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# Optimal 1-rectifiable transports.

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**Motivation:** understand how optimal transport with economy of scales leads to optimal channel networks

- Stationary formulation (Monge and Kantorovich):

$$\inf \{ \mathcal{F}(\lambda) , -\operatorname{div} \lambda = \mu^+ - \mu^- \} \quad (\lambda \text{ transport flux measure})$$

- Dynamic formulation (Brenier and Benamou):

$$\inf \left\{ \int_0^T \mathcal{F}'(V\rho, \rho) dt , \frac{\partial \rho}{\partial t} + \operatorname{div}(V\rho) = 0 , \rho(0) = \mu^+ , \rho(T) = \mu^- \right\}$$

( $\rho(t)$  mass density at time  $t$  ,  $V$  the speed)

# Goals

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- Two main mathematical issues:

1) Functionals  $\mathcal{F}$  for which optimal measures  $\lambda$  are one dimensional ?

2) Functionals  $\mathcal{F}'$  for which optimal  $\rho(t)$  are discrete measures ?

- A model case has been widely studied

$$\mathcal{F}(\lambda) = \int_S \theta^\alpha dH^1 \quad \text{if } \lambda = \theta \tau_S H^1 \llcorner S \quad (+\infty \text{ otherwise}) \quad (1)$$

where  $0 \leq \alpha < 1$ .

**Remark:**  $\alpha = 1$  gives [Monge-Kantorovich](#) problem (the one-rectifiability constraint disappears after relaxation)

- J.R Banavar and All.: Universality classes of optimal channel networks, Science, 1996
- Irrigation problems: J.M. Morel, V.Caselles, M. Bernot (probability on curves)
- Q. Xia, B. Hardt:  $W^\alpha$  Monge distance (via completion)
- G.Buttazzo, F. Santambrogio, E.Stepanov, GMT point of view

# Observation in the 2D-case

Connection with Mumford-Shah image segmentation problem holds for:

$d = 2$  (static case) or for  $d = 1$  (dynamic case)

Let:

- $\Omega \subset \mathbb{R}^2$  bounded with smooth boundary  $\Gamma$
- $\mu^+, \mu^-$  densities on  $\Gamma$
- $u_0 : \Gamma \rightarrow \mathbb{R}$  is a primitive of  $f := \mu^+ - \mu^-$

Then for  $\lambda \in \mathcal{M}(\mathbb{R}^2; \mathbb{R}^2)$  supported in  $\bar{\Omega}$

$$(i) \quad -\operatorname{div} \lambda = f \iff \exists u \in \operatorname{BV}(\Omega) : u = u_0 \text{ on } \Gamma, \lambda = (-\partial_2 u, \partial_1 u) .$$

$$(ii) \quad \lambda = \theta H^1 \llcorner S \iff \nabla u = 0 \text{ a.e. in } \Omega \setminus S, [u] = \theta \text{ on } S.$$

Thus  $\inf\{\mathcal{F}_\alpha(\lambda) \mid -\operatorname{div} \lambda = f, \operatorname{spt} \lambda \subset \bar{\Omega}\}$  is equivalent to:

$$\inf \left\{ \int_{S_u} [u]^\alpha dH^1 \mid u \in \operatorname{SBV}(\Omega), u = u_0 \text{ on } \Gamma, \nabla u = 0 \text{ a.e.} \right\}$$

**Remark:** Truncation of  $u$  (piecewise constant)  $\Leftrightarrow$  Removing loops in  $\lambda$ .

# Mass transport, economy of scales and speed

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- **Monge transport** G. Monge was motivated by transporting earth from an area to an other one, “the price of the transport of a single molecule being”

(i) “proportional to its weight and” (ii) “to the distance that one makes it covering”

hence the price of the total transport is proportional to the sum of the products of the molecules each multiplied by the distance covered.

**Remark:** Assumption (i) says that many molecules can be transported in a single “convoy”, the cost of which is proportional to the number of transported molecules. Then molecules follow a straight line (with a constant speed).

- **Economy of scales** In contrast the marginal cost of the transport decreases when the transported mass increases. We will assume that “the price of the transport of one molecule” is a **concave function**  $g(m)$  of “its weight”  $m$  ( typically  $g(m) = m^\alpha$  with  $0 < \alpha < 1$ ). This changes drastically the structure of optimal solutions: it is economic to group the transported masses as long as possible : each “molecule” will not follow a straight line any more and the optimal strategy has to be described in a time-space setting.

# Mass transport, economy of scales and speed

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- **Speed of molecules** In general, transport at high speed is much more expensive. We admit that “the price of the transport of one molecule” is a **convex function  $f$  of the velocity  $V$**  , typically:

$$f(V) = A + BV + CV^p \quad \text{with } p > 1, \quad A, B \text{ and } C \geq 0.$$

- $f(0) = A > 0$  means that “parking” has a cost (which is not absurd from the economical point of view).
- $B = f'(0) > 0$  in contrast favors stationary masses.
- $\alpha = 1$  and  $A = B = 0, C = 1$  yields time formulation for  $p$  Wasserstein (see Benamou-Brenier)
- $\alpha < 1, p = 1$  and  $A = B = 0, C = 1$  yields time formulation for for the irrigation problem studied by Xia and Morel.

# Example 1

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Consider two masses  $M > 0$  and  $m > 0$  located at time 0 at the same point  $x_0$  to be transported respectively to points  $x_1$  and  $x_2$  at time  $T = 1$ . In other words  $\rho(0) = \mu^+ = (M + m)\delta_{x_0}$  and  $\rho(1) = \mu^- = M\delta_{x_1} + m\delta_{x_2}$ .

We look only for two phases optimal dynamics:

- a) For  $[0, t_c]$  the two masses are transported together from  $x_0$  toward a point  $x_c$  following a kinematic law  $y_0(t)$ .
- b) For  $t \in [t_c, 1]$ , they are transported separately from  $x_c$  toward  $x_1$  and  $x_2$  following respectively the kinematic laws  $y_1(t)$  and  $y_2(t)$ .

The cost of such a transport is:

$$\int_0^{t_c} f(\|\dot{y}_0(t)\|)g(M+m) dt + \int_{t_c}^1 f(\|\dot{y}_1(t)\|)g(M) dt + \int_{t_c}^1 f(\|\dot{y}_2(t)\|)g(m) dt \quad (2)$$

to be minimized with respect to the junction time and place  $(t_c, x_c) \in \mathbb{R} \times \mathbb{R}^d$  and the kinematic laws  $y_0(t)$ ,  $y_1(t)$  and  $y_2(t)$  which are subjected to the constraints

$$y_0(0) = x_0, \quad y_0(t_c) = y_1(t_c) = y_2(t_c) = x_c, \quad y_1(1) = x_1, \quad y_2(1) = x_2. \quad (3)$$

By the convexity of  $f$ , the velocities of the different convoys are constant.

# Example 1

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Thus by (2)(3), we have to minimize

$$F = t_c g(M + m) f\left(\frac{\|x_c - x_0\|}{t_c}\right) + (1 - t_c) g(M) f\left(\frac{\|x_c - x_1\|}{1 - t_c}\right) + (1 - t_c) g(m) f\left(\frac{\|x_c - x_2\|}{1 - t_c}\right)$$

with respect to  $(t_c, x_c)$

We draw a time-space representation of the optimal transport for

$$d = 1, \quad M = 1, \quad m = 0.5, \quad x_0 = 0, \quad x_1 = 1, \quad x_2 = 0.5$$

showing the effects of the parameters  $\alpha, A, B, p$

(  $g$  and  $f$  are defined by  $g(m) = m^\alpha$  and  $f(V) = A + BV + V^p$  )

# Numerics for example 1

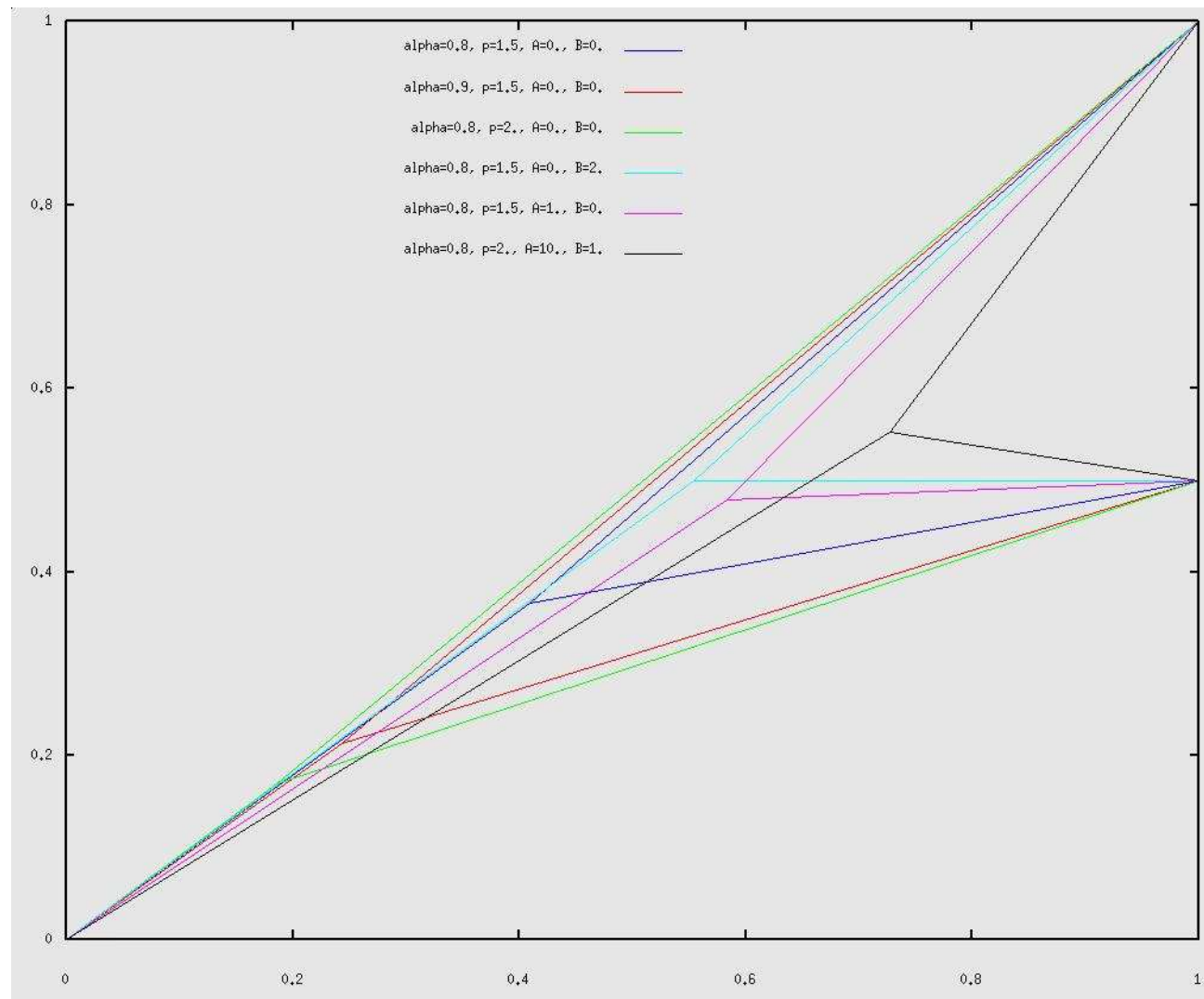


Figure 1: Optimal transport depends on  $p, \alpha, A$  and  $B$ .



## Example 2: Transport of a line density to a Dirac

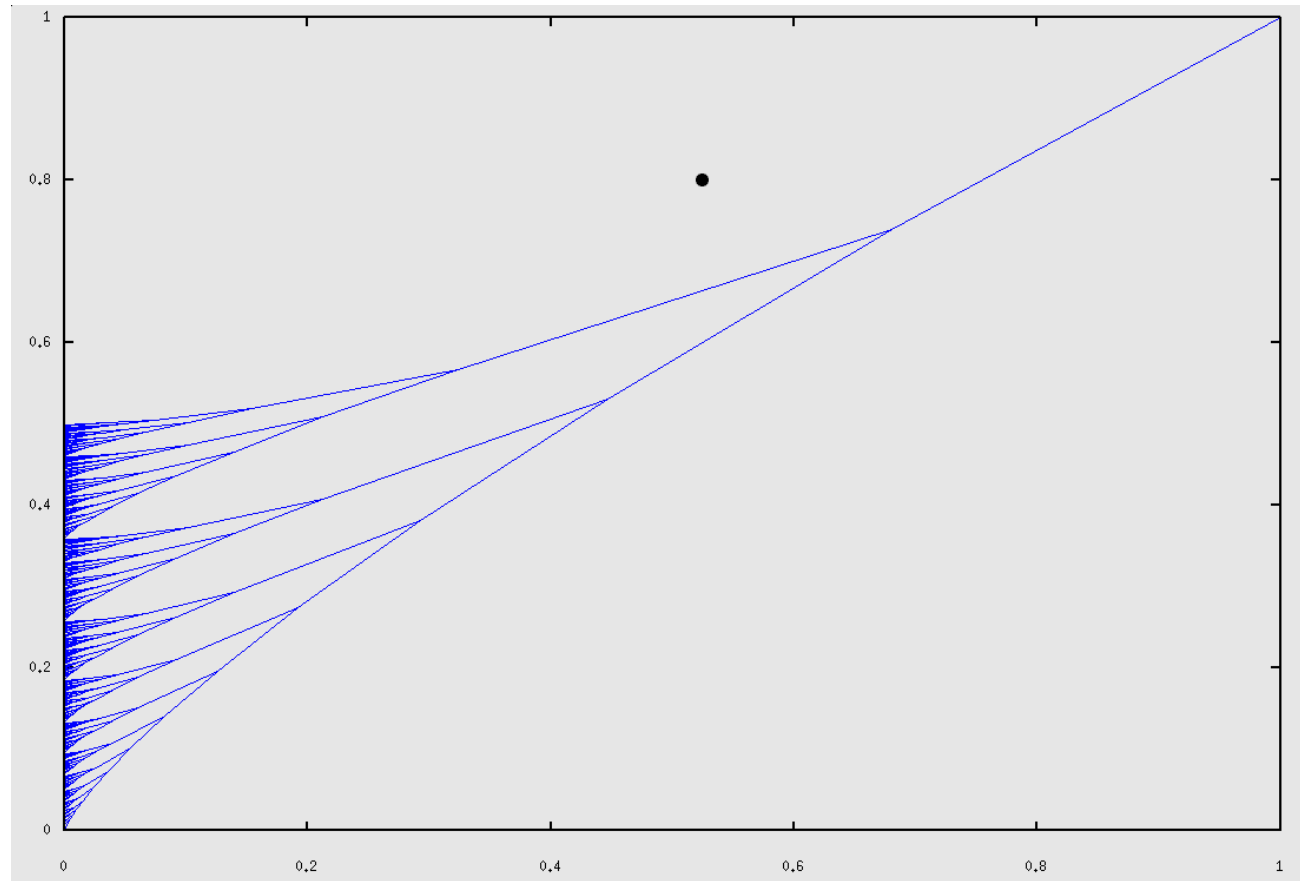


Figure 2: Auto-similar construction of optimal transport  $p = 2, \alpha = 0.9$

# Test for $\alpha$ closed to 1

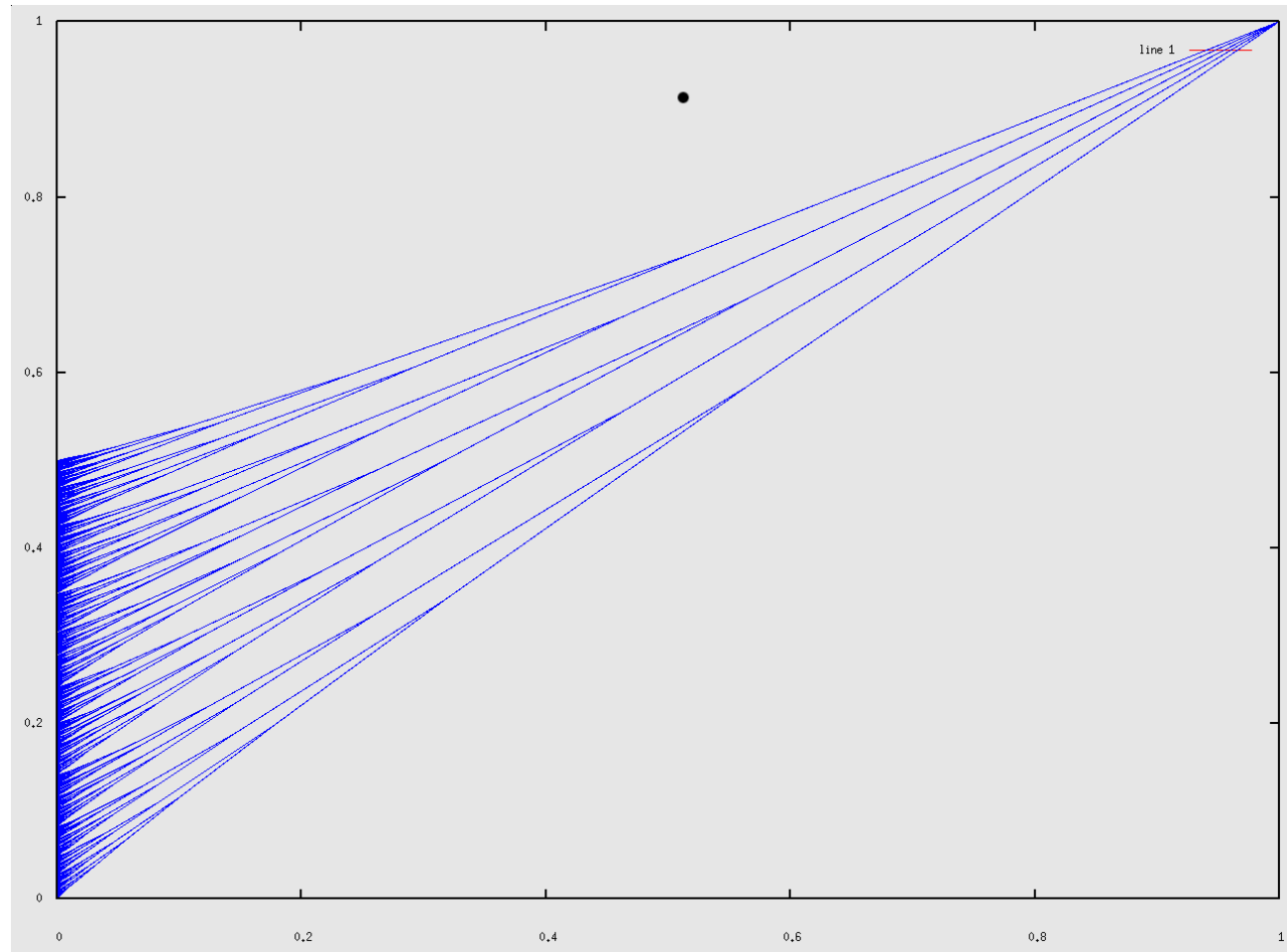


Figure 3: Transport of a line density to a Dirac,  $p = 6, \alpha = 0.9$

# OUTLINE

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$$\mathcal{F}(\lambda) := \int_S g(\theta) h(\tau_S) dH^1$$

where:

- $S$  is a 1-rectifiable subset of  $\mathbb{R}^d$ ,  $\tau_S$  a unit tangent vector
- $g : \mathbb{R}^+ \rightarrow [0, +\infty]$  is **concave**, monotone increasing and  $g'(0+) = +\infty$ .
- $h : \mathbb{R}^d \rightarrow [0, +\infty]$  is convex l.s.c., 1-homogeneous

Note that no lower semicontinuity result is known (except in dimension 2 via Mumford-Shah functional). We will mix different techniques

**1- Removing loops**

**2- Probability on curves and Smirnov decomposition of transport measures.**

**3- Intensity function and 1-rectifiability Theorem**

**4- Tightness results and lower semicontinuity of  $\mathcal{F}$ .**

**5- Application to dynamic formulations**

**6- Optimality conditions and approximation**

# 1- Transport and loops

- **Transport measures:** A transport on  $\mathbb{R}^d$  is a vector measure  $\lambda \in \mathcal{M}(\mathbb{R}^d, \mathbb{R}^d)$  such that  $\operatorname{div} \lambda = \mu^+ - \mu^- \in \mathcal{M}(\mathbb{R}^d)$  (thus  $\int \mu^+ = \int \mu^-$ ).

It is called **1-rectifiable** if of the kind  $\lambda = \theta \tau_S \llcorner H^1 \llcorner S$  (for a suitable 1-rectifiable subset  $S$ )

The weak convergence of transports is defined by (Flat norm convergence)

$$\lambda_n \rightharpoonup \lambda \iff \lambda_n \xrightarrow{*} \lambda, \operatorname{div} \lambda_n \xrightarrow{*} \operatorname{div} \lambda.$$

- **Sub-transport:**  $\lambda'$  is a sub-transport of  $\lambda$  if there exists suitable Borel functions:  $\xi, \alpha, \beta : \mathbb{R}^d \rightarrow [0, 1]$  such that

$$\lambda' = \xi \lambda, \quad \operatorname{div}(\lambda') = \alpha \mu^+ - \beta \mu^-$$

- **Loops:** It is a sub-transport such that  $\operatorname{div} \lambda' = 0$  ( i.e.  $\alpha = \beta = 0$ )

**OBSERVATION:** As  $g$  monotone  $\searrow$ , we have

$$\text{For all } \xi : \mathbb{R}^d \rightarrow [0, 1], \quad \mathcal{F}(\xi \lambda) \leq \mathcal{F}(\lambda). \quad (4)$$

## 1.2 Sub- transports and loops

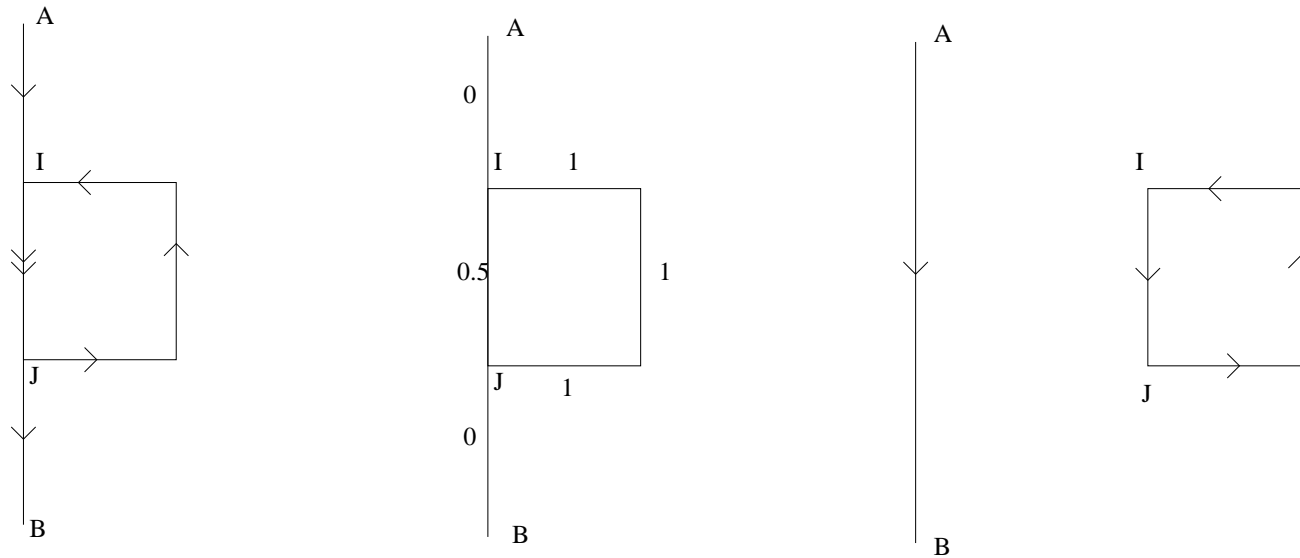


Figure 4: Removing loops:  $\lambda$ ,  $\xi$  and decomposition of  $\lambda$

## 1.3 Removing loops

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If loops are allowed, no hope for coerciveness

$$(*) \quad \mathcal{F}(\lambda_n) \leq C, \quad \operatorname{div} \lambda_n = \mu^+ - \mu^- \quad \Rightarrow \quad \sup_n \int |\lambda_n| < +\infty.$$

We will therefore **remove loops** by using :

**LEMMA 1:** Given a transport measure  $\lambda$ , there exists a (non unique) Borel function  $\xi : \mathbb{R}^d \rightarrow [0, 1]$  such that:  $\xi \lambda$  is loop free and  $\operatorname{div}(1 - \xi)\lambda = 0$

**Proof:** We consider a minimizer for problem:

$$\inf \left\{ \int \xi |\lambda| : \xi \in L^\infty_{|\lambda|}(\mathbb{R}^d; [0, 1]), \operatorname{div}(1 - \xi)\lambda = 0 \right\}.$$

The property (\*) will be reached for loop-free 1-rectifiable transports thanks to:

**LEMMA 2:** Assume that  $\lambda = \theta \tau_S H^1 \llcorner S$  is loop-free and satisfies  $\operatorname{div} \lambda = \mu^+ - \mu^-$  where  $\int \mu^+ = \int \mu^- = M$ . Then:  $0 \leq \theta \leq M$

Then by the concavity of  $g$ :  $g(\theta) \geq \theta \frac{g(M)}{M}$ .

## 2- Space of curves and probabilities

We assume that transport takes place in a **convex compact subset**  $\Omega$ .

- $X_\Omega$  will denote the space of equivalent classes of Lipschitz **oriented** curves  $\gamma : [0, 1] \mapsto \Omega$

where  $\gamma \sim \tilde{\gamma}$  means that  $\tilde{\gamma} = \gamma \circ \theta$  for a strictly increasing bijection of  $[0, 1]$ .

- $X_\Omega$  is a complete separable metric space (see Buttazzo) with

$$d(\gamma_1, \gamma_2) := \inf_{\theta} \sup_{t \in [0, 1]} |\gamma_1(t) - \gamma_2(\theta(t))| ,$$

and convergence in  $(X_\Omega, d)$  implies Hausdorff convergence of the images.

- $X_\Omega$  is not locally compact but for every  $l > 0$  the following subset is compact

$$K_l := \{ \gamma : L(\gamma) \leq l \} \quad , \quad L(\gamma) := \int_0^1 |\dot{\gamma}(t)| dt .$$

- To every  $\gamma \in X_\Omega$ , we can associate a line transport measure (supported in  $\Omega$ )

$$\langle \lambda_\gamma, \phi \rangle := \int_0^1 \phi(\gamma(t)) \cdot \dot{\gamma}(t) dt \quad , \quad \operatorname{div} \lambda_\gamma = \delta_{\gamma(0)} - \delta_{\gamma(1)} .$$

**Remark:** The map  $\gamma \in X_\Omega \mapsto \lambda_\gamma \in \mathcal{M}(\Omega; \mathbb{R}^d)$  is not continuous (no control on  $\dot{\gamma}$ ).

## 2.2 Representation of transports through measures on $X_\Omega$

To every finite positive Borel measures  $p$  on  $X_\Omega$ , we may associate the weighted transport denoted  $\lambda(p)$  ( or  $\int \lambda_\gamma p(d\gamma)$ ) defined by

$$\langle \lambda(p), \phi \rangle := \int_{X_\Omega} \left( \int_0^1 \phi(\gamma(t)) \cdot \dot{\gamma}(t) dt \right) p(d\gamma). \quad (5)$$

$$\operatorname{div}(\lambda(p)) = e_0^\#(p) - e_1^\#(p),$$

where for  $i \in \{0, 1\}$ ,  $e_i^\#(p)$  denotes the image of  $p$  by the continuous map  $e_i : \gamma \mapsto \gamma(i)$ .

**Notion of complete decomposition:** we say that the decomposition (5) is **complete** (or simply that  $p$  is **complete**) if:

$$(i) \quad \operatorname{spt}(p) \subset \{\gamma \in X_\Omega : \gamma \text{ is simple}\}$$

$$(ii) \quad \int_{\mathbb{R}^d} |\lambda(p)| = \int_{X_\Omega} |\lambda_\gamma| p(d\gamma) = \int_{X_\Omega} L(\gamma) p(d\gamma)$$

$$(iii) \quad \int_{\mathbb{R}^d} |\operatorname{div} \lambda(p)| = \int_{X_\Omega} |\operatorname{div} \lambda_\gamma| p(d\gamma) = 2 \int_{X_\Omega} p(d\gamma)$$



## 2.3 Example of complete decompositions

**Remark** The following localized inequalities (always true) become equalities

$$|\lambda(p)| \leq \int_{X_\Omega} |\lambda_\gamma| p(d\gamma) \quad , \quad |\operatorname{div}(\lambda(p))| \leq e_0^\#(p) + e_1^\#(p).$$

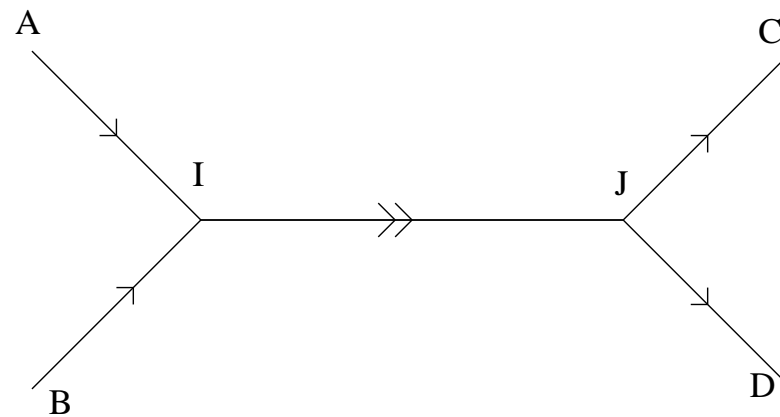


Figure 5: A transport to be decomposed

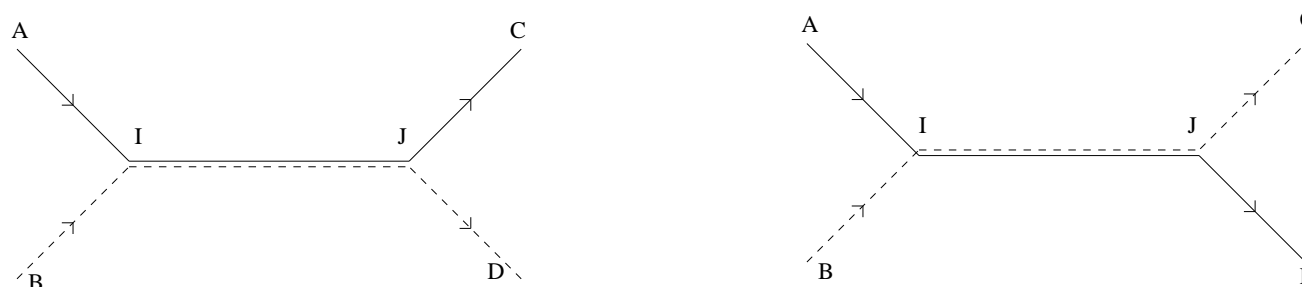


Figure 6: Two complete decompositions

## 2.4 Reformulation of optimal transport problem

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It is tempting to reformulate the optimal transport problem as

$$\inf \left\{ \mathcal{F}(\lambda(p)) : p \in \mathcal{M}_+(X_\Omega), e_0^\#(p) = \mu^+, e_1^\#(p) = \mu^- \right\}.$$

Several mathematical questions arise:

1) Are all transports of the form  $\lambda(p)$  and with  $p$  complete ??

Answer: YES if  $\lambda$  is loop-free (consequence of Smirnov Thm)

2) Tightness: does  $\mathcal{F}(\lambda(p_n)) \leq C$  imply that  $p_n$  is tight ?

Answer: Yes but only if  $p_n$  is complete

3) Does  $p_n \rightharpoonup p$  imply that  $\lambda(p_n) \rightharpoonup \lambda(p)$  ?

Answer: No in general but OK if  $L(\gamma)p_n(d\gamma)$  is tight !

4) Alternative expression for  $\mathcal{G}(p) := \mathcal{F}(\lambda(p))$

Needs to check that  $\mathcal{G}(p) < +\infty \Rightarrow \lambda(p)$  is 1- rectifiable

5) Lower semicontinuity of  $\mathcal{G}$  ?

OK if  $g(t)/t$  is monotone decreasing

## 2.5 Smirnov decomposition

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**THM.** Let  $\mu^+, \mu^-$  be two positive measures ( with  $\int \mu^+ = \int \mu^-$  ).  
Let  $\lambda \in \mathcal{M}(\Omega; \mathbb{R}^d)$  be a transport such that  $\operatorname{div} \lambda = \mu^+ - \mu^-$ .

If  $\lambda$  is loop-free , then it admits a complete decomposition  $\lambda = \lambda(p)$  for a suitable  $p \in \mathcal{M}_+(X_\Omega)$ .

Futhermore, we have  $\mu^+ = e_0^\#(p)$  ,  $\mu^- = e_1^\#(p)$  so that

$$\int \mu^+ = \int \mu^- = \int_{X_\Omega} p(d\gamma)$$

**Proof** By SMIRNOV , any transport measure  $\lambda$  can be decomposed in the form:

$\lambda = \lambda(p) + \lambda^0$  , where:

- i)  $\operatorname{div} \lambda^0 = 0$  (  $\lambda^0$  accounts the loops)
- ii)  $\int |\lambda| = \int |\lambda(p)| + \int |\lambda^0| = \int L(\gamma) p(d\gamma) + \int |\lambda^0|$
- iii)  $\int |\operatorname{div} \lambda| = \int |\operatorname{div}(\lambda(p))| = 2p(X_\Omega)$

The first equality in ii) and the strict convexity of Euclidean norm implies that  $\lambda(p) = \xi \lambda$  for  $\xi \in [0, 1]$ . As  $\lambda$  is loop free  $\xi = 1$  and  $\lambda^0 = 0$ .

### 3. Intensity functions and 1-rectifiability Theorem

Given  $p \in \mathcal{M}_+(X_\Omega)$ , we define  $\theta_p(x)$  (simple intensity) and  $i_p(x)$  (total intensity) the Borel functions:

$$\theta_p(x) := p(\{x \in \gamma([0, 1])\}) = \int_{X_\Omega} 1_{\{x \in \text{Im}(\gamma)\}} p(d\gamma) ,$$

$$i_p(x) := \int_{X_\Omega} \#(\{t \in [0, 1] : \gamma(t) = x\}) p(d\gamma) .$$

- $i_p(x) \geq \theta_p(x)$  but  $\theta_p(x) = 0 \Rightarrow i_p(x) = 0$  !
- $\theta_p(x) \leq p(X_\Omega)$  and by Smirnov, we deduce Lemma 2.

**Lemma 3** The function  $(x, p) \in \Omega \times \mathcal{M}_+(X_\Omega) \mapsto \theta_p(x)$  is upper semicontinuous.

We introduce also the **intensity measure**

$$\mu_p(B) := \int_{X_\Omega} |\lambda_\gamma|(B) p(d\gamma) = \int_{X_\Omega} \left( \int_{\gamma^{-1}(B)} |\dot{\gamma}(t)| dt \right) p(d\gamma).$$

Then  $\int \mu_p = \int L(\gamma) p(d\gamma)$  ( $\geq \int |\lambda(p)|$ ) with equality if  $p$  is complete).

## 3.2 The functional $\mathcal{G}(p)$

We set  $\beta(t) := \frac{g(t)}{t}$  and then

$$\mathcal{G}(p) := \int_{X_\Omega} \left( \int_0^1 \beta(\theta_p(\gamma(t))) h(\dot{\gamma}(t)) dt \right) p(d\gamma) .$$

**Justification** If  $p$  is supported on a simple curve  $\gamma_0$  i.e.  $p = \theta \delta_{\gamma_0}$ , then denoting  $S = \gamma_0([0, 1])$ , we obtain  $\theta_p = i_p = \theta$  on  $S$  and

$$\lambda(p) = \theta \tau_S H^1 \llcorner S \quad , \quad \mathcal{G}(p) = \theta \int_0^1 \beta(\theta) h(\dot{\gamma}_0(t)) = \int_S g(\theta) h(\tau_S) dH^1 = \mathcal{F}(\lambda(p)) .$$

**LEMMA 4.** Assume that  $g$  is concave  $\nearrow$  and that  $h$  is convex l.s.c. positively one-homogeneous. Then the functional  $p \in \mathcal{M}_+(X_\Omega) \mapsto \mathcal{G}(p)$  is lower semicontinuous (for the weak convergence of measures)

### 3.3 Rectifiability result

**LEMMA 5** (1-rectifiability) Let  $p \in \mathcal{M}_+(X_\Omega)$  such that:  $\mu_p(\Omega) < +\infty$  and  $i_p > 0$   $\mu_p$  a.e. . Then there exists a 1-rectifiable subset  $S$  such that:

$$\mu_p = i_p(x) H^1 \llcorner S .$$

**COROLLARY** Assume that  $\beta(t) \nearrow +\infty$  as  $t \searrow 0$  (with  $h$  coercive). Then

$$\mathcal{G}(p) < +\infty \Rightarrow \lambda(p) = \theta \tau_S H^1 \llcorner S , \text{ with } S \text{ rectifiable , } 0 \leq \theta(x) \leq i_p(x) \text{ on } S .$$

**Proof:** Observe that  $x \in \gamma([0, 1]) \Rightarrow \liminf_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_{\gamma^{-1}(B(x, \varepsilon))} |\dot{\gamma}(t)| dt \geq 1 .$

By Fatou  $\liminf_{\varepsilon \rightarrow 0} \frac{\mu_p(B(x, \varepsilon))}{2\varepsilon} \geq \theta_p(x)$ . Let  $S_k := \{\theta_p > 1/k\}$ .

Then  $H^1(S_k) < +\infty$  and  $\cup S_k = \{\theta_p > 0\} = S \cup S'$  with  $S$  one-rectifiable.

$\mu_p$  has no contribution on the purely unrectifiable part  $S'$ . Applying Fubini with  $H^1 \llcorner S(dx) \otimes p(d\gamma)$  leads to  $\mu_p = i_p H^1 \llcorner S$ .

Eventually as  $|\lambda(p)| \leq \mu_p$ ,  $\lambda(p) = \xi \mu_p$  where  $\xi$  parallel to  $\tau_S$  by the divergence condition.

### 3.4 Relation between $\mathcal{F}$ and $\mathcal{G}$

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**THM** Let  $p \in \mathcal{M}_+(X_\Omega)$  a finite measure. Assume that  $\beta(t) = g(t)/t$  is nonincreasing with  $\beta'(0+) = +\infty$ . Then

$$(i) \quad \mathcal{F}(\lambda(p)) \leq \mathcal{G}(p) \quad \text{for every } p$$

$$(ii) \quad \mathcal{F}(\lambda(p)) = \mathcal{G}(p) \quad \text{if } p \text{ is complete}$$

**Hint:**  $\mathcal{G}(p) < +\infty$  implies that  $\theta_p > 0$  on the support of  $\mu_p$ . Then we apply the rectifiability result to  $\lambda(p)$ .

**COROLLARY** As every transport can be substituted with another one loop-free with lower energy (and also with complete decomposition):

$$\inf \{ \mathcal{F}(\lambda) : \operatorname{div} \lambda = \mu^+ - \mu^- \} = \inf \{ \mathcal{G}(p) : e_0^\#(p) = \mu^+, e_1^\#(p) = \mu^- \},$$

$$\operatorname{Argmin} \mathcal{F} = \{ \lambda(p) : p \in \operatorname{Argmin} \mathcal{G} \}$$

STILL NEEDS TO PROVE CONVERGENCE OF MINIMIZING SEQUENCES  $(p_n)$   
for  $\mathcal{G}$  in  $\mathcal{M}_+(X_\Omega)$  !

## 4. Tightness results and lower semicontinuity of $\mathcal{F}$

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We already know that loop-free minimizing sequence of transports are uniformly bounded in variation hence weakly compact in  $\mathcal{M}(\Omega; \mathbb{R}^d)$ .

As regards minimizing sequences  $(p_n)$  in  $\mathcal{M}_+(X_\Omega)$ , we need to check the **Prokhorov's tightness criterium**.

- **Complete minimizing  $(p_n)$  are precompact**

Assume that  $\int \mu^+ = \int \mu^- = 1$ . If  $p_n$  is complete,  $\theta_{p_n} \leq 1$  (by Lemma 2) and  $g(\theta_{p_n}) \geq g(1) \theta$  (by the concavity of  $g$ ). Thus

$$\mathcal{G}(p_n) \geq g(1) \int_{X_\Omega} L(\gamma) p_n(d\gamma) = g(1) \int_{X_\Omega} \mu_{p_n}.$$

For every  $l$ , the compact set  $K_l = \{L(\gamma) > l\}$  satisfies  $p_n(K_l) \leq Cl$ .

$\Rightarrow$  **EXISTENCE OF SOLUTIONS**

**Argmin  $\mathcal{G}$  is non empty** (by the l.s.c. of  $\mathcal{G}$ )

**Argmin  $\mathcal{F}$  is non empty** (by the equivalence between  $\mathcal{F}$  and  $\mathcal{G}$ )

**Remark** If  $g$  is strictly  $\searrow$ , then optimal transports are loop-free.



## 4.2 Reinforced compactness

- Why need reinforced compactness ?

It is natural to ask whether or not

$$p_n \rightharpoonup p \quad \Rightarrow \quad \lambda(p_n) \rightharpoonup \lambda(p)$$

The answer is:

**NO** if we know merely that  $\sup_n \int L(\gamma) p_n(d\gamma) < +\infty$

**YES** if the sequence  $\{L(\gamma) p_n\}$  is tight.

**Hint:** The map  $\Lambda : \gamma \in X_\Omega \mapsto \lambda_\gamma \in \mathcal{M}(\Omega; \mathbb{R}^d)$  is not continuous.

Ex.: Let  $p_n := \frac{1}{n+1} \delta_{\gamma_n}$ ,  $\gamma_n(t) := (1 + n^{-1}) \exp(2i\pi n t)$ . Then  $p_n \rightharpoonup 0$  whereas  $\lambda(p_n) \rightharpoonup \lambda_{\gamma_1}$  (factor  $(1 + n^{-1})$  in order to have simple curves)

However  $\Lambda$  is **continuous on all compact**  $K_l$  and the contribution from  $X_\Omega \setminus K_l$  is controlled by

$$\int_{X_\Omega \setminus K_l} \|\lambda_\gamma\| p_n(d\gamma) \quad \left( = \int_{X_\Omega \setminus K_l} L(\gamma) p_n(d\gamma) \right)$$

## 4.3 Sub-transport estimate

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- **More on sub-transport**

Let  $p \in \mathcal{M}_+(X_\Omega)$  and  $E$  a Borel subset. Then  $\lambda(p \llcorner E)$  is a sub-transport of  $\lambda(p)$ .

Thus as  $g$  is **non decreasing**:  $\mathcal{F}(\lambda(p \llcorner E)) \leq \mathcal{F}(\lambda)$ . Besides if  $p$  is complete, so is  $p \llcorner E$  and  $\theta_{p \llcorner E} \leq p(E)$  (by Lemma 2). Now exploiting  $\beta \searrow$

**Lemma 6:** For every Borel subset  $E \subset X_\Omega$  and every complete  $p \in \mathcal{M}_+(X_\Omega)$ , there holds:

$$\mathcal{G}(p) \geq G(p_E) \geq h_{\min} \beta(p(E)) \int_E L(\gamma) p(d\gamma).$$

Applying to  $E = K_l$ , we get

$$(**) \sup_n G(p_n) < +\infty \Rightarrow \{L(\gamma) p_n\} \text{ tight} \Rightarrow \lambda(p_n) \rightharpoonup \lambda(p) \text{ whenever } p_n \rightharpoonup p$$

## 4.4 Lower semicontinuity of $\mathcal{F}$

- A weak statement for the l.s.c of  $\mathcal{F}$

We keep the previous assumptions on  $g, h$ .

**THM** Let  $(\lambda_n)$  be a sequence of **loop-free** transport measures such that:  
 $\lambda_n \rightharpoonup \lambda$  ,  $\operatorname{div} \lambda_n \rightharpoonup \operatorname{div} \lambda$  .  
Then  $\liminf_n \mathcal{F}(\lambda_n) \geq \mathcal{F}(\lambda)$  .

**Proof** There exists a sequence of complete measures  $p_n$  such that  $\lambda_n = \lambda(p_n)$ . As  $\operatorname{div} \lambda_n$  is upperbounded in variation so is  $p_n$  and we may assume  $p_n \rightarrow p$ . Then, as we know that  $G$  is l.s.c.:

$$\liminf \mathcal{F}(\lambda_n) = \liminf \mathcal{G}(p_n) \geq \mathcal{G}(p) \geq \mathcal{F}(\lambda(p)).$$

By the implication (\*\*):  $\lambda(p)$  coincides with  $\lambda$  (at least when  $\mathcal{F}(\lambda_n)$  is bounded)

## 4.5 A case excluding loops

Let  $\Sigma := \{z \in S^{d-1} : h(z) < +\infty\}$  and assume

$$(***) \quad \exists z^* \in S^{d-1} \quad : \quad z^* \cdot z < 0 \quad \forall z \in \Sigma$$

**Then  $\mathcal{F}(\lambda) < +\infty$  implies that  $\lambda$  has no loop**

**Proof** A transport  $\lambda = \theta \tau_S H^1 \llcorner S$  with  $\mathcal{F}(\lambda) < +\infty$  satisfies  $\tau_S \in \Sigma$  a.e. on  $S$ . Thus  $\int_S \theta \tau_S \cdot z^* dH^1 < 0$  : incompatible with  $\operatorname{div} \lambda = 0$  which forces  $\int \lambda = 0$ .

**CONCLUSION: Under (\*\*\*) ,  $\mathcal{F}$  is lower semicontinuous**

**Model example:**

- $z \in \mathbb{R}^d$  substituted with  $(z, t) \in \mathbb{R}^d \times \mathbb{R}_+$  (space and time).
- $f : \mathbb{R}^d \rightarrow \mathbb{R}_+$  is convex l.s.c. (function of the speed)

$$h(z, t) = \begin{cases} t f\left(\frac{z}{t}\right) & \text{if } t > 0 \\ f^\infty(z) & \text{if } t = 0 \\ +\infty & \text{if } t < 0 \end{cases}$$

Then (\*\*\*) holds for  $p > 1$ . If  $p = 1$ , loops can occur in slices of time  $\{t^*\} \times \mathbb{R}^d$  (meaning that the speed in the loop is infinite !)

## 5. Application to dynamic formulations

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- The transport of discrete measures is described at time  $t \in [0, T]$  by:

$$\rho_t = \sum_i c_i(t) \delta_{x_i(t)} \quad (\text{fonction from } [0, T] \text{ to } \mathcal{M}_+(\Omega))$$

$$\sigma_t = \sum_i c_i(t) \dot{x}_i(t) \delta_{x_i(t)} \quad (\text{"momentum" function from } [0, T] \text{ to } \mathcal{M}_+(\Omega; \mathbb{R}^d))$$

- We associate the measure  $\lambda = (\sigma, \rho) \in \mathcal{M}(\Omega \times \mathbb{R}; \mathbb{R}^{d+1})$  by setting:

$$\langle \rho, \varphi(x, t) \rangle = \int_0^T \left( \int_{\Omega} \varphi(x, t) \rho_t(dx) \right) dt, \quad \langle \sigma, \phi(x, t) \rangle = \int_0^T \left( \int_{\Omega} \phi(x, t) \cdot \sigma_t(dx) \right) dt.$$

- As far as finite speeds are considered:  $\sigma \ll \rho$  and  $V(x, t)$  denotes the Radon-Nikodym density. The kinematic constraints (including the one at the junctions) and boundary conditions  $\rho(0) = \mu^+$ ,  $\rho(T) = \mu^-$  reduce to:

$$(\text{dyn}) \quad \frac{\partial \rho}{\partial t} + \text{div}_x(\rho V) = \mu^+ \otimes \delta_0 - \mu^- \otimes \delta_T \quad \text{as distributions on } \mathbb{R}^{d+1}$$

**Remark** In fact under (dyn) and condition  $\int_{\mathbb{R}^{d+1}} |V| \rho(dxdt) < +\infty$ ,  $\rho$  is of the form  $\int_0^T \rho_t(dx) \otimes dt$  where the map  $t \mapsto \rho_t \in \mathcal{M}(\mathbb{R}^d)$  is continuous (weak topology) (see Ambrosio).

## 5.2 Parametrized curves in space-time

Let us argue on a situation with finitely many masses and junction points (the number of points may change in the process !)  $\rho(t) = \sum c_i(t) \delta_{x_i(t)}$ .

- Each trajectory  $(x_i(t), t) : t \in [0, T]$  can be seen as an oriented curve  $\gamma_i : s \in [0, 1] \mapsto \mathbb{R}^{d+1}$ , with a Lipschitz ↗ parametrization  $t = t(s)$  such that  $t(0) = t_i^-, t(1) = t_i^+$ .
- The measure  $\lambda = (\sigma, \rho)$  is supported on the one dimensional subset  $S = \cup_i S_i$ ,  $S_i = \text{Im}(\gamma_i)$  and has the form  $\lambda = \theta \tau H^1|_S$  where  $\tau = (\tau_x, \tau_t)$  is a unit tangent vector to  $S$  with  $\tau_t > 0$ .

The ratio  $\frac{|\tau_x|}{\tau_t}$  represents then the velocity.

- A simple change of variables on the graph  $S_i$  shows that

$$\int_{t_i^-}^{t_i^+} g(c_i(t)) f(\dot{x}_i) dt = \int_{S_i} f\left(\frac{\tau_x}{\tau_t}\right) g(c_i(t)) \tau_t H^1(dx)$$

so that summing over  $i$ , the total cost coincides with  $\mathcal{F}(\lambda) = \int_S h(\tau) g(\theta) dH^1$ , where  $\theta(x, t)$  is roughly  $\sum \{c_i(t) : x_i(t) = x\}$  and  $h$  is the function defined on  $\mathbb{R}^{d+1}$  by:  $h(\tau) := f\left(\frac{\tau_x}{\tau_t}\right) \tau_t$  if  $\tau_t > 0$ ,  $h(\tau) = +\infty$  otherwise.

## 5.3 Reformulation of the problem

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We end up with the variational problem:

$$\inf \left\{ \int_0^T \mathcal{F}'(V\rho, \rho) dt, \frac{\partial \rho}{\partial t} + \operatorname{div}(V\rho) = 0, \rho(0) = \mu^+, \rho(T) = \mu^- \right\}$$

where:

$$\mathcal{F}'(V\rho, \rho) = \int_S f\left(\frac{\tau_x}{\tau_t}\right) g(\theta(x, t)) \tau_t dH^1 = \int_S g(\theta) h(\tau_S) dH^1.$$

**EXISTENCE follows from the previous section**

**Remark** The fact that  $\sigma \ll \rho$  is forced if  $f$  has superlinear growth. In this case we know also that loops are also ruled out.

The case  $f(z) = |z|$  is more delicate !!

## 6. Optimality conditions and approximation

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The curve representation allows a very easy derivation of first order optimality conditions:

Let  $\lambda = \lambda(p)$  an optimal transport. Recall that  $p$  is complete ( $\theta_p = i_p$ ) and satisfies  $e_1^\sharp(p) = \mu^+$ ,  $e_0^\sharp(p) = \mu^-$ .

We consider a deformation map:  $\Psi_\varepsilon(x) = x + \varepsilon V(x)$  where  $V(x)$  is a smooth vector field such that:  $V = 0$  on  $\text{spt } \mu^+ \cup \text{spt } \mu^-$ .

Then set  $p_\varepsilon = \Psi_\varepsilon^\sharp(p)$ . Notice that for all  $\gamma$ ,  $\theta_{p_\varepsilon}(\Psi_\varepsilon \circ \gamma) = \theta_p(\gamma)$ , there holds:

$$\begin{aligned} 0 &= \lim_{\varepsilon \rightarrow 0} \frac{\mathcal{G}(p_\varepsilon) - G(p)}{\varepsilon} = \int_{X_\Omega} \int_0^1 \left( \beta(\theta_p)(\gamma) \frac{h(\dot{\gamma} + \varepsilon \nabla V(x) \dot{\gamma}) - h(\dot{\gamma})}{\varepsilon} \right) dt p(d\gamma) \\ &= \int_{X_\Omega} \int_0^1 (\beta(\theta_p)(\gamma(t)) \dot{\gamma}(t) \otimes \nabla h(\gamma) \nabla V(\gamma)) dt p(d\gamma) \\ &= \int_S g(\theta(x)) \frac{\tau \otimes \nabla h(\tau)}{h(\tau)} \cdot \nabla V(x) dH^1 \end{aligned}$$

$\implies$

$$\text{div} \left( g(\theta) \frac{\tau \otimes \nabla h(\tau)}{h(\tau)} H^1 \llcorner S \right) = 0 .$$



## 6.2 Approximation by viscosity

We consider Stokes flow with small viscosity ( $\varepsilon$ ) on a bounded domain  $\Omega \subset \mathbb{R}^3$  and add a non linear potential  $V : \mathbb{R}^3 \rightarrow \mathbb{R}^+$

$$\inf \left\{ \int_{\Omega} (V(u) + \varepsilon |\nabla u|^2) : \operatorname{div} u = \mu^+ - \mu_- \right\}$$

Choose  $V(u) := |u|^p$  where  $0 \leq p < 1$  (so that it is better to concentrate the flow)

**Remark:**  $p = 0$  is equivalent to find the optimal shape of the set  $\{u \neq 0\}$

**CONJECTURE:** As  $\varepsilon \rightarrow 0$ , the minimizers  $u_\varepsilon$  (in  $H^1(\Omega; \mathbb{R}^3)$ ) concentrate on 1-dimensional subsets:  $u_\varepsilon \rightharpoonup \lambda$  (in the sense of measures) where  $\lambda$  solves

$$\inf \left\{ \int_S \theta^\alpha dH^1 : \lambda = \theta \tau_S H^1 \llcorner S : \operatorname{div} \lambda = \mu^+ - \mu_- \right\}$$

The scaling law is deduced from the 2D profile equation on a disk  $D_r$ :

$$-\Delta \varphi + |\varphi|^{p-2} \varphi = cte, \quad \varphi \in H_0^1(D_r)$$

and its asymptotic as  $r \rightarrow 0$  (see GB, Dubs, Seppecher, CRAS and M3AS (1997))

We obtain

$$\alpha = \frac{2}{3-p}$$