LAMA Université de Savoie

Shape analysis of eigenvalues

Relaxation phenomena

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Benasque, August 23-September 4, 2009

Generic problem

$$\min_{|\Omega|=m} F(\lambda_1(\Omega), ..., \lambda_k(\Omega))$$

$\Omega \subseteq \mathbb{R}^N$ open, $|\Omega|$ the Lebesgue measure,

$$\lambda_1(\Omega) \leq \lambda_2(\Omega) \leq ... \leq \lambda_k(\Omega)$$

the first k eigenvalues of the Laplacian with some b.c.
Questions :

- existence of a solution : Ω
- properties of Ω coming from optimality : regularity, symmetry, convexity,...is it the ball ?
- numerical computations

Examples

$$\begin{cases} -\Delta u = & \lambda u \text{ in } \Omega \\ u = & 0 \ \partial\Omega \end{cases}$$

$$0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_k \leq \rightarrow +\infty$$

Variation definition :

$$\lambda_1(\Omega) = \min_{u \in H_0^1(\Omega), u \neq 0} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx}$$

$$\lambda_k(\Omega) = \min_{S_k \in H_0^1(\Omega)} \max_{u \in S_k} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx}$$

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Rayleigh 1877

The solution of

$$\min_{|\Omega|=m} \lambda_1(\Omega)$$

is the ball.

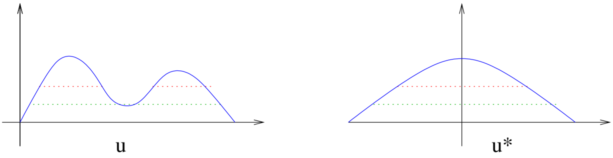
Rearrangement by symmetrization

Faber and Krahn 1921-1923
For

$$u \in H_0^1(\Omega), u \geq 0$$

we build

$$\{u^* > c\} = \text{ball of volume } |\{u > c\}|$$



Then

$$\int_\Omega u^2 dx = \int_{\text{ball}} |u^*|^2 dx$$

$$\int_\Omega |\nabla u|^2 dx \geq \int_{\text{ball}} |\nabla u^*|^2 dx$$

$$\lambda_1(\Omega) = \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx} \geq \frac{\int_{\text{ball}} |\nabla u^*|^2 dx}{\int_{\text{ball}} |u^*|^2 dx} \geq \lambda_1(\text{ball})$$

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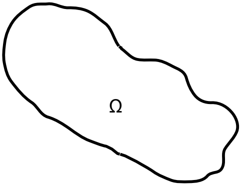
Variational approach

B. Freitas

Assume Ω is a minimizing open set for λ_1. Then Ω is symmetric in all directions :



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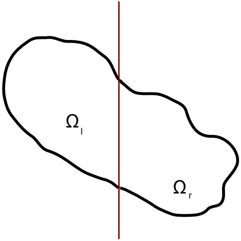

$$\lambda_1(\Omega) = \frac{\int_{\Omega_l} |\nabla u_l|^2 dx + \int_{\Omega_r} |\nabla u_r|^2 dx}{\int_{\Omega_l} u_l^2 dx + \int_{\Omega_r} u_r^2 dx} \geq$$

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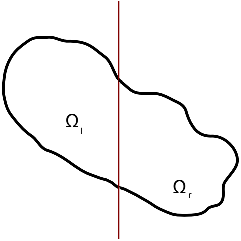

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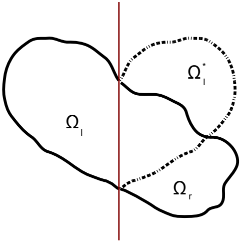

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$$\frac{\int_{\Omega_l} |\nabla u_l|^2 dx}{\int_{\Omega_l} u_l^2 dx} = \frac{\int_{\Omega_l} |\nabla u_l|^2 dx + \int_{\Omega_l^*} |\nabla u_l^*|^2 dx}{\int_{\Omega_l} u_l^2 dx + \int_{\Omega_l^*} |u_l^*|^2 dx} \geq \lambda_1(\Omega_l \cup \Omega_l^*)$$

Ω is symmetric

So $\Omega_l \cup \Omega_l^*$ is also optimal!
Equalites hold above, thus u_l has two analytic extensions u_r and u_l^*!

Conclusion :

- Ω is symmetric with respect to the line.
- $\Longrightarrow \Omega$ is a union of annuli
- 1D analysis $\Longrightarrow \Omega$ is a ball

Other eigenvalues

- $\min_{|\Omega|=m} \lambda_2(\Omega)$: two balls of volume $\frac{m}{2}$, Faber-Krahn 1923
- $\min \frac{\lambda_1(\Omega)}{\lambda_2(\Omega)}$: ball, Ashbaugh-Benguria 1993
- $\min_{|\Omega|=m} \lambda_3(\Omega)$: conjecture : ball in 2D/3D, 3 equal balls in higher dimension...
- $\min_{|\Omega|=m} \lambda_4(\Omega)$: conjecture : two balls of different measures in 2D/3D
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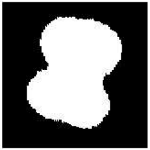
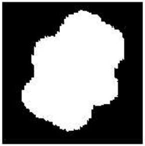


k=5

k=6

k=7



k=8

Existence of a solution

Theorem Buttazzo-Dal Maso
Let D be bounded and open. If F is increasing in each variable and l.s.c., the problem

$$\min_{|\Omega|=c, \Omega \subset D} F(\lambda_1(\Omega), ..., \lambda_k(\Omega))$$

has a solution.

Examples :

- $F(\lambda_1, .., \lambda_k) = \lambda_1 + \lambda_2$
- $F(\lambda_1, .., \lambda_k) = \lambda_k$
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Idea of the proof

- $(\Omega_n)_n$ minimizing sequence
- $\Omega_{n_k} \longrightarrow \mu$ in an appropriate sense : γ-convergence
- $\lambda_j(\Omega_{n_k}) \longrightarrow \lambda_j(\mu)$ the spectrum follows the geometry

$$\begin{cases} -\Delta u + \mu u = & \lambda u \text{ in } D \\ u \in & H_0^1(D) \cap L^2(D, \mu) \end{cases}$$

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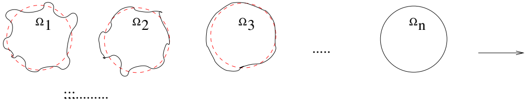
Global minimization in $\mathbb{R}^N$

concentration-compactness principle for functions P.L. Lions : 3 possibilities

- compacntess :

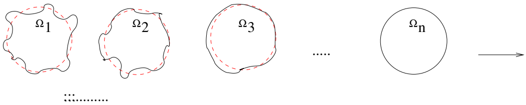


by translation one can concentrate the mass

Intuitively :

concentration-compactness principle for functions P.L. Lions : 3 possibilities

- compactness :



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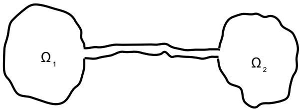
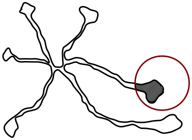
- dichotomy :



"two distancing pieces"

- vanishing :



nowhere mass concentration

Theorem B.

Let $R_\Omega := (-\Delta)^{-1} : L^2(\mathbb{R}^N) \to H_0^1(\Omega) \subset L^2(\mathbb{R}^N)$.

- compactness : $\exists y_k \in \mathbb{R}^N$, $\exists \mu$ t.q.

$$R_{\Omega_{n_k}+y_k} \longrightarrow R_\mu, \ \text{ in } \ \mathcal{L}(L^2(\mathbb{R}^N))$$

- dichotomy : $\exists \Omega_k^1, \Omega_k^2$, t.q.

$$d(\Omega_k^1, \Omega_k^2) \to +\infty$$

$$\Omega_k^1 \cup \Omega_k^2 \subseteq \Omega_{n_k}$$

$$\liminf_{n\to\infty} |\Omega_k^i| > 0$$

$$\|R_{\Omega_{n_k}} - R_{\Omega_k^1} - R_{\Omega_k^2}\|_{\mathcal{L}(L^2(\mathbb{R}^N))} \to 0$$

Theorem B.
Let $R_\Omega := (-\Delta)^{-1} : L^2(\mathbb{R}^N) \to H^1_0(\Omega) \subset L^2(\mathbb{R}^N)$.

- compactness : $\exists y_k \in \mathbb{R}^N$, $\exists \mu$ t.q.

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Application to λ_3

B. Henrot

$$\min_{|\Omega|=m} \lambda_3(\Omega)$$

- if compactness, we construct μ and by monotonicity $\Longrightarrow$ existence of a solution
- if dichotomy, we replace the minimizing sequence by $\Omega_k^1 \cup \Omega_k^2$! The optimum consists on three balls...

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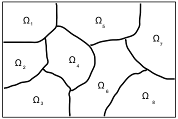
Optimal partitions



$$\min\{\sum_{i=1}^{N}\lambda_1(\Omega_i) : \Omega_i\cap\Omega_j=\emptyset,\ \Omega_i\subseteq D\}$$

Conjecture Van den Berg (1981), Caffarelli-Lin (2007) :

$$\min\sum_{i=1}^{N}\lambda_1(\Omega_i)\simeq N^2\lambda_1(H), \tag{1}$$

where H is the regular hexagon of surface equal to 1 in $\mathbb{R}^2$.

Results

- general existence of quasi-open sets B. Buttazzo Henrot 1998
- regularity Conti, Terracini, Verzini 2003, Caffarelli, Lin 2007
- qualitative properties, Bonnaillie-Noël, Helffer, Hoffmann-Ostenhof, Vial 2006-2009
- numerical computations by Chang, Lin, Lin, Lin 2004

Analysis based on measures

Bourdin, B., Oudet 2009

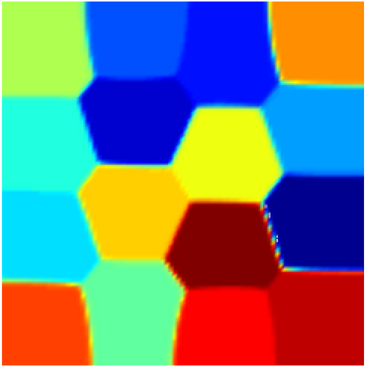
Theorem

- $$\min_{\Omega_i} \sum_{i=1}^{N} \lambda_{k_i}(\Omega_i)$$

  has a solution in the family of open sets

- $$\min_{\Omega_i \text{ open}} \sum_{i=1}^{N} \lambda_{k_i}(\Omega_i) = \lim_{C\to\infty} \min_{\sum \varphi_i = 1} \sum_{i=1}^{N} \lambda_{k_i}(C\varphi_i dx)$$

Numerical results



16 cells, non periodical

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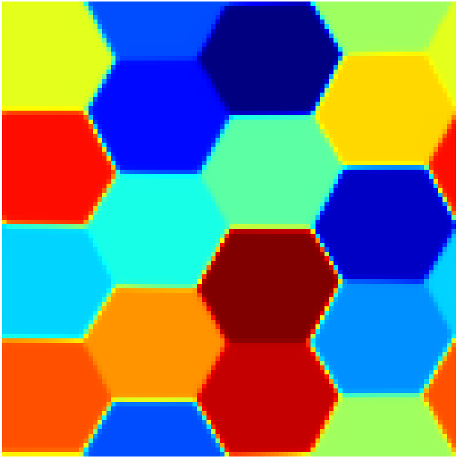

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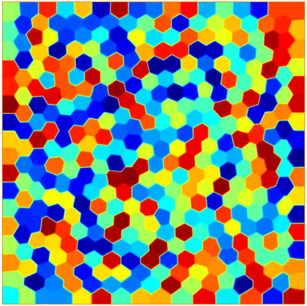
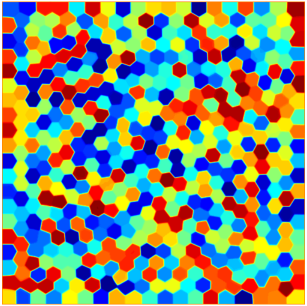


384 cells

Numerical results



512 cells

Robin boundary conditions

For $\beta > 0$

$$\left\{ \begin{array}{rll} -\Delta u = & \nu u & \text{in} \quad \Omega \\ \frac{\partial u}{\partial n} + \beta u = & 0 & \partial\Omega \end{array} \right.$$

Rayleigh quotioent :

$$\nu_1(\Omega) = \min_{u \in H^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx + \beta \int_{\partial\Omega} |u|^2 d\mathcal{H}^{N-1}}{\int_\Omega u^2 dx}$$

The ball minimizes ν_1 : Bossel ($\mathbb{R}^2$) 1986, Daners ($\mathbb{R}^N$) 2006 among Lip-domains.

Proof : "dearrangement" technique using C^1-regularity of the solution up to the boundary and density of C^2 domains in Lip-domains.

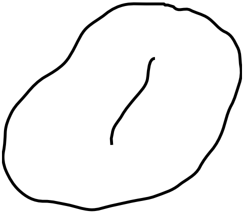
Uniqueness : B. Daners 2009 for p-Laplacian and Lip-domains.

Question : What is the most general class where the isoperimetric inequality is valid ?

Variational formulation

B. Giacomini (Brescia), objective : give the relaxed formulation on any domain.

Motivation :



cracked set Ω

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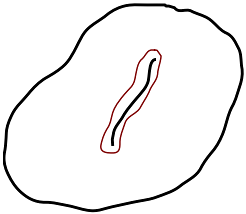

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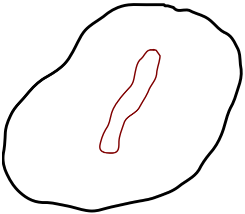

Ω_ε

For $\varepsilon \to 0 \Longrightarrow \nu_1(\Omega_\varepsilon) \to \nu_1$, where

$$\begin{cases} -\Delta u = & \nu u \text{ in } \Omega \\ \frac{\partial u}{\partial n} + \beta^* u = & 0 \ \partial\Omega \end{cases}$$

and

$$\beta * \geq \beta!$$

If it shrinks "without oscillations" then $\beta = \beta^*$.
In this case, the ball is still minimizer ! If $\beta^* > \beta$, it is not clear...

Robin eigenvalues on non smooth domains

Daners 1995, Daners-Dancer 1996, Arendt-Warma 2003 : use the Mazya space
Completion of $C^1(\overline{\Omega})$ in the norm $\|\cdot\|_{H^1(\Omega)} + \|\cdot\|_{L^2(\partial\Omega)}$: subspace in $H^1(\Omega) \times L^2(\partial\Omega, \mathcal{H}^{N-1})$.

- is well defined for every domain
- compact embedding in $L^2(\Omega)$, so spectrum of eigenvalues
- no cracks
- the trace operator is not well defined. The zero function may have the trace equal to 1 !

Idea : take the first eigenfunction of the Robin problem and extend it by zero. The (square of) the new function seen in $\mathbb{R}^N$ has a distributional derivative

$$Du^2 = \nabla u^2 dx|_{\Omega} + u^2 \nu \mathcal{H}^{N-1}|_{\partial\Omega}.$$

So $u^2 \in SBV(\mathbb{R}^N)$!

$$BV(\mathbb{R}^N) = \{v \in L^1(\mathbb{R}^N) : \ Dv \text{ is a finite Radon measure}\}.$$

There exists J_v is a $(N-1)$-rectifiable set such that Du admits the following representation for every Borel set $B \subseteq \mathbb{R}^N$:

$$Dv(B) = \int_B \nabla v \, dx + \int_{J_v \cap B} (v^+ - v^-)\nu_v \, d\mathcal{H}^{N-1} + D^c u(B),$$

$$SBV(\mathbb{R}^N) = \{u \in BV(\mathbb{R}^N) : D^c u = 0\}$$

Variational formulation

$$\min_{u^2 \in SBV(\mathbb{R}^N), |\{u \neq 0\} = m|} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2 dx + \beta \int_{J_u} (|u^+|^2 + |u^-|^2) d\mathcal{H}^{N-1}}{\int_{\mathbb{R}^N} u^2 dx}$$

Theorem (B. Giacomini 2009)

$$\nu_1(ball_m) \leq \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2 dx + \beta \int_{J_u} (|u^+|^2 + |u^-|^2) d\mathcal{H}^{N-1}}{\int_{\mathbb{R}^N} u^2 dx}$$