \text{in} & \Sigma_s,
\end{array}
$$

which can be approximated by

$$
\begin{array}{lll}
\eta = \sum_{i=1}^{N} q_i(t)\,\zeta_i(x) = q(t)\cdot\zeta & \text{with} & \zeta = (\zeta_1, \cdots, \zeta_N)^T, \\
v = q'\cdot\zeta\,\vec{e}_2 & \text{on} & \Sigma_s, \\
q'' + Bq' + Aq = \displaystyle\int_{\Gamma_s} p\,\zeta & \text{in} & (0, T).
\end{array}
$$

Stokes equation coupled with a finite dimensional elastic structure

$$
\begin{aligned}
&v' - \nu\Delta v + \nabla p = f\,\chi_{\mathcal{O}} \quad \text{and} \quad \operatorname{div} v = 0 && \text{in} \quad Q,\\
&v = 0 && \text{on} \quad \Sigma_0,\\
&v = q' \cdot \zeta\, \vec{e}_2 && \text{on} \quad \Sigma_s,\\
&v(0) = v_0 && \text{in} \quad \Omega,\\
&q'' + Bq' + Aq = \int_{\Gamma_s} p\,\zeta && \text{in} \quad (0, T),\\
&q(0) = q_0 \qquad \text{and} \qquad q'(0) = q_1 && \text{in} \quad \mathbb{R}^N.
\end{aligned}
$$

With e.g.

$$
-\beta\zeta_{i,xx} + \alpha\zeta_{i,xxxx} = \lambda_i\,\zeta_i, \quad \text{and} \quad BC \quad \text{and} \quad \int_{\Gamma_s} \zeta_i = 0.
$$

Theorem.

If $v_0 \in \mathbf{L}^2(\Omega)$, $\operatorname{div} v_0 = 0$, $q_0 \in \mathbb{R}^N$, $q_1 \in \mathbb{R}^N$, $v_0 \cdot n = q_1 \cdot n$ on Γ_s, $v_0 \cdot n = 0$ on Γ_0, then there exists a control $f \in L^2(0, T; L^2(\mathcal{O}))$, such that the solution (v, q, q') to SE + FDBE obeys

$$(v(T), q(T), q'(T)) = (0, 0, 0).$$

Carleman inequality for the adjoint system

$$
\begin{aligned}
&-\phi' - \Delta\phi + \nabla\pi = 0 \quad \text{and} \quad \operatorname{div}\phi = 0 && \text{in} \quad Q,\\
&\phi = 0 && \text{on} \quad \Sigma_0,\\
&\phi = -r' \cdot \zeta\, \vec{e}_2 && \text{on} \quad \Sigma_s,\\
&\phi(T) = \phi_T && \text{in} \quad \Omega,\\
&r'' - B^T r' + Ar = \int_{\Gamma_s} \pi\, \zeta && \text{in} \quad (0, T),\\
&r(T) = r_{0,T} \qquad \text{and} \qquad r'(T) = r_{1,T} && \text{in} \quad \mathbb{R}^N.
\end{aligned}
$$

We have to take different weight functions

$$\eta(x) > 0 \quad \text{for all } x \in \overline{\Omega},$$
$$\eta(x) = C_{\Gamma_e} \quad \text{and } \partial_n \eta \le 0 \quad \text{for } x \in \Gamma_e,$$
$$|\nabla \eta(x)| > 0 \quad \text{for all } x \in \overline{\Omega \setminus \mathcal{O}_0},$$

with $\mathcal{O}_0 \subset\subset \mathcal{O}$,

$$\beta(x,t) = \frac{e^{5\lambda m \|\eta\|_\infty/4} - e^{\lambda(\eta + m\|\eta\|_\infty)}}{t^k(T-t)^k}.$$

$$\xi(x,t) = \frac{e^{\lambda(\eta + m\|\eta\|_\infty)}}{t^k(T-t)^k}, \quad m > 1, \quad k = 4.$$

Conjugate the Heat operator $e^{s\beta}(\partial_t - \Delta)e^{-s\beta}\phi$.

$$
\begin{aligned}
& s^{-1}\int_Q \xi^{-1}\, e^{-2s\beta}\,(|\phi'|^2+|\Delta\phi|^2) \\
& +s\lambda^2\int_Q \xi\; e^{-2s\beta}\,|\nabla\phi|^2 + s^3\lambda^4\int_Q \xi^3\, e^{-2s\beta}\,|\phi|^2 \\
& +s\lambda\int_{\Sigma_s} \xi\, e^{-2s\beta}\,|\partial_n\phi|^2 \;+ s^3\lambda^3\int_{\Sigma_s} \xi^3\, e^{-2s\beta}|\phi|^2 \\
& \leq C\left\{\int_Q e^{-2s\beta}|\nabla\pi|^2 + s^3\lambda^4\int_{\mathcal{O}\times(0,T)} \xi^3\, e^{-2s\beta}\,|\phi|^2 \right. \\
& \qquad \left. +\int_{\Sigma_s} e^{-2s\beta}|\phi'|^2 + \int_0^T e^{-2s\beta}\left|\int_{\Gamma_s}\pi\,\zeta\right|^2\right\},
\end{aligned}
$$

$$
\begin{aligned}
& s^{-1} \int_Q \xi^{-1}\, e^{-2s\beta}\, (|\phi'|^2 + |\Delta\phi|^2) \\
& + s\lambda^2 \int_Q \xi\ e^{-2s\beta}\, |\nabla\phi|^2 + s^3\lambda^4 \int_Q \xi^3\, e^{-2s\beta}\, |\phi|^2 \\
& + s\lambda \int_{\Sigma_s} \xi\, e^{-2s\beta}\, |\partial_n \phi|^2 \\
& + \int_0^T e^{-2s\beta}(|r''|^2 + |r|^2) \ + s^3\lambda^3 \int_0^T \xi^3\, e^{-2s\beta} |r'|^2 \\
& \leq C \Big\{ \int_Q e^{-2s\beta} |\nabla\pi|^2 + s^3\lambda^4 \int_{\mathcal{O}\times(0,T)} \xi^3\, e^{-2s\beta}\, |\phi|^2 + \int_0^T e^{-2s\beta} |r|^2 \\
& \qquad + \int_0^T e^{-2s\beta} \Big| \int_{\Gamma_s} \pi\, \zeta \Big|^2 \Big\},
\end{aligned}
$$

After eliminating the pressure gradient, we arrive at

$$
\begin{aligned}
&LHS \\
&\le C \left\{ s^2 \lambda^2 \int_{\mathcal{O}\times(0,T)} e^{-2s\beta} |\pi|^2 + s^3 \lambda^4 \int_{\mathcal{O}\times(0,T)} \xi^3 \; e^{-2s\beta} |\phi|^2 \right. \\
&\qquad \left. + \int_0^T e^{-2s\beta} |r|^2 + s^{1/2} \int_0^T |\xi^*|^{1/2} e^{-2s\beta^*} \|\pi(t)\|^2_{H^{1/2}(\Gamma)} \right\},
\end{aligned}
$$

with

$$
\xi^*(t) = \min_{\overline{\Omega}} \xi(\cdot, t) = \frac{e^{\lambda(\eta^* + m\|\eta\|_\infty)}}{t^k (T-t)^k}, \quad \eta^*(t) = \min_{\overline{\Omega}} \eta(\cdot, t),
$$

$$
\beta^*(t) = \max_{\overline{\Omega}} \beta(\cdot, t) = \frac{e^{5\,\lambda m \|\eta\|_\infty / 4} - e^{\lambda(\eta^* + m\|\eta\|_\infty)}}{t^k (T-t)^k}.
$$

Trace estimate of the pressure

$$\int_0^T |\xi^*|^{1/2} e^{-2s\beta^*} \|\pi(t)\|^2_{H^{1/2}(\Gamma)}.$$

We set

$$\mu(t) = |\xi^*|^{1/4} e^{-s\beta^*}, \quad \phi^* = \mu\,\phi, \quad \pi^* = \mu\,\pi.$$

The equation satisfied by $r^* = \mu\, r$ is not similar to that satisfied by r. The usual proof (FC-G-I-P, I-T, B-O) relies on an estimate of π^* in $L^2(0,T;H^1(\Omega))$.

We introduce $\phi_e = P\phi$, $\phi_s = (I-P)\phi$, $\pi = \pi_e + \pi_s$. We do not estimate directly π^* in $L^2(0,T;H^1(\Omega))$. We eliminate π_s in the equation. We estimate ϕ_e^*, ϕ_s^*, π_e^*, and r. We use

$$\mu\,\pi_s = \gamma_s N(\mu r'' \cdot n).$$

We show that $\phi_e = P\phi$, $\phi_s = (I-P)\phi$, π_e, obey

$$-\phi_e' - A\phi_e = (-A)PL(r'\chi_{\Gamma_s}), \quad \phi_e(T) = \phi_T,$$

$$(I-P)\phi = \phi_s = (I-P)L(r'\chi_{\Gamma_s}),$$

$$r'' - B^T r' + Ar = -\int_{\Gamma_s} \pi_e \zeta - \int_{\Gamma_s} \gamma_s N(r''\cdot\zeta)\zeta,$$

$$(I+K)r'' - B^T r' + Ar = -\int_{\Gamma_s} \pi_e \zeta,$$

$$r(T) = r_{0,T} \qquad \text{and} \qquad r'(T) = r_{1,T}\,,$$

with

$$K = K^* \geq 0, \quad Kr'' = \int_{\Gamma_s} \gamma_s N(r''\cdot\zeta)\zeta.$$

We first obtain an estimate of ϕ_e, π_e and r. The estimate for π_s is derived from the one of r.

After eliminating the pressure trace, we arrive at

$$LHS$$

$$\leq C\left\{s^2\,\lambda^2\int_{\mathcal{O}\times(0,T)}\xi^2\,e^{-2s\beta}|\pi|^2+s^3\lambda^4\int_{\mathcal{O}\times(0,T)}\xi^3\,e^{-2s\beta}\,|\phi|^2 + \int_0^T \mu^2|r|^2\right\}.$$

Estimate of $\int_0^T \mu^2|r|^2$. Let us choose

$$0=T_0<T_1<T_2<\cdots<T_\ell<T_{\ell+1},$$

such that μ is monotone on the intervals (T_j, T_{j+1}).

Let E be the vector space of solutions to system satisfied by (ψ, π, r). We introduce the following subspace of E :

$$E_i = \Big\{(\psi, p, r) \in E \mid r(T_j) = 0 \quad \text{for all } 1 \leq j \leq \ell\Big\}.$$

The space E_i is of finite codimension and

$$\int_0^T \mu^2 \, |r|^2 \leq C \int_0^T \frac{e^{3\lambda(\eta + m\|\eta\|_\infty)}|_{\Gamma_s}}{t^{3k}(T-t)^{3k}} \; e^{-2s\beta}|_{\Gamma_s} \, |r'|^2.$$

if $(\psi, p, r) \in E_i$.

With a compactness argument we arrive at

$$LHS \leq C(s,\lambda)\left\{\int_{\mathcal{O}\times(0,T)} \xi^3\, e^{-2s\beta}\,|\phi|^2 + \int_{\mathcal{O}\times(0,T)} \xi^2\, e^{-2s\beta}\,|\pi|^2\right\}.$$

Next step.

i – Use two controls to obtain a null controllability result with a boundary control. (Gonzales Burgos - Guerrero - Puel, 2008).

ii – Prove that the second control is regular enough to be removed.

Control of the Fluid – Solid structure model with two controls

$$
\begin{array}{lll}
v' - \nu \Delta v + \nabla p = \chi_{\mathcal{O}} f, \quad \operatorname{div} v = \theta\, g & \text{in} & Q, \\
v = 0 & \text{on} & \Sigma_0, \\
v = q' \cdot \zeta\, \vec{e}_2 & \text{on} & \Sigma_s, \\
v(0) = v_0 & \text{in} & \Omega, \\
q'' + Bq' + Aq = \displaystyle\int_{\Gamma_s} p\, \zeta & \text{in} & (0, T), \\
q(0) = q_0 \qquad \text{and} \qquad q'(0) = q_1 & \text{in} & \mathbb{R}^2.
\end{array}
$$

where $\theta \in C_0^\infty(\mathcal{O})$, $\theta|_{\mathcal{O}_1} = 1$, $\theta \geq 0$, $\mathcal{O}_1 \subset\subset \mathcal{O}$. $\int_\Omega \theta\, g = 0.$

Minimize

$$J_\varepsilon(v,q,f,g) = \frac{1}{2\varepsilon}\|Pv(T)\|^2_{H^{-1}(\Omega)} + \frac{1}{2\varepsilon}|A^{1/2}q(T)|^2 + \frac{1}{2\varepsilon}|q'(T)|^2 + \frac{1}{2}\int_Q (\xi^{-3}\, e^{2s\beta}\, |f|^2 + \theta\, \xi^{-2}\, e^{2s\beta}\, |g|^2),$$

where (v,q,f,g) solve the state equation.

Introduce the adjoint system

$$
\begin{aligned}
&-\phi' - \nu\Delta\phi + \nabla\pi = 0 \quad \text{and} \quad \operatorname{div}\phi = 0 && \text{in} \quad Q,\\
&\phi = 0 && \text{on} \quad \Sigma_0,\\
&\phi = -r' \cdot \zeta\,\vec{e}_2 && \text{on} \quad \Sigma_s,\\
&\phi(T) = \frac{1}{\varepsilon}(-P\Delta)^{-1} P v_\varepsilon(T) && \text{in} \quad \Omega,\\
&r'' - B^T r' + Ar = \int_{\Gamma_s} \pi\,\zeta && \text{in} \quad (0,T),\\
&r(T) = -\frac{1}{\varepsilon} q_\varepsilon(T), \qquad r'(T) = -\frac{1}{\varepsilon} q'_\varepsilon(T) && \text{in} \quad \mathbb{R}^2.
\end{aligned}
$$

We first have

$$
f_\varepsilon = \xi^3\, e^{-2s\beta}\,\phi_\varepsilon \quad \text{and} \quad g_\varepsilon = -\xi^2\, e^{-2s\beta}\,\pi_\varepsilon + \frac{1}{\int_{\mathcal{O}}\theta} \int_{\mathcal{O}} \xi^2\, e^{-2s\beta}\,\pi_\varepsilon\,\theta\, dx.
$$

and

$$\frac{1}{\varepsilon}\|Pv_\varepsilon(T)\|^2_{H^{-1}(\Omega)} + \frac{1}{\varepsilon}|q_\varepsilon(T)|^2 + \frac{1}{\varepsilon}|q'_\varepsilon(T)|^2$$

$$+ \int_{\mathcal{O}\times(0,T)} \xi^3\, e^{-2s\beta}\, |\phi_\varepsilon|^2 + \int_{\mathcal{O}\times(0,T)} \xi^2\, e^{-2s\beta}\;\theta\, |\pi_\varepsilon|^2$$

$$= \int_\Omega \phi_\varepsilon(0)\, Pv_0 - r_\varepsilon(0)\cdot q_0 - r'_\varepsilon(0)\cdot q_1.$$

With the observability inequality, we have

$$\|\phi_\varepsilon(0)\|^2_{L^2} + |r_\varepsilon(0)|^2 + |r'_\varepsilon(0)|^2$$

$$\le c \int_{\mathcal{O}\times(0,T)} \xi^3\, e^{-2s\beta}\, |\phi_\varepsilon|^2 + c\int_{\mathcal{O}_1\times(0,T)} \xi^2\, e^{-2s\beta}\, |\pi_\varepsilon|^2.$$

From which we deduce the null controllability of the Stokes system with two controls, or with a boundary control.

Regularity of the control v_ε.

$$\|\theta\, g_\varepsilon\|_{H^1(0,T;H^1(\Omega))} \leq C.$$

Thus

$$\theta\, g \in H^1(0,T;H^1(\Omega)).$$

The equation

$$\operatorname{div} z = \theta\, g \quad \text{in } \mathcal{O}, \qquad z = 0 \text{ on } \partial\mathcal{O}_0,$$

admits a solution in $H^1(0,T;H_0^2(\mathcal{O}))$. Its extension $\tilde{z}$ by zero belongs to $H^1(0,T;H_0^2(\Omega))$ and

$$\tilde{z}(0) = \tilde{z}(T) = 0.$$

We set $v = Z + \tilde{z}$. The function (Z,p) is solution to the null controllability problem with one control equal to

$$f - (\partial_t - \nu\Delta)\tilde{z}.$$

References

H. Beirao da Veiga, On the existence of strong solutions to a coupled fluid-structure evolution system, JMFM 6 (2004), 21-52.

V. Barbu, Stabilization of a plane channel flow by wall normal controllers, Nonlinear Analysis 67 (2007), 2573-2588.

M. Boulakia and A. Osses, Local null controllability of a two-dimensional fluid-structure interaction problem, ESAIM COCV, 2008.

S. Chen, R. Triggiani, Characterization of fractional powers of certian operators arising in elastic systems and applications, JDE 88 (1990), 279-293.

C. Fabre, G. Lebeau, Prolongement unique des solutions de l'equation de Stokes, Comm. Partial Differential Equations 21 (1996), no. 3-4, 573-596.

A. V. Fursikov, Stabilization of the 3D Navier-Stokes system by feedabck boundary control, Disc. Cont. Dyn. Syst. 10 (2004), 289-314.

C. Grandmont, Existence of weak solutions for unsteady interaction of a viscous fluid with an elastic plate, 2007, preprint.

M. Gonzalez-Burgos, S. Guerrero and J.-P. Puel, Local exact controllability to the trajectories of the Boussinesq system via a fictitious control on the divergence equation, Communications on Pure and Applied Analysis 8 (2009), 311–333.

J.-L. Lions, E. Zuazua, Approximate controllability of a hydro-elastic coupled system, ESAIM COCV 1 (1995), 1-15.

A. Osses, J.-P. Puel, Approximate controllability for a hydro-elastic model in a rectangular domain, ISNM 133 (1999), 231-243.

A. Osses, J.-P. Puel, Approximate controllability of a linear model in solid-fluid interaction, ESAIM COCV 4 (1999), 497-513.

O. Imanuvilov, T. Takahashi, Exact controllability of a fluid-rigid body system, J. Math. Pures Appl., 87 (2007), 408-437.

J.-P. Raymond, Stokes and Navier-Stokes equations with nonhomogeneous boundary conditions, Ann. I. H. Poincaré – AN 24 (2007) 921-951.

J.-P. Raymond, Feedback stabilization of a fluid–structure model, preprint 2008.

J.-P. Raymond, M. Vanninathan, Null controllability of a heat - solid structure model, A.M.O., 59 (2009), 247-273.

J.-P. Raymond, M. Vanninathan, Null controllability of a fluid - solid structure model, preprint, 2009.