Some shape optimization problems with a polygonal solution

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Introduction

We work in a particular class of plane convex sets:

$$\mathcal{A} := \{K \text{ convex set in } \mathbb{R}^2, s(K) = O, P(K) = 2\pi\}.$$

where $s(K)$ denotes the Steiner point of K and $P(K)$ its perimeter.

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- $\mathcal{A}$ is compact
- $\mathcal{A}$ is "convex" (for the Minkowski sum)
- What is the boundary of $\mathcal{A}$? Does it contain only polygons?

The farthest convex set

Let C_0 be a given convex set in $\mathcal{A}$.
Find the "farthest convex set" of C_0 in $\mathcal{A}$, i.e. one which satisfies

$$d(K, C_0) = \max\{d(C, C_0),\ C \in \mathcal{A}\}.$$

where d stands for a given distance among convex sets, e.g. the Hausdorff distance or the L^2 distance.

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Theorem [Existence] For any suitable distance, there exists at least one farthest convex set in the class $\mathcal{A}$.

The support function(1)

Let K be a plane convex set.
The support function h_K of K is defined by:

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The Steiner point $s(K)$ of the convex set is defined by:

$$s(K) = \frac{1}{\pi} \int_0^{2\pi} h_K(\theta) e^{i\theta}\, d\theta\,.$$

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The polygons are also well characterized

$$K \text{ is a polygon} \iff h_K'' + h_K = \sum_{j=1}^{n} a_j \delta_{\theta_j}$$

where $a_1, a_2, \ldots, a_n$ and $\theta_1, \theta_2, \ldots, \theta_n$ denote the lengths of the sides and the angles of the corresponding outer normals.

Examples

the equilateral triangle T:

$$h_T(\theta) = \begin{cases} \dfrac{2\pi}{3\sqrt{3}} \cos(\theta - \pi/3) & 0 \leq \theta \leq 2\pi/3 \\ \dfrac{2\pi}{3\sqrt{3}} \cos(\theta - \pi) & 2\pi/3 \leq \theta \leq 4\pi/3 \\ \dfrac{2\pi}{3\sqrt{3}} \cos(\theta - 5\pi/3) & 4\pi/3 \leq \theta \leq 2\pi\,. \end{cases}$$

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The line segments are particular convex sets.
If Σ_α designate the segment $[-i\frac{\pi}{2}e^{i\alpha}, i\frac{\pi}{2}e^{i\alpha}]$, its support function is given by

$$h_\alpha(\theta) := \frac{\pi}{2}|\sin(\theta - \alpha)|$$

which satisfies ${h_\alpha}'' + h_\alpha = \pi(\delta_\alpha + \delta_{\pi+\alpha})$.

Support function and distances

The Hausdorff distance can be defined using the support functions:

$$d_H(K, L) = \|h_K - h_L\|_\infty.$$

We can also define a L^p distance (Mc Clure and Vitale) by

$$d_p(K, L) := \left(\int_0^{2\pi} |h_K - h_L|^p \, d\theta \right)^{1/p} .$$

We will use here only the L^2 distance.

A geometric inequality

Theorem Let K be any plane convex set with its Steiner point at the origin. Then

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The first inequality is due to P. Mc Mullen. It implies that the diameter of $\mathcal{A}$ is less than $\pi/2$.

Idea of the proof

We introduce $F(K) := \min h_K + \max h_K$ and a line L which go through O and a point where h_K is minimum.

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- Let S be the segment orthogonal to L and for any convex K introduce $K_t := tK + (1-t)S$. We prove that $F(K) < F(S) \Rightarrow F(K_t) < F(S)\ \forall t > 0$.

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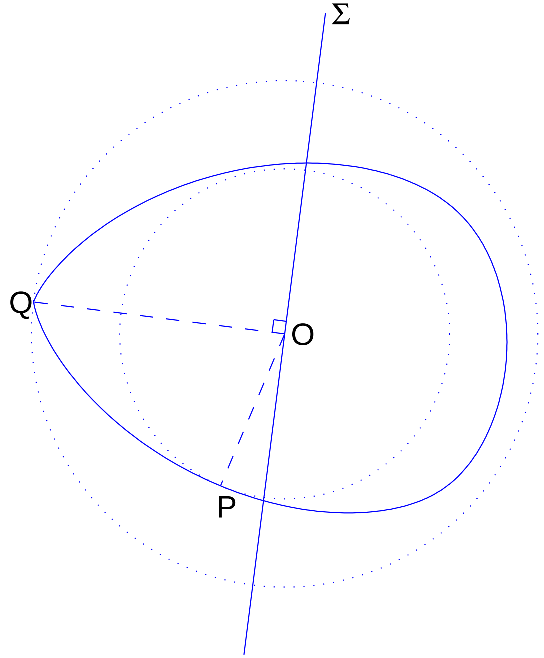
- It suffices to prove that S is a local minimum.

The farthest convex set (Hausdorff)

Theorem [farthest convex set for Hausdorff distance]
If C is a given convex set in the class $\mathcal{A}$, then the convex set K_C for which

$$d_H(C, K_C) = \max\{d_H(C, K) : K \in \mathcal{A}\}$$

is a segment.



For the L^2 distance

We come back to the L^2 distance

$$Q(K) = \int_0^{2\pi} (h_K - h_C)^2 \, d\theta = \int_0^{2\pi} h_K^2 - 2h_K h_C + h_C^2 \, d\theta$$

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More generally, we consider functionals J like

$$J(K) := \int_0^{2\pi} a\, h_K^2 + b\, {h'_K}^2 + c\, h_K + d\, h'_K \, d\theta$$

where a and b are nonnegative bounded functions of θ, one of them being positive almost everywhere. The functions c, d are assumed to be bounded.

A general result

Theorem Let J be a functional defined by

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where a, b, c, d satisfy the above conditions. Then every local maximizer of the functional J within the class $\mathcal{A}$ is either a segment or a triangle.

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Corollary The farthest convex set for the L^2 distance is either a segment or a triangle.

The optimality condition (1)

Let K_0 be a (local) maximizer of some functional J defined on the class $\mathcal{A}$, h_0 be its support function and S_{h_0} the support of the measure $h_0'' + h_0$.

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First order condition:
$\exists \xi_0 \in H^1(\mathbb{T})$, $\xi_0 \leq 0$, and $\mu_1, \mu_2, \mu_3 \in \mathbb{R}$ such that

$$\xi_0 = 0 \text{ on } S_{h_0},$$

and $\forall v \in H^1(\mathbb{T})$,

$$\langle J'(h_0), v \rangle = \langle \xi_0 + {\xi_0}'', v \rangle + \int_0^{2\pi} v(\mu_1 + \mu_2 \cos\theta + \mu_3 \sin\theta) d\theta \, .$$

The optimality condition (2)

Second order condition: Moreover, if $v \in H^1(\mathbb{T})$ is such that $\exists \lambda \in \mathbb{R}$ which satisfies

$$\begin{cases} v'' + v \geq \lambda({h_0}'' + h_0) \\ v \geq \lambda h_0 \\ \langle \xi_0 + {\xi_0}'', v \rangle + \int_0^{2\pi} v(\mu_1 + \mu_2 \cos\theta + \mu_3 \sin\theta) d\theta = 0. \end{cases}$$

then

$$\langle J''(h_0), v, v \rangle \leq 0\,.$$

Sketch of the proof of the main theorem

We follow ideas by T. Lachand-Robert, M.Peletier and J. Lamboley, A. Novruzi.

We want to prove that the support S_0 of $h_0'' + h_0$ does not contain more than 3 points.

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We want to prove that the support S_0 of $h_0'' + h_0$ does not contain more than 3 points.

Assume, for a contradiction, that S_0 contains at least four points $\theta_1 < \theta_2 < \theta_3 < \theta_4$. We solve the four differential equations

$$\begin{cases} v_i'' + v_i = \delta_{\theta_i} & \theta \in (\theta_1 - \varepsilon, \theta_4 + \varepsilon) \\ v_i(\theta_1 - \varepsilon) = v_i(\theta_4 + \varepsilon) = 0, & \end{cases}$$

Sketch of the proof (2)

We choose four numbers λ_i, $i = 1, \ldots, 4$ such that the three following conditions hold, where we denote by v the function defined by $v = \sum_{i=1}^{4} \lambda_i v_i$:

$$v'(\theta_1 - \varepsilon) = v'(\theta_4 + \varepsilon) = 0, \quad \int_0^{2\pi} v \, d\theta = 0 \, .$$

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Then v is admissible for the second order condition and we check that

$$\left\langle J''(h_0), v, v \right\rangle > 0$$

which is a contradiction.

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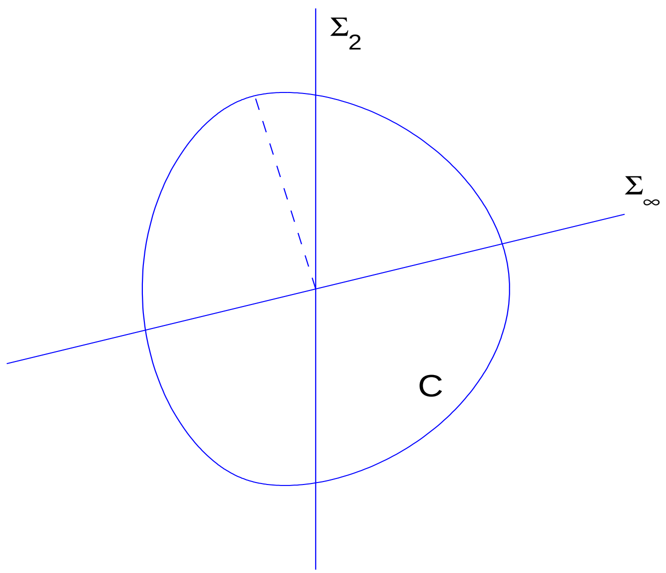

Conclusion and Generalization

The following results is mainly due to J. Lambolley and A. Novruzi. Let

$$J(K) := \int_0^{2\pi} G(h, h')\, d\theta$$

a general functional that we want to maximize.

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We have the general result:

G is convex in h' $\Longrightarrow$ every (local) maximizer is a polygon.