Anisotropic Total Variation, Cheeger sets and Applications.

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We look for a salient object in an image I.

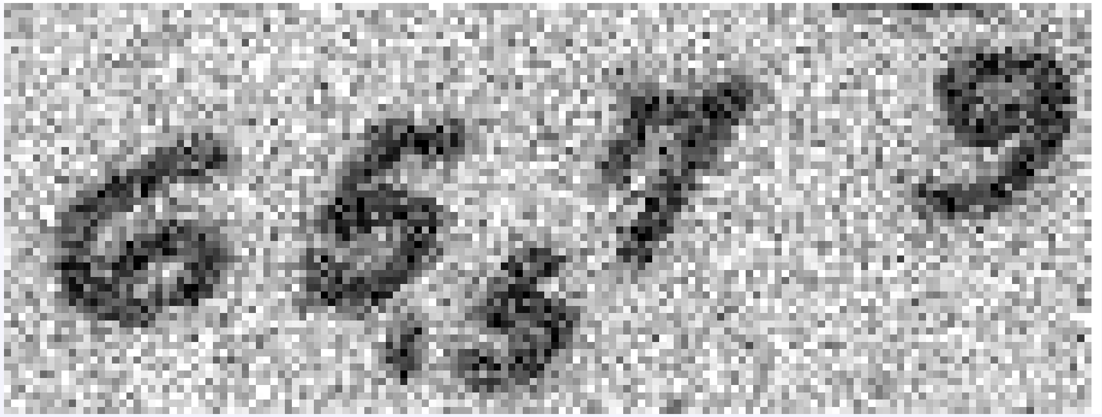


Figure 1: A noisy image

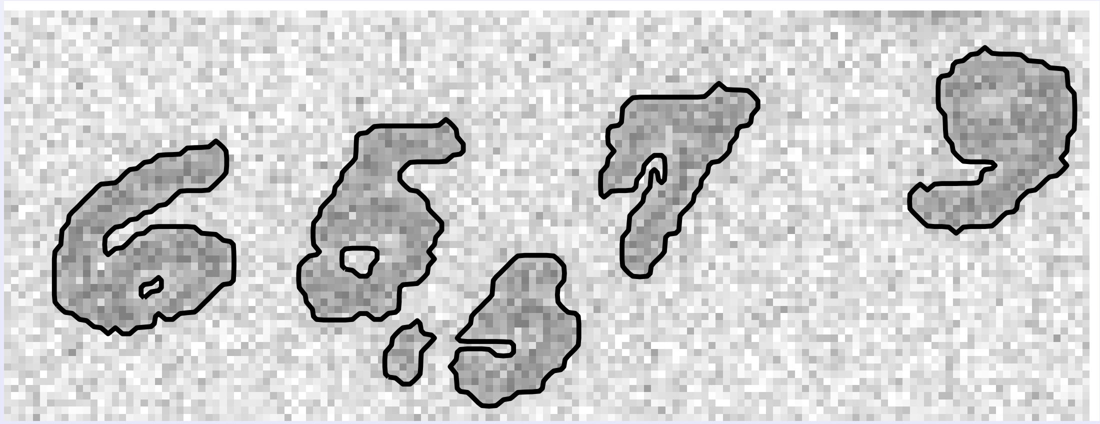


Figure 2: The contours computed using GAC's model

Introduction: Example of Active Contours

We look for a closed curve $C : [0, L(C)] \to \mathbb{R}^2$, $C = \partial^* E$, which minimizes the Riemann metric (GAC's model) plus an inflating force

$$\min_{E \subseteq \Omega} P_g(E) - \mu |E| = \int_{\partial^* E} g(x) d\mathcal{H}^{N-1} - \mu |E|$$

where the coefficient g depends on the image gradient:

$$g(x) = \frac{\mathbf{1}}{\mathbf{1} + |\nabla(\mathbf{G}_\sigma * \mathbf{I})(\mathbf{x})|^{\mathbf{2}}}.$$

This problem was first approached with a level set formulation leading to the PDE (V.C., R. Kimmel and G. Sapiro, IJCV, 1997):

$$\frac{\partial u}{\partial t} = |\nabla u| \text{div} \left(g(I) \frac{\nabla u}{|\nabla u|} \right) + \mu |\nabla u|.$$

Introduction: Example of Active Contours

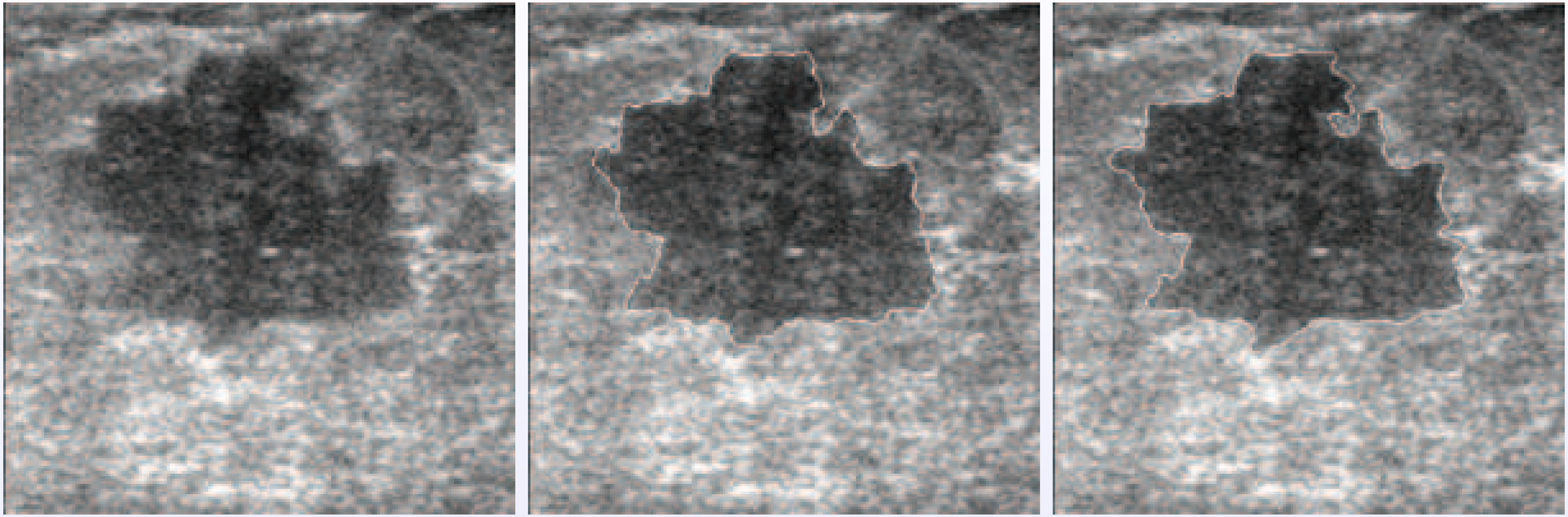


Figure 3: Segmenting the tumor of a breast echographic image (M. Alemán, L. Álvarez and V.C., 2005.)

Introduction: Example of Active Contours

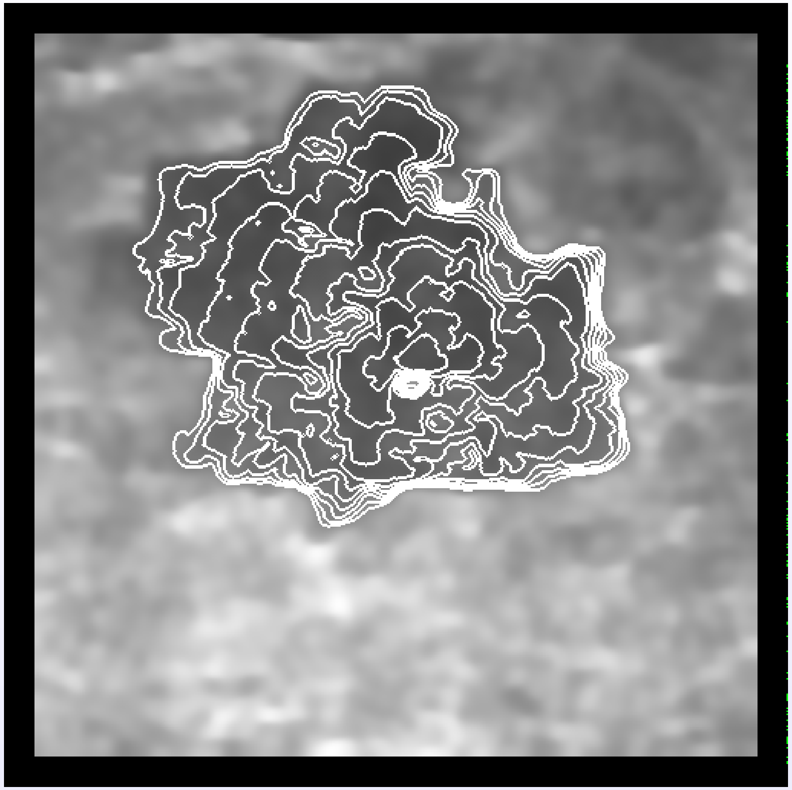


Figure 4: Several steps of the evolution of the curve.

Introduction: Region Based Active Contours

Inspired by the Mumford-Shah model, Chan-Vese proposed a region based data attachment term:

$$\min_{E\subseteq\Omega, c_1, c_2\in\mathbb{R}} P_g(F) + \lambda \int_E (I(x) - c_1)^2\, dx + \lambda \int_{\Omega\setminus E} (I(x) - c_2)^2\, dx, \quad (1)$$

where $\lambda > 0$.

Iterative solution: If the set E is fixed, then the minimum of the energy with respect to $c_1, c_2 \in \mathbb{R}$ gives us the values

$\overline{c}_1 = \frac{\int_E I(x)\, dx}{|E|}$ and $\overline{c}_2 = \frac{\int_{\Omega\setminus E} I(x)\, dx}{|\Omega\setminus E|}$, and then the problem is

$$\min_{E\subseteq\Omega} P_g(E) + \lambda \int_E ((I(x) - c_1)^2 - (I(x) - c_2)^2)\, dx.$$

Introduction: Edge linking problems

Let Γ be a set of curves (or surfaces) which can be computed by en edge detector (i.e. by thresholding the modulus of the gradient of the image).
Let $d_\Gamma(x)$ be the distance to Γ:

$$\min_{E \subseteq \Omega} P_{d_\Gamma}(E) - \mu|E| = \int_{\partial^* E} d_\Gamma(x) d\mathcal{H}^{N-1} - \mu|E|$$

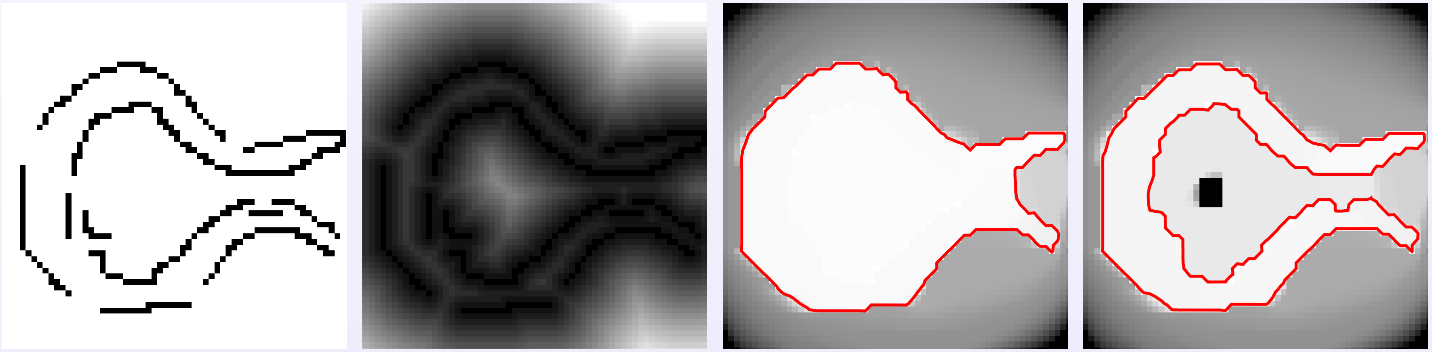


Figure 5: Segmentation and edge linking with barriers.

The total variation with respect to an anisotropy

$\phi : \Omega \times \mathbb{R}^N \to [0, \infty)$ is a *metric integrand* if ϕ is a Borel function satisfying the conditions:

$$\begin{aligned}
&\text{for a.e. } x \in \Omega, \text{ the map } \xi \in \mathbb{R}^N \to \phi(x,\xi) \text{ is convex,} \\
&\phi(x, t\xi) = |t|\phi(x,\xi) \qquad \forall x \in \Omega, \quad \forall \xi \in \mathbb{R}^N, \quad \forall t \in \mathbb{R}, \\
&\exists \Lambda > 0 \text{ s.t. } 0 \le \phi(x,\xi) \le \Lambda \|\xi\| \qquad \forall x \in \Omega, \quad \forall \xi \in \mathbb{R}^N.
\end{aligned} \tag{2}$$

Let

$$\mathcal{K}_\phi =:= \{\sigma \in X_\infty(\Omega) : \phi^0(x, \sigma(x)) \le 1 \text{ for a.e. } x \in \Omega, \, [\sigma \cdot \nu^\Omega] = 0\}.$$

Let $u \in L^1(\Omega)$. We define the ϕ-total variation of u in Ω as

$$\int_\Omega |Du|_\phi := \sup \left\{ \int_\Omega u \operatorname{div} \sigma \, dx : \sigma \in \mathcal{K}_\phi \right\}, \tag{3}$$

We set $BV_\phi(\Omega) := \{u \in L^1(\Omega) : \int_\Omega |Du|_\phi < \infty\}$.

The total variation with respect to an anisotropy

Let $\Omega \subset\subset Q \subseteq \mathbb{R}^N$ open bounded with Lipschitz boundary. Let $\phi : Q \times \mathbb{R}^N \to \mathbb{R}$ be a metric integrand continuous and coercive in a neighborhood of $Q \setminus \Omega$.

For $f \in L^2(\Omega)$, $h \in L^\infty(\Omega)$, $h(x) > 0$ a.e. in Ω, with $\int_\Omega 1/h\,dx < \infty$, $\lambda > 0$, let us consider the energy functional

$$\mathcal{E}_{\phi,\lambda}(u) := \int_\Omega |Du|_\phi + \frac{1}{2\lambda}\int_\Omega h(u-f)^2\,dx + \int_{\partial\Omega} \phi(x,\nu^\Omega)|u|\,d\mathcal{H}^{N-1}. \quad (4)$$

The E-L equation related to $\mathcal{E}_{\phi,\lambda}(u)$ is

$$hu - \lambda \mathrm{div}\,(\sigma) = hf \quad (5)$$

where $\sigma(x) \in \partial_\xi \phi(x, Du))$, i.e. $\phi^0(x,\sigma(x)) \leq 1$,
$\sigma \cdot Du = |Du|_\phi$,
$[\sigma \cdot \nu^\Omega] \in \mathrm{sign}(-u)\phi(x,\nu^\Omega(x))\ \mathcal{H}^{N-1}$-a.e..

ATV: Existence of solutions

Theorem

(i) The energy functional is lower semicontinuous.

(ii) There is a unique solution of the problem

$$(Q)_\lambda : \quad \min_{u \in BV_\phi(\Omega) \cap L^2(\Omega, h\, dx)} \quad \mathcal{E}_{\phi,\lambda}(u). \tag{6}$$

(iii) Assume that $f \in L^\infty(\Omega)$. There is a unique solution $u \in BV_\phi(\Omega) \cap L^2(\Omega, h\, dx)$ of the Euler-Lagrange equation. Moreover the solution $u \in L^\infty(\Omega)$ and minimizes $\mathcal{E}_{\phi,\lambda}(u)$.

Proof of (i) uses results of Amar - Bellettini and Bellettini - Bouchité - Fragalà.

ATV: relation with the geometric problem

Proposition Let $u_\lambda \in BV_\phi(\Omega) \cap L^2(\Omega, h\,dx)$ be the solution of the variational problem

$$\min_{u \in BV_\phi(\Omega)} \int_\Omega |Du|_\phi + \frac{1}{2\lambda} \int_\Omega h(u-1)^2\,dx + \int_{\partial\Omega} \phi(x, \nu^\Omega)|u|\,d\mathcal{H}^{N-1} \tag{7}$$

Then $0 \le u_\lambda \le 1$.
Let $E_s := \{u_\lambda \ge s\}$, $s \in (0,1]$. Then for any $s \in (0,1]$ we have

$$P_\phi(E_s) - \frac{(1-s)}{\lambda}|E_s|_h \le P_\phi(F) - \frac{(1-s)}{\lambda}|F|_h \tag{8}$$

for any $F \subseteq \Omega$.

Comment: Type of problem for active contours or edge linking problem.

Proof:

(A) Coarea formula:

$$\int_\Omega |Du|_\phi + \int_{\partial\Omega} \phi(x,\nu^\Omega)|u|\,d\mathcal{H}^{N-1} = \int_0^\infty P_\phi(\{u>s\})\,ds,$$

(B) Multiplying the E-L equation by u and integrating by parts :

$$\int_\Omega \sigma\cdot Du + \int_{\partial\Omega} \phi(x,\nu^\Omega)|u|\,d\mathcal{H}^{N-1} = \lambda\int_\Omega (1-u)uh\,dx.$$

(C) Multiplying the E-L equation by χ_{E_s} and integrating by parts :

$$\int_\Omega \sigma\cdot D\chi_{E_s} - \int_{\partial\Omega} [\sigma\cdot\nu^\Omega]\chi_{E_s}\,d\mathcal{H}^{N-1} = \lambda\int_\Omega (1-u)h\chi_{E_s}\,dx.$$

Proof:

(D)

$$\int_\Omega |Du|_\phi + \int_{\partial\Omega} \phi(x,\nu^\Omega)|u|\,d\mathcal{H}^{N-1} = \int_\Omega \sigma\cdot Du + \int_{\partial\Omega} \phi(x,\nu^\Omega)|u|\,d\mathcal{H}^{N-1}$$

$$\text{by } (B) = \lambda \int_\Omega (1-u)uh\,dx = \lambda \int_0^\infty \int_\Omega (1-u)h\chi_{E_s}\,dx\,ds$$

$$\text{by } (C) = \int_0^\infty \int_\Omega \sigma\cdot D\chi_{E_s}\,ds - \int_0^\infty \int_{\partial\Omega} [\sigma\cdot\nu^\Omega]\chi_{E_s}\,d\mathcal{H}^{N-1}\,ds.$$

$$\leq \int_0^\infty \int_\Omega |D\chi_{E_s}|_\phi\,ds + \int_0^\infty \int_{\partial\Omega} \phi(x,\nu^\Omega(x))\chi_{E_s}\,d\mathcal{H}^{N-1}\,ds =$$

$$= \int_0^\infty \int_{\mathbb{R}^N} |D\chi_{E_s}|_\phi\,ds = \int_0^\infty P_\phi(E_s)\,ds.$$

$$\text{by } (A) = \int_\Omega |Du|_\phi + \int_{\partial\Omega} \phi(x,\nu^\Omega)|u|\,d\mathcal{H}^{N-1}$$

Proof:

Conclusion:

$$P_\phi(E_s) = \int_\Omega \sigma \cdot D\chi_{E_s} - \int_{\partial\Omega} [\sigma \cdot \nu^\Omega]\chi_{E_s}\, d\mathcal{H}^{N-1}.$$

Let $F \subseteq \Omega$ be a set of ϕ-finite perimeter. Multiply (E-L) by $\chi_F - \chi_{E_s}$:
Then

$$P_\phi(F) - P_\phi(E_s) \geq -\int_\Omega \operatorname{div} \sigma\, (\chi_F - \chi_{E_s})\, dx$$

$$= \lambda \int_\Omega (1-u)h(\chi_F - \chi_{E_s}) = \lambda \int_\Omega ((1-s) + (s-u))h(\chi_F - \chi_{E_s}).$$

Since $(s-u)(\chi_F - \chi_{E_s}) \geq 0$, we have

$$P_\phi(F) - P_\phi(E_s) \geq \lambda(1-s) \int_\Omega h(\chi_F - \chi_{E_s}) = \lambda(1-s)(|F|_h - |E_s|_h).$$

The ϕ-Cheeger set

Proposition Let $\alpha^{-1}, \beta^{-1} > \frac{1}{\|h\chi_\Omega\|_*}$. Then $\{u_\alpha \geq \|u_\alpha\|_\infty\} = \{u_\beta \geq \|u_\beta\|_\infty\}$, and

$$\frac{P(\{u_\alpha \geq \|u_\alpha\|_\infty\})}{|\{u_\alpha \geq \|u_\alpha\|_\infty\}|_h} = \text{Cheeger constant.} \tag{9}$$

The set $\{u_\alpha \geq \|u_\alpha\|_\infty\}$ is the maximal ϕ-Cheeger set of Ω.

In the Euclidean case, if Ω is convex, the Cheeger set is unique (C-Chambolle-Novaga, Alter-C).

Example: the GAC model, the edge linking problem

Consider the problem

$$\min_{u\in U} \frac{\|u-f\|_U^2}{2} + \lambda J_g(u), \quad \text{where} \quad J_g(u) = \sum_{0\le i,j<N} g_{i,j}|(\nabla u)_{i,j}|. \tag{10}$$

We solve the dual problem

$$\min_{w\in U} \frac{\|w-\lambda^{-1}f\|_U^2}{2} + \frac{1}{\lambda} J_g^*(w), \tag{11}$$

$$J_g^*(w) = \begin{cases} 0, & \text{if } w \in \mathcal{K}_g \\ +\infty, & \text{otherwise} \end{cases} \quad \text{with} \quad \mathcal{K}_g = \{-\mathrm{div}\xi : |\xi_{i,j}| \le g_{i,j}\ \forall i,j\}. \tag{12}$$

Therefore (11) is a projection over the set $\mathcal{K}_g$

$$\min_{w\in\mathcal{K}_g} \|w-\lambda^{-1}f\|_U^2. \tag{13}$$

Numerical scheme

Note that any solution $w \in \mathcal{K}_g$, must satisfy $w_{i,j} = -\mathrm{div}(g_{i,j} p_{i,j})$ with $|p_{i,j}| \leq 1$.

Introducing the Lagrange multipliers $\alpha_{i,j}$ for the constraint we obtain the functional

$$\mathcal{F}(p, \alpha) = \sum_{0 \leq i,j < N} |\mathrm{div}(gp)_{i,j} + b_{i,j}|^2 + \sum_{0 \leq i,j < N} \alpha_{i,j}(|p_{i,j}|^2 - 1).$$

Numerical scheme:

$$p^{n+1} = \frac{p^n + \tau \left\{ g \nabla [\mathrm{div}(gp^n) + \lambda^{-1} f] \right\}}{1 + \tau |g \nabla [\mathrm{div}(gp^n) + \lambda^{-1} f]|} \tag{14}$$

Solution: $u = f + \lambda h^{-1} \mathrm{div}(gp)$

Cheeger sets for the euclidean perimeter

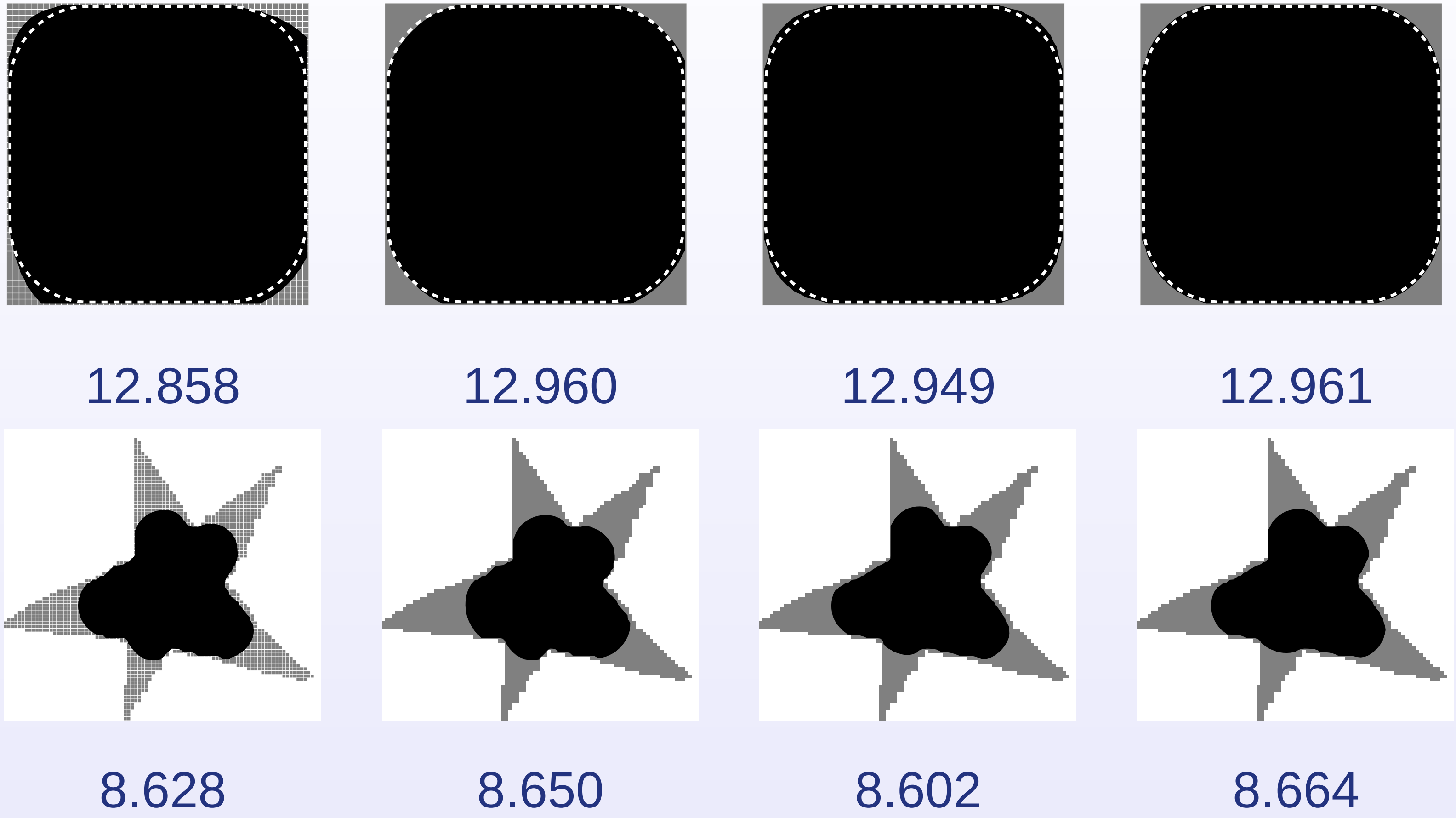
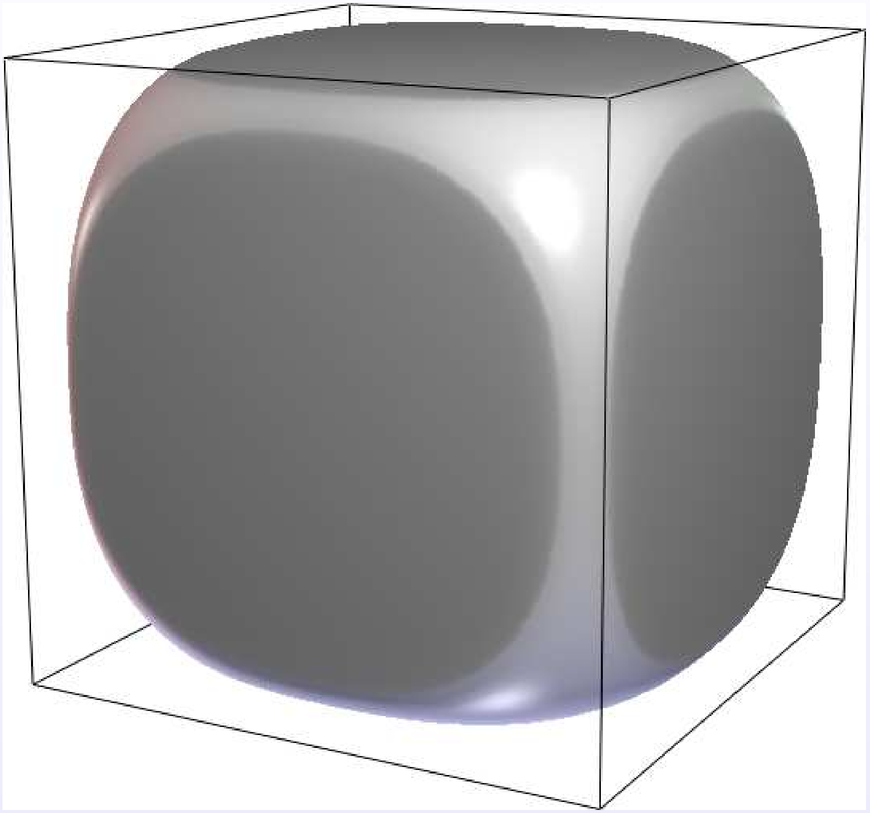


Figure 6: $\phi(x,\xi) = |\xi|$



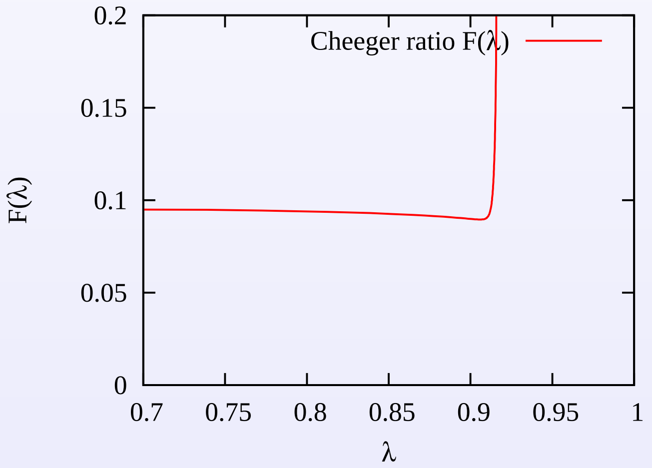


Figure 7: Cheeger set of a cube.

The GAC model and edge linking

Take $\phi(x,\xi) = g(x)|\xi|$. Solve

$$\min_{u \in BV_\phi(\Omega)} \int_\Omega g(x)|Du| + \frac{1}{2\lambda}\int_\Omega h(u-1)^2\,dx + \int_{\partial\Omega} \phi(x,\nu^\Omega)|u|\,d\mathcal{H}^{N-1}$$

For active contours:

$$g(x) = \frac{\mathbf{1}}{\mathbf{1} + |\nabla(\mathbf{G}_\sigma * \mathbf{I})(\mathbf{x})|^{\mathbf{2}}}.$$

For edge linking Γ is a set of edges and

$$g(x) = d_\Gamma(x).$$

Examples: GAC model, edge linking

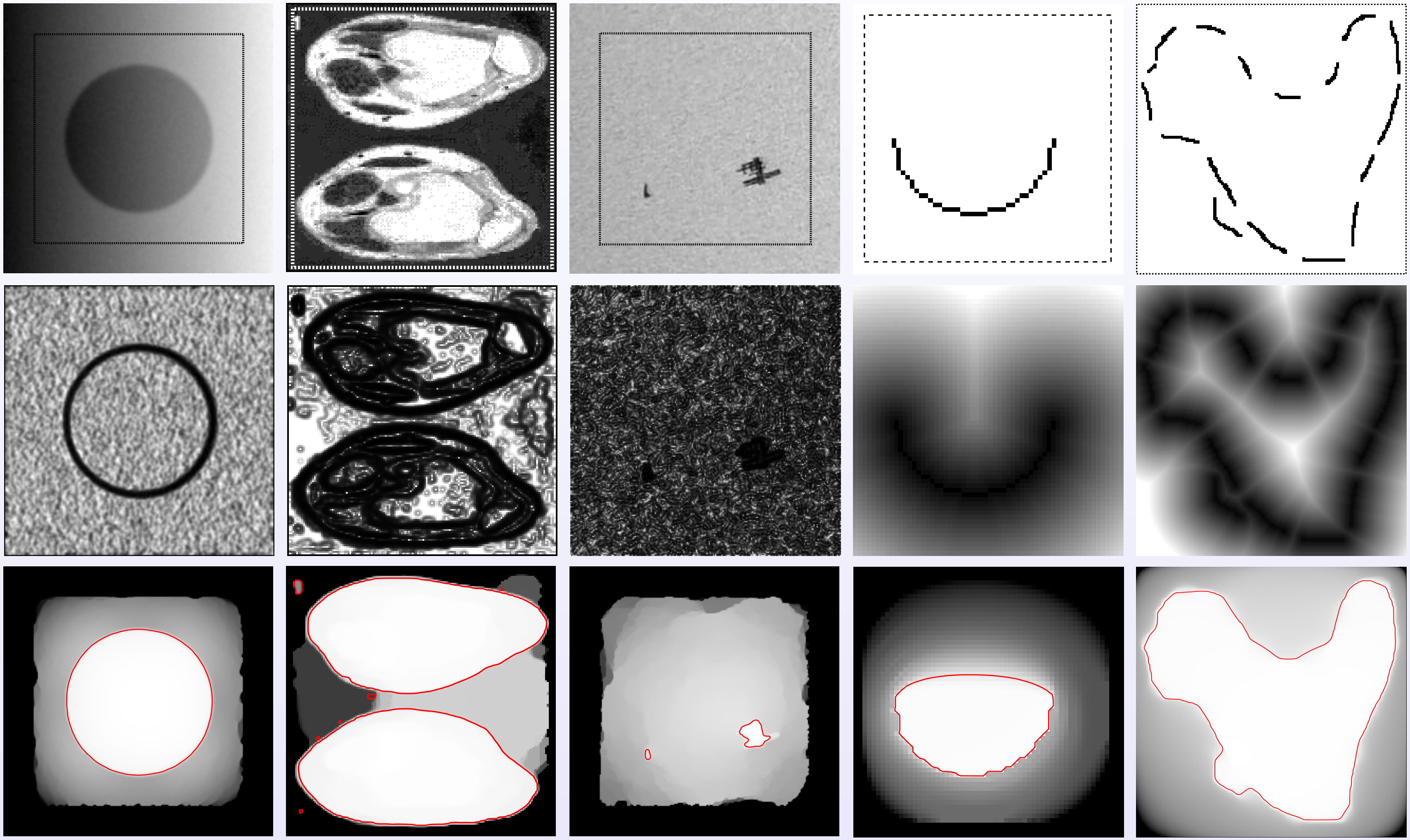


Figure 8: GAC, linking

Linking

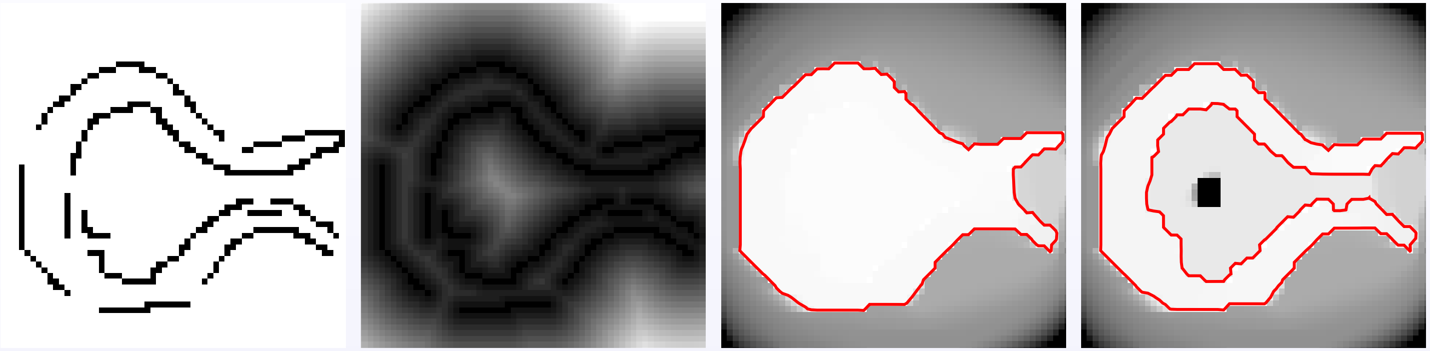


Figure 9: Segmentation and edge linking with barriers.

GAC Cheeger sets

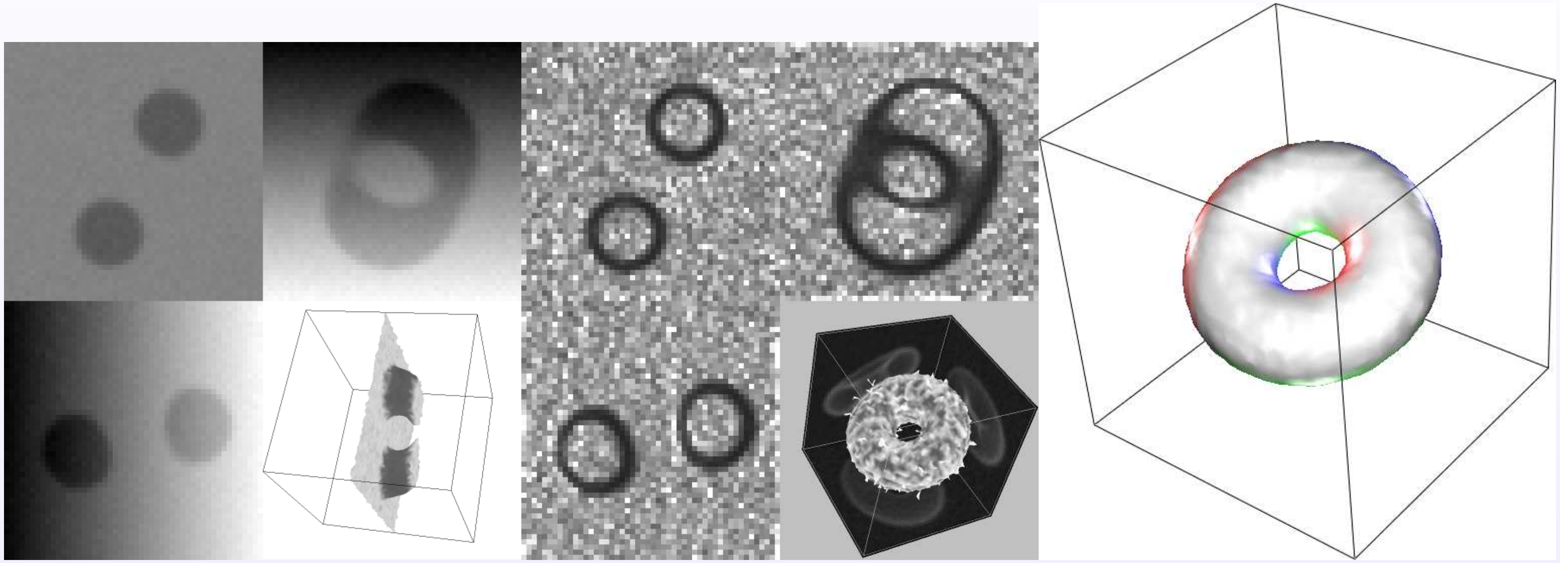


Figure 10: Pipeline for computing gradient Cheegers, applied to a synthetic 3D image. From left to right: slices of the original image I, slices of the metric $g = \frac{1}{|\nabla I|}$, gradient Cheeger.

Segmentation: examples of images

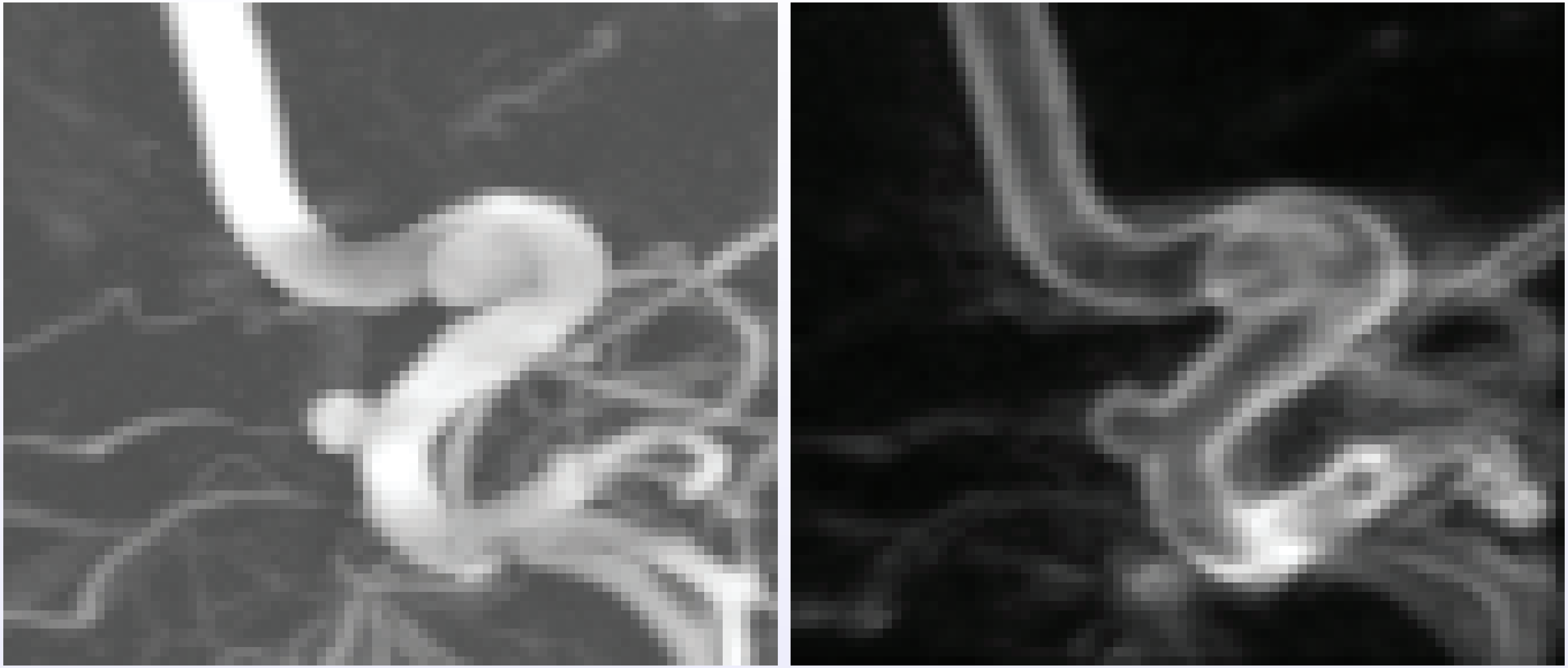


Figure 11: a) projection of image b) projection of gradient

Difficulties of thresholding

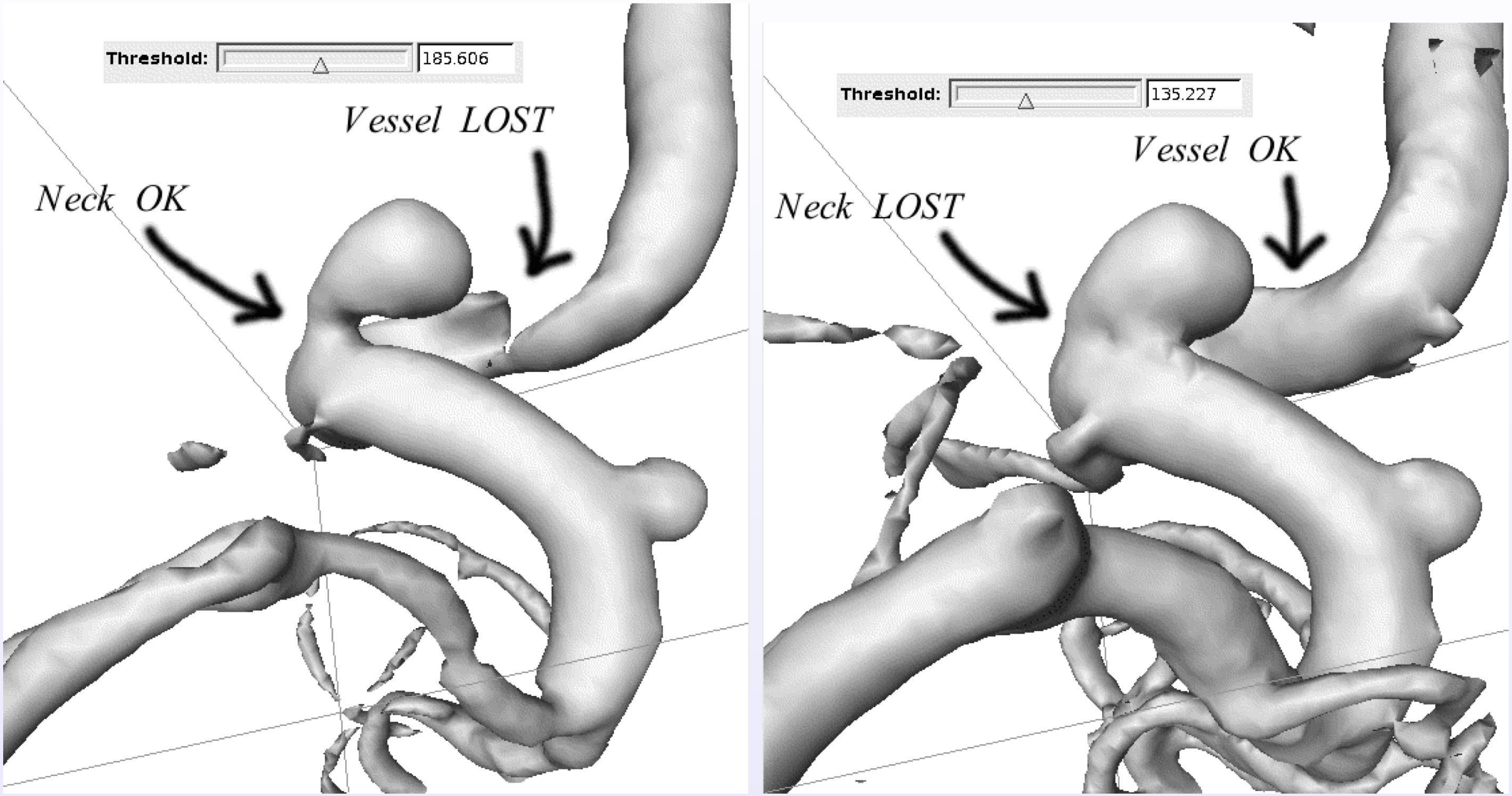


Figure 12: Problem of loss of contrast

Results

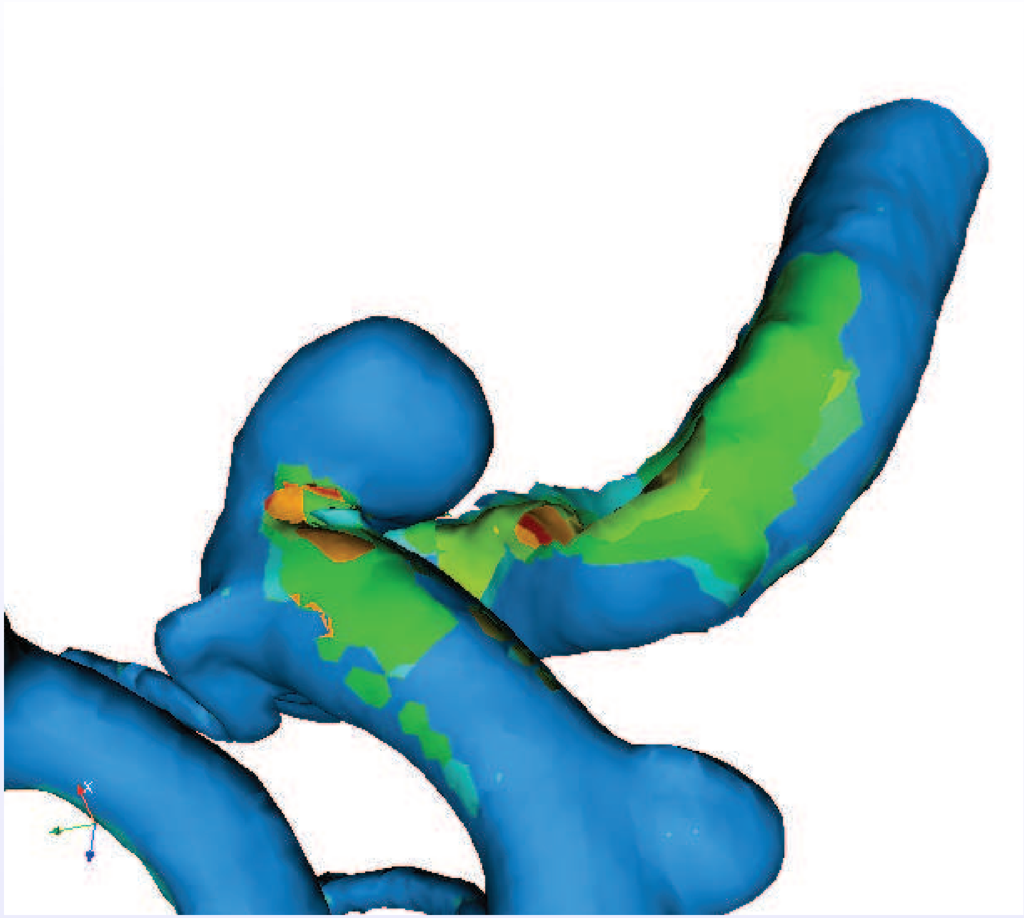
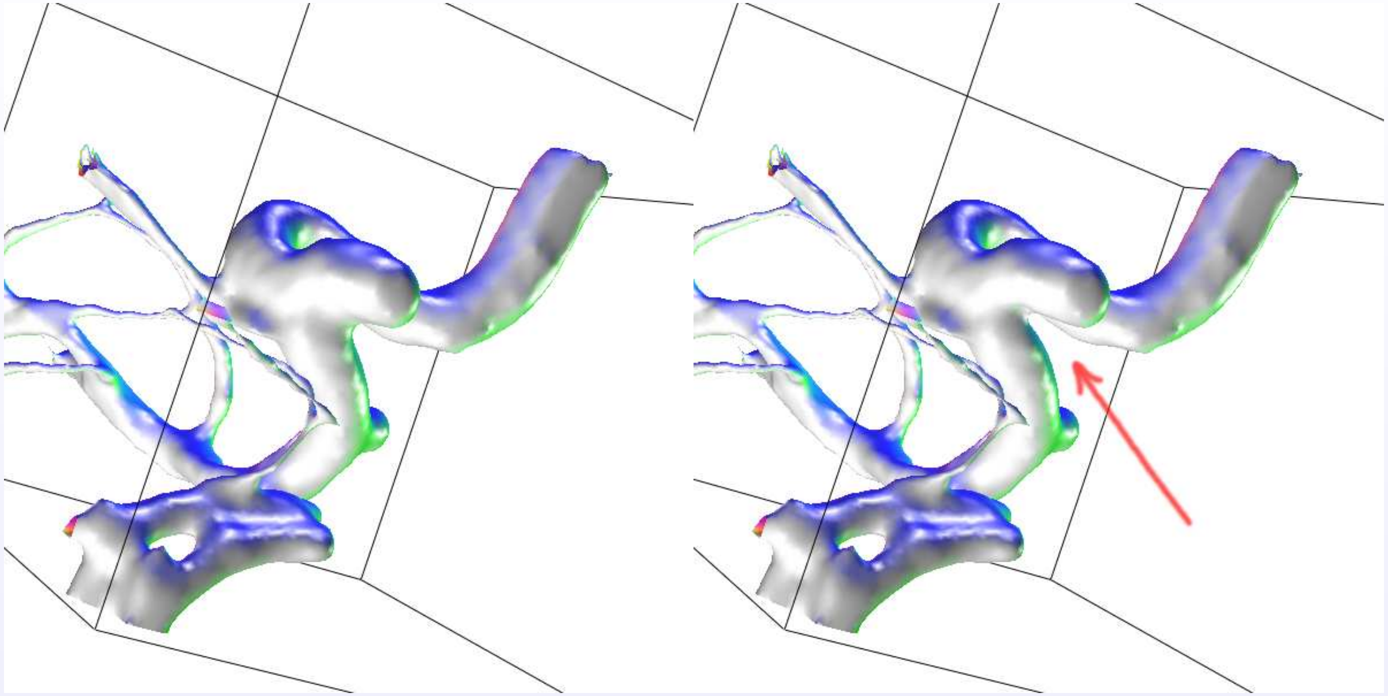


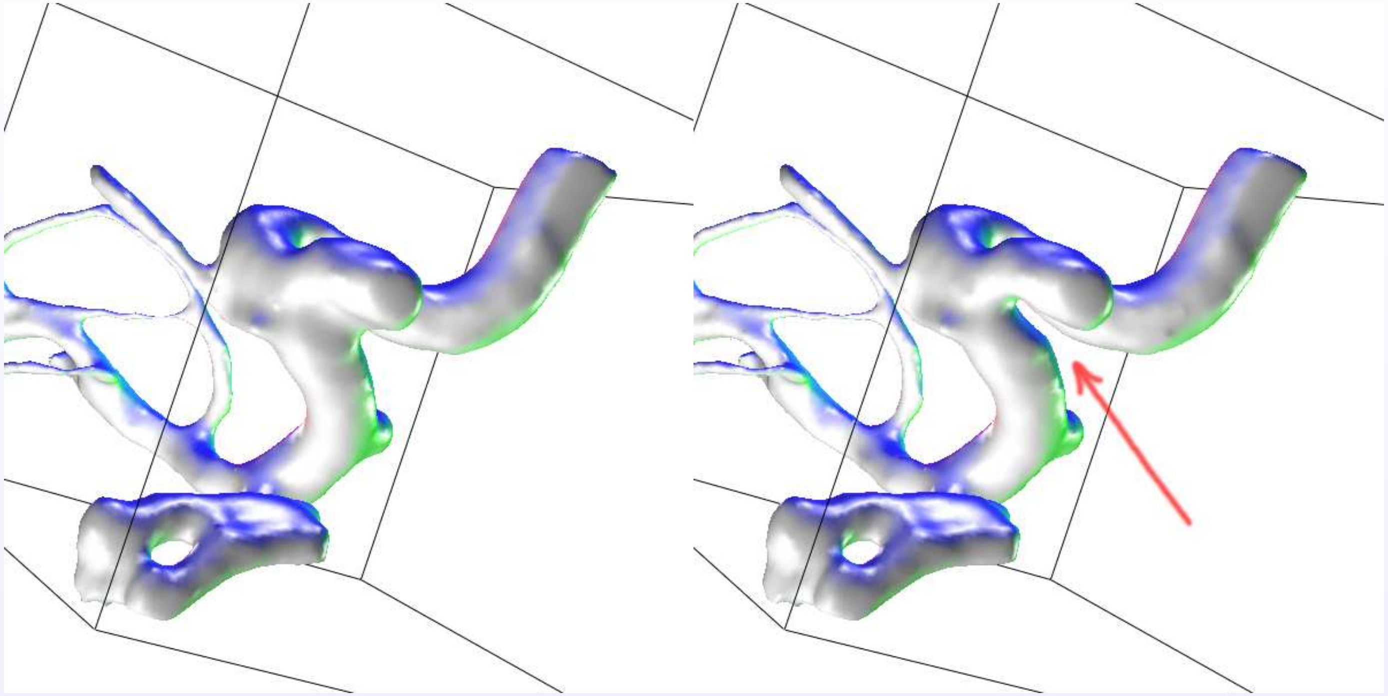
Figure 13: Result of edge detection in CTA

Results



Global distance Cheegers of the “cta” image. Left: Cheeger of the whole image domain. Right: Cheeger of the whole image domain minus some manually selected voxels at the neck of the aneurism.

Results



Global gradient Cheegers of the “cta” image. Left: Cheeger of the whole image domain. Right: Cheeger of the whole image domain minus some manually selected voxels at the neck of the aneurism.

Results

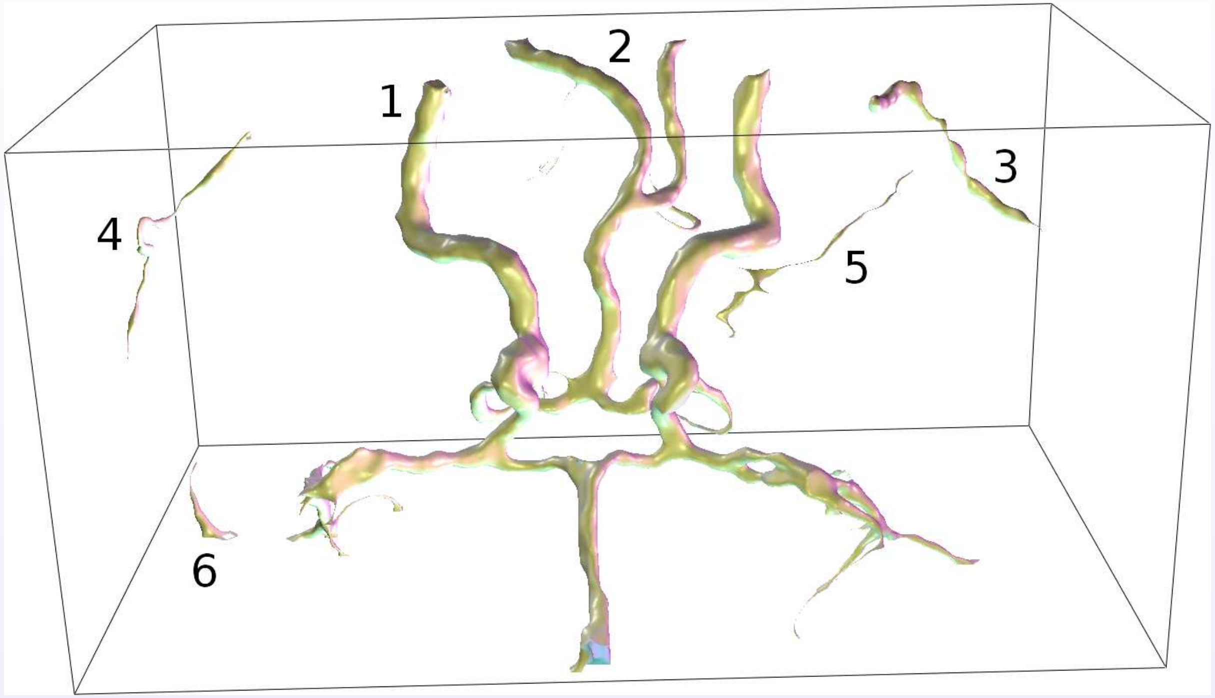


Figure 14: This Figure displays the best six local distance Cheegers.

Results

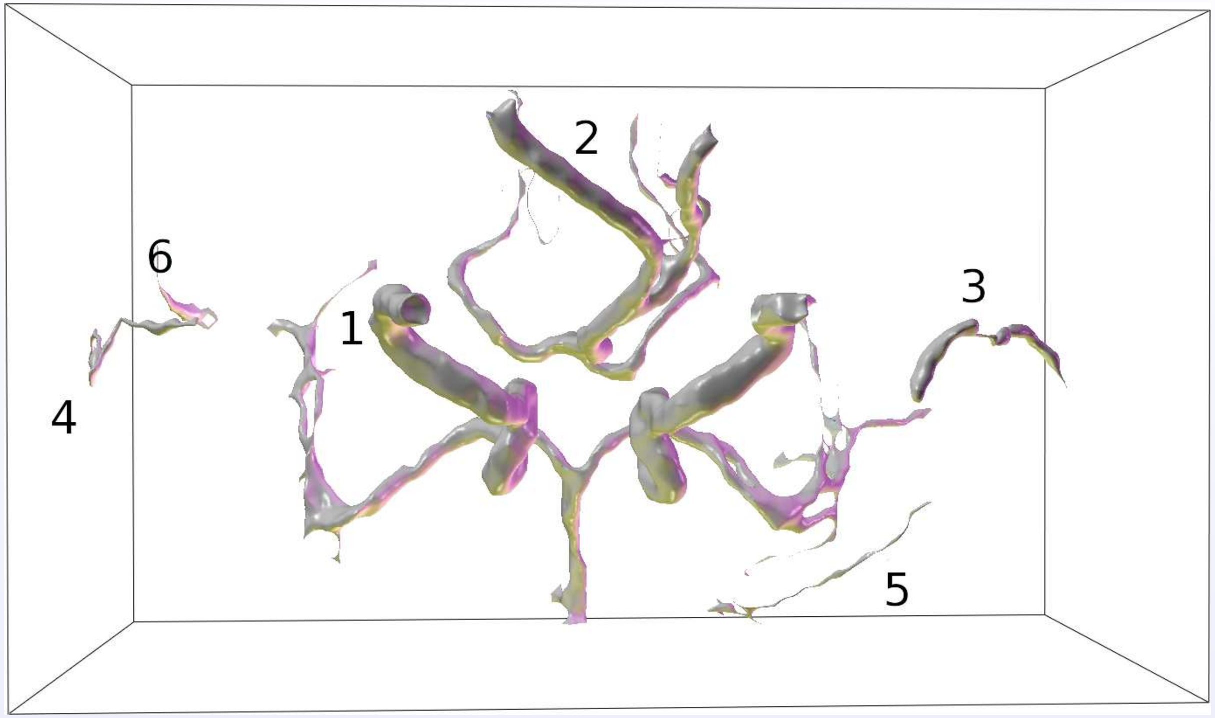


Figure 15: This Figure displays the best six local distance Cheegers, labelled, from a different point of view.

Anisotropic diffusion for colorization

Give a gray level image I. Extract from I the vector field of directions of level lines:

$$V(x) = \frac{\nabla I(x)}{\sqrt{\epsilon + |\nabla I(x)|^2}}$$

Consider the matrix

$$A(x) = V(x)^{\perp} \otimes V(x)^{\perp}.$$

We may also take

$$A(x) = V(x)^{\perp} \otimes V(x)^{\perp} + \frac{\epsilon}{\sqrt{1 + |\nabla I(x)|^2}} V(x) \otimes V(x).$$

Then we define $\phi(x, \xi) = |A(x)\xi|$ and solve

$$\min_{u \in BV_\phi(\Omega)} \int_\Omega \phi(x, \xi) + \frac{1}{2\lambda} \int_\Omega (u - f)^2 \, dx + \int_{\partial\Omega} \phi(x, \nu^\Omega)|u| \, d\mathcal{H}^{N-1}$$

where f is the data (manually colored part of the image).

Anisotropic diffusion for colorization

Anisotropic diffusion for interpolation

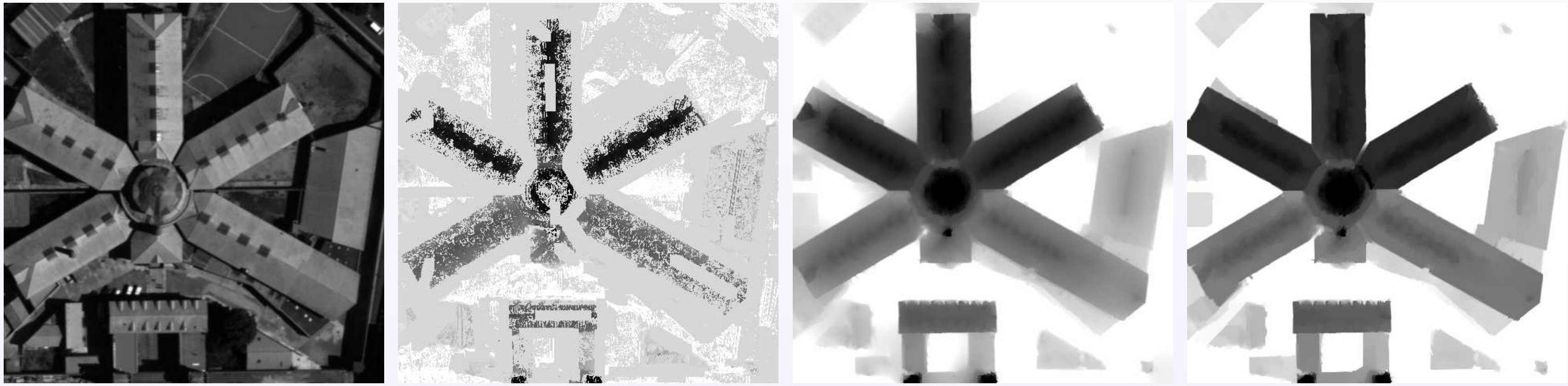


Figure 17: Disparity interpolation in an urban digital elevation model. reference image, incomplete data set (30% of the image), interpolation with the minimal surface interpolation (RMSE 0.239) and with the proposed algorithm (RMSE 0.190).

Anisotropic total variation in stereo problems

Stereo: Convexification of the disparity computation in rectified images (Chambolle-Pock-Schoenemann-Graber-Bischof-Cremers)

Problem: given a pair of stereo images $I_L(x)$, $I_R(x)$ which has been rectified (corresponding epipolars are horizontal lines)

FIND $u(x)$ such that $I_L(x) \approx I_R(x + (u(x), 0))$.

MINIMIZE

$$\lambda \int_\Omega |Du| + \int_\Omega |I_L(x) - I_R(x + (u(x), 0))|^2 \tag{15}$$

Write $\rho(x, u(x))) = |I_L(x) - I_R(x + (u(x), 0))|^2$.
Is a nonlinear, nonconvex optimization problem. It can be convexified adding an additional variable $\phi(x, t) = \chi_{\{u \geq t\}}(x)$.

Assume that u takes values in $[a, b]$.

Anisotropic total variation in stereo problems

Then

$$\int_\Omega |Du| = \int_\Omega \int_a^b |D\phi|$$

$$\begin{aligned}\int_\Omega \rho(x,u(x)) &= \int_\Omega \int_a^b \rho(x,t)\delta(u(x)-t)\,dx\,dt \\ &= \int_\Omega \int_a^b \rho(x,t)|\partial_t \phi(x,t)|\end{aligned}$$

The problem can be written as

$$\int_\Omega \int_a^b |D\phi| + \rho(x,t)|\partial_t \phi(x,t)|.$$

BC: $\phi(x,a) = 1$, $\phi(x,b) = 0$.