Asymptotic Analysis and Optimization in Metamaterials: Heterogeneous Layers

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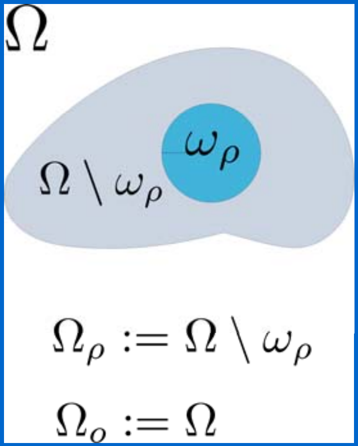
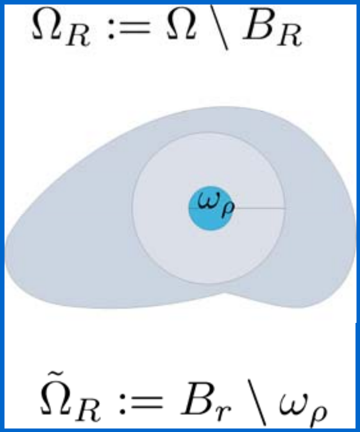
Cloaking: problem formulation

(Mathematical treatment: Uhlmann, Kohn, Bouchitte, Felbacq, Vogelius….)



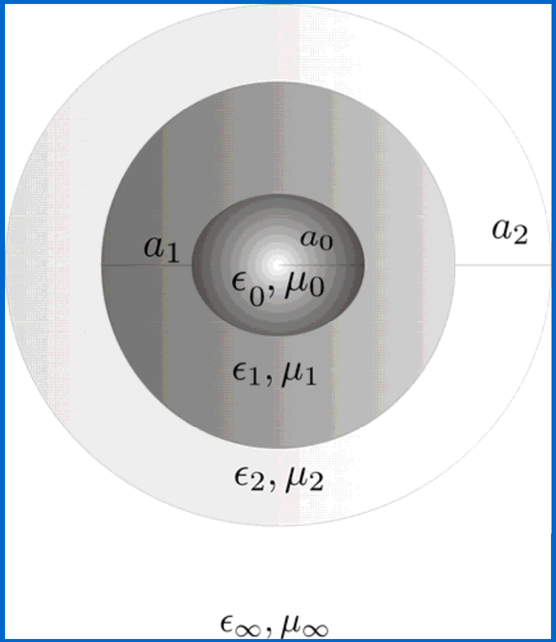
Object to be cloaked: here a ball

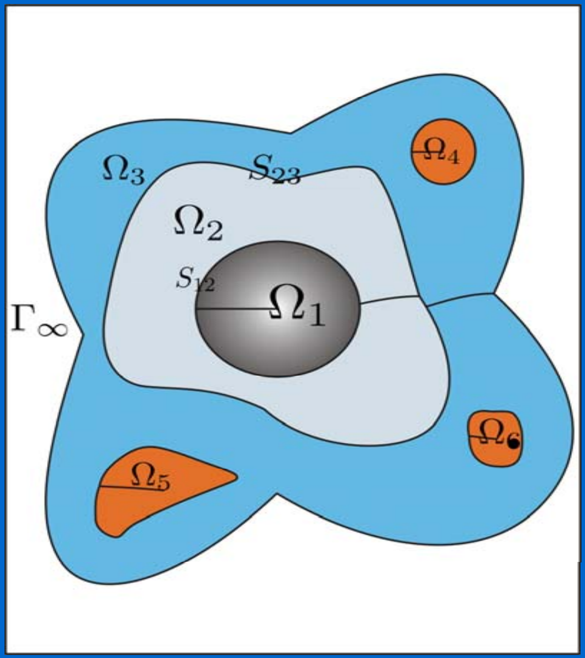


Object to be cloaked: Claoking layer

Desing material properties in the coating layer $\tilde{\Omega}_R$ such that the scattering of an incident 'wave' is 'minimal'. The ultimate, goal complete extinction, is reminiscent of a controllability problem

Turning cloaking into PDE-constrained optimization problem



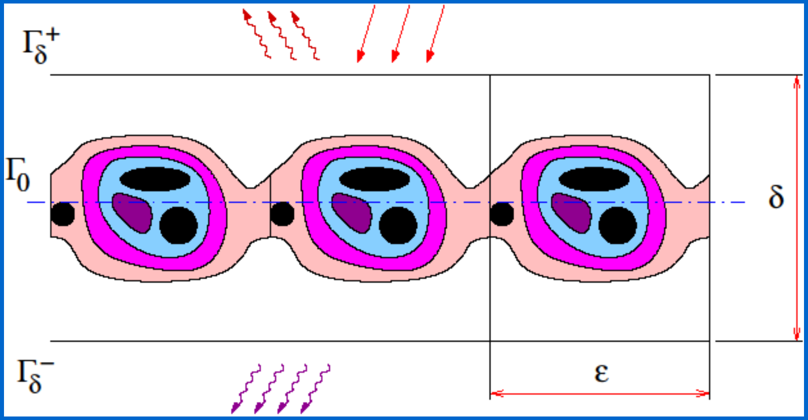


Spherical geometry
can also be handled exactly
using moment theory

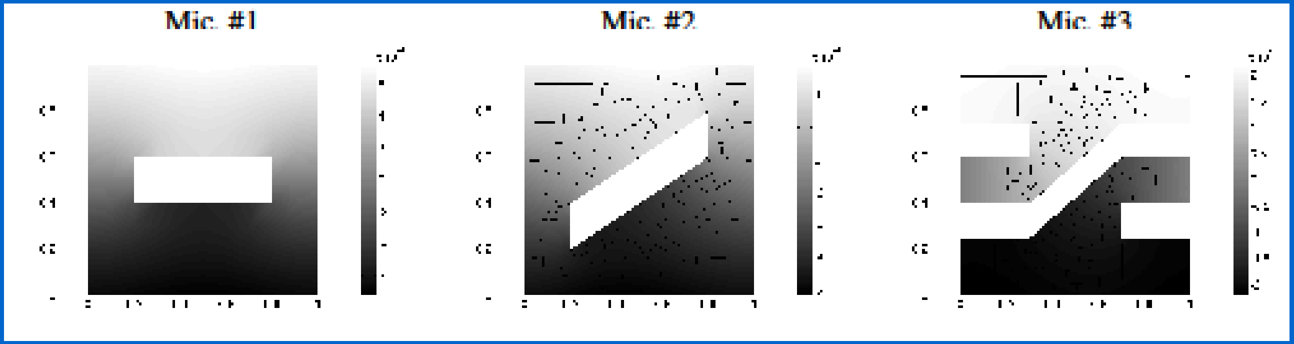
General heterogeneous
material layers and inclusion
no explicit solution

Layers: microstructures

i.e. design such microstructures in order to achieve cloaking or the appearnace of band-gaps! (see alsoTuesday's session)



General microstructure in periodic cells: change shapes and properties



Examples in FEM-realizazion

Time harmonics for elastodynamics

$$\mathbf{F}(x,t) = \mathbf{f}(x)e^{i\omega t}\ ,$$

where $\mathbf{f} = (f_i), i = 1,2,3$ is its local amplitude and ω is the frequency. We consider a displacement field with local magnitude $\mathbf{u}^\varepsilon$

$$\mathbf{U}^\varepsilon(x,\omega,t) = \mathbf{u}^\varepsilon(x,\omega)e^{i\omega t}\ ,$$

Th steady periodic response of the medium, by the displacement field $\mathbf{u}^\varepsilon$, satisfies the following boundary value problem:

$$\begin{aligned} -\omega^2\rho^\varepsilon\mathbf{u}^\varepsilon - \text{ div } \sigma^\varepsilon = \rho^\varepsilon\mathbf{F} \quad &\text{in } \Omega, \\ \mathbf{u}^\varepsilon = 0 \quad &\text{on } \partial\Omega, \end{aligned}$$

Piezoelectricity: strong form (shape analysis of time varying problems: Novotny, Perla-Menzal, Sokolowski, G.L. 2009)

Similarly, we can consider a piezoelectric field with magnitudes $(\mathbf{u}^\varepsilon, \varphi^\varepsilon)$

$$\tilde{\mathbf{u}}^\varepsilon(x,\omega,t) = \mathbf{u}^\varepsilon(x,\omega)e^{i\omega t}\,, \qquad \tilde{\varphi}^\varepsilon(x,\omega,t) = \varphi^\varepsilon(x,\omega)e^{i\omega t}\,.$$

and the corresponding system of equations

$$\begin{aligned} -\omega^2\rho^\varepsilon\mathbf{u}^\varepsilon - \ \mathrm{div}\ \sigma^\varepsilon &= \rho^\varepsilon\mathbf{f} \quad \text{in } \Omega, \\ -\ \mathrm{div}\ \mathbf{D}^\varepsilon &= q \quad \text{in } \Omega, \\ \mathbf{u}^\varepsilon &= 0 \quad \text{on } \partial\Omega, \\ \varphi^\varepsilon &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

$$\begin{aligned} \sigma^\varepsilon_{ij} &= c^\varepsilon_{ijkl}e_{kl}(\mathbf{u}^\varepsilon) - g^\varepsilon_{kij}\partial_k\varphi^\varepsilon, \\ D^\varepsilon_k &= g^\varepsilon_{kij}e_{kl}(\mathbf{u}^\varepsilon) + d^\varepsilon_{kl}\partial_l\varphi^\varepsilon. \end{aligned}$$

Homogenized mass tensor

To simplify the notation we introduce the *eigen-momentum* $\mathbf{m}^r = (m_i^r)$,

$$\mathbf{m}^r = \int_{Y_2} \rho^2 \phi^r. \tag{1}$$

The effective mass of the homogenized medium is represented by mass tensor $\mathbf{M}^* = (M_{ij}^*)$, which is evaluated as

$$M_{ij}^*(\omega^2) = \frac{1}{|Y|}\int_Y \rho\delta_{ij} - \frac{1}{|Y|}\sum_{r\geq 1} \frac{\omega^2}{\omega^2 - \lambda^r} m_i^r m_j^r \ ; \tag{2}$$

Homogenized elasticity tensor

The elasticity coefficients are computed just using the same formula as for the perforated matrix domain, thus being independent of the inclusions material:

$$C^*_{ijkl} = \frac{1}{|Y|} \int_{Y_1} c^1_{pqrs} e^y_{rs}(\mathbf{w}^{kl} + \Pi^{kl}) e_{pq}(\mathbf{w}^{ij} + \Pi^{ij}) \ ,$$

where $\Pi^{kl} = (\Pi^{kl}_i) = (y_l \delta_{ik})$ and $\mathbf{w}^{kl} \in \mathbf{H}^1_\#(Y_1)$ are the corrector functions satisfying

$$\int_{Y_1} c^1_{pqrs} e^y_{rs}(\mathbf{w}^{kl} + \Pi^{kl}) e^y_{pq}(\mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathbf{H}^1_\#(Y_1) \ .$$

Homogenized Helmholtz-type elasticity

The *global (homogenized) equation* of the homogenized medium, here presented in its differential form, describes the macroscopic displacement field $\mathbf{u}$:

$$\omega^2 M^*_{ij}(\omega) u_j + \frac{\partial}{\partial x_j} C^*_{ijkl} e_{kl}(\mathbf{u}) = M^*_{ij}(\omega) f_j \ ,$$

$M(\omega)^*$ and C^* can be viewed as objects of optimization in order to influence the functionality of the structure! Analogous situation for piezo-electric material!

Metamaterial: e.g. negative mass!

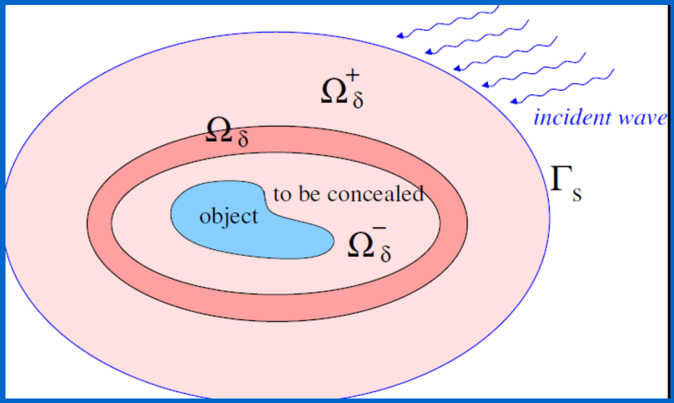
Major goal: describe and control the mapping $\omega \longrightarrow M(\omega)$ in order to influence wave propagation. In particular, if

• all eigenvalues of $M(\omega)$ are negative, *no wave propagation is admitted*

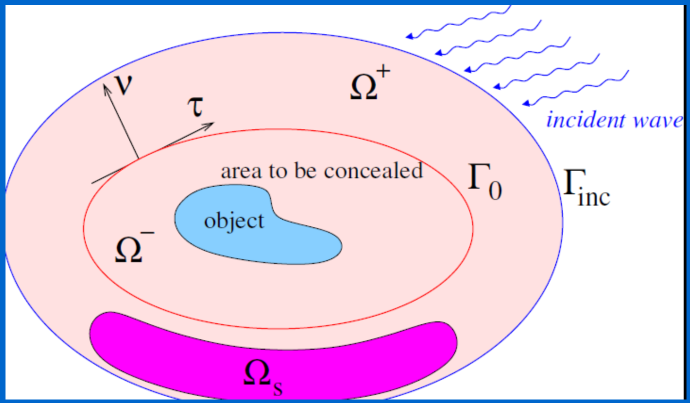
• there positive and negative eigenvalues, *only polarized waves admittes*

Thus, the desired occurance of band-gaps becomes a matter of PDE-constrained optimization linked with homogenization!

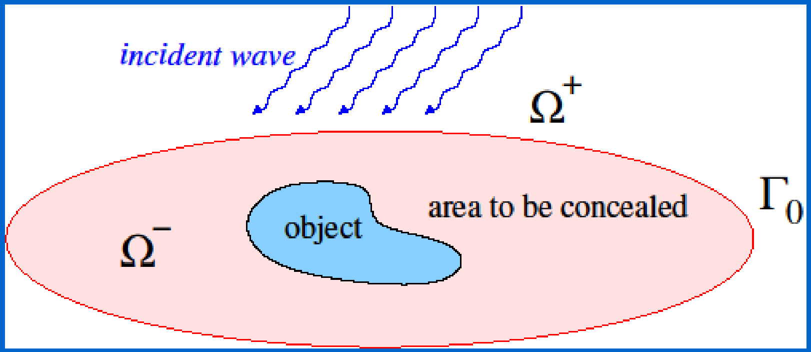
Three cloaking scenarios as cartoons



Non homogenized $\delta - \varepsilon$ level



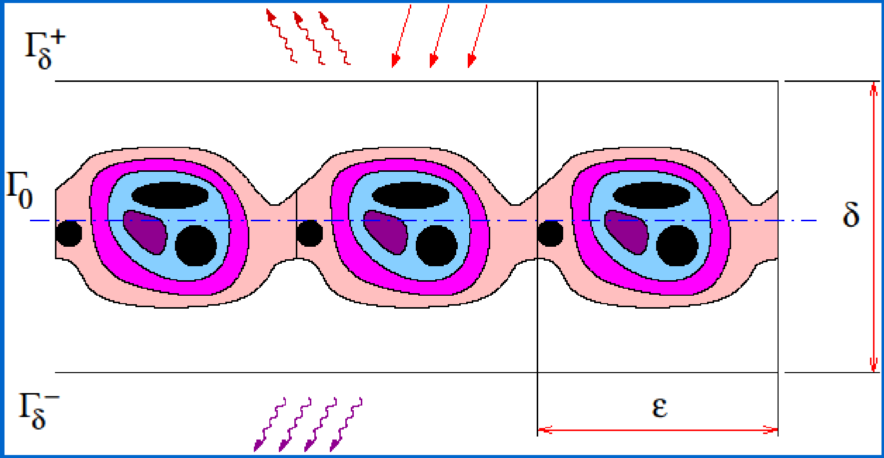
homogenized bi-domain level



homogenized singke-domain level

Notation

domain:	parameters of the medium:	description:
$\Omega_\delta^+, \Omega^+$	β_0^+, μ_0^+	constant
$\Omega_\delta^- \setminus \Omega_c, \Omega^- \setminus \Omega_c$	β_0^-, μ_0^-	const.
Ω_c	β, μ	pm const.
$\Omega_{\varepsilon\delta}$	$\beta^{\varepsilon\delta}, \mu^{\varepsilon\delta}$	pw const.



Cost function

scattered field in Ω^+ given as $u^{sc} = u - u^{inc}$. A physically reasonable measure of the cloaking effect is the extinction function defined as:

$$Q^{ext}_{\Gamma_s}(u^{inc}, u^{sc}) = \frac{2}{|\mathbf{d}\cdot\partial\Omega_c|}\text{real}\int_{\Gamma_s}\{\mathbf{n}\cdot\mathbf{d}u^{inc}\overline{u^{sc}} + \frac{\mathbf{i}\gamma}{k^{inc}}u^{sc}\overline{u^{inc}}\},$$

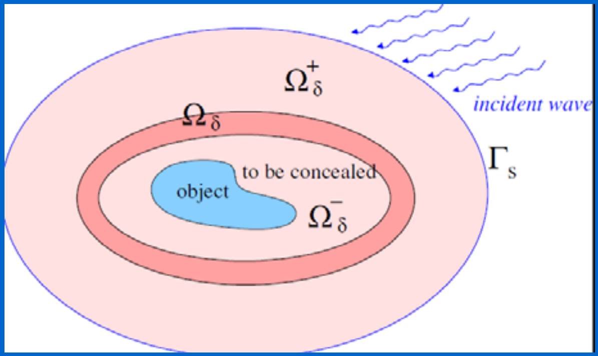
The non-homogenized problem $\delta = \varepsilon h$

$$\frac{1}{\beta_0^+}\nabla^2 u^{\delta+} + \omega^2 \mu_0 u^{\delta+} = 0 \qquad \text{in } \Omega_\delta^+ ,$$

$$\nabla \cdot \left(\frac{1}{\beta}\nabla u^{\delta-}\right) + \omega^2 \mu u^{\delta-} = 0 \qquad \text{in } \Omega_\delta^- ,$$

$$\nabla \cdot \left(\frac{1}{\beta}\nabla u^{\delta}\right) + \omega^2 \mu u^{\delta} = 0 \qquad \text{in } \Omega_\delta ,$$

$$\partial_n(u^{\delta+} - u^\delta) = 0 \text{ on } \Gamma_\delta^+ ,$$
$$\partial_n(u^{\delta-} - u^\delta) = 0 \text{ on } \Gamma_\delta^- ,$$
$$u^{\delta+} - u^\delta = 0 \text{ on } \Gamma_\delta^+ ,$$
$$u^{\delta-} - u^\delta = 0 \text{ on } \Gamma_\delta^- ,$$
$$\partial_n u^{sc} - \gamma u^{sc} = 0 \text{ on } \partial\Omega,$$



Optimization on the δ level

The corresponding optimization problem can be treated as a *material optimization problem* as follows.

$$\begin{cases} \min\limits_{\beta,\mu} Q_{\Omega_s}(u^{inc}, u^{sc}) \text{ s.t.} \\ (u^{\delta+}, u^{\delta-}, u^{\delta}) \text{ satisfies system above} \\ (\beta,\mu) \in \mathcal{U}_{\text{ad}} \ , \end{cases}$$

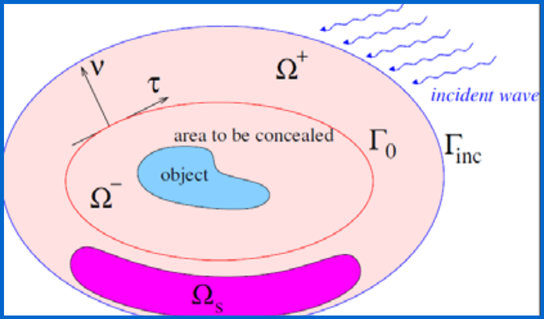
However, the tensors can now be characterized by homogenization. They carry the parameters of the inclusions, voids etc. in the coeffcients!

The limiting bi-domain problem

$$\frac{1}{\beta_0^+}\nabla^2 u^+ + \omega^2\mu_0 u^+ = 0 \quad \text{in } \Omega^+,$$

$$\nabla\cdot\left(\frac{1}{\beta}\nabla u^-\right) + \omega^2\mu u^- = 0 \quad \text{in } \Omega^-,$$

$$-\partial_\nu u^+ = \partial_n^+ u^+ = -\mathrm{i}\omega\beta_0^+ g^0 \quad \text{on } \Gamma_0,$$



$$\partial_\nu u^- = \partial_n^- u^- = \mathrm{i}\omega\beta_0^- g^0 \quad \text{on } \Gamma_0,$$

$$\partial_\tau\left(A\partial_\tau u^0 + \mathrm{i}\omega B g^0\right) + \omega^2\mu\varrho^* u^0 = 0 \quad \text{on } \Gamma_0,$$

$$\mathrm{i}\omega B\partial_\tau u^0 + \omega^2 F g^0 = -\frac{\mathrm{i}\omega}{\delta_0}(u^+ - u^-) \quad \text{on } \Gamma_0.$$

$$\partial_n u^{sc} - \gamma u^{sc} = 0 \text{ on } \partial\Omega.$$

The far-field optimization problem

$$\begin{cases} \min\limits_{A,B,F,\beta,\mu} Q_{\Omega_s}(u^{inc}, u^{sc}) \text{ s.t.} \\ (u^+, u^-, u^0) \text{ satisfies system above} \\ (A, B, F, \beta, \mu) \in \mathcal{U}_{\text{ad}} \ , \end{cases}$$

This problem is open:

here we take coeeficients of a Laplace-Beltrami operator on the boundary as controls!

we can regard the cloaking condition as an exact controllability constraint!

$$\mathbf{i}\omega\beta_0 g^0 = \partial_n^- u^- = \partial_n^+ u^+ = -k_n^+ u^{inc} \ ,$$

The simplest boundary-coefficient control problem

$$\min_{A,B,F} \Psi(g^0, A, B, F) := \|k_n^+ u^{inc} + \mathrm{i}\omega\beta_0 g^0\|_{\Gamma_0} \text{ , s.t.}$$

$$\nabla\left(\frac{1}{\beta}\nabla u\right) + \omega^2 \mu u = 0 \quad \text{in } \Omega^- \text{ ,}$$

$$\partial_\nu u = bfi\omega\beta_0^- g^0 \quad \text{on } \Gamma_0 \text{ ,}$$

$$\partial_\tau\left(A\partial_\tau u^0 + \mathrm{i}\omega B g^0\right) + \omega^2\mu\varrho^* u^0 = 0 \quad \text{on } \Gamma_0 \text{ ,}$$

$$\mathrm{i}\omega B\partial_\tau u^0 + \omega^2 F g^0 = -\frac{\mathrm{i}\omega}{\delta_0}(u^{inc} - u) \quad \text{on } \Gamma_0 \text{ .}$$