Controllability of a transport equation in singular limit

O. Glass

Laboratoire Jacques-Louis Lions,
Université Pierre et Marie Curie

&

CEREMADE,
Université Paris-Dauphine

Benasque, Aug. 26, 2009.

I. Introduction

- We consider a 1-D transport equation

$$y_t + My_x = 0 \text{ in } [0,T] \times [0,L],$$

  with $M \in \mathbb{R} \setminus \{0\}$.

- Standard controllability problem: given $T > 0$, y_0 and y_1 in some function space, can we find a solution from y_0 at $t = 0$ to y_1 at $t = T$ by choosing ad hoc boundary conditions?

- This equation is (trivially) controllable for $T > L/|M|$ and not controllable for $T < L/|M|$.

- Question. What can be said about the controllability of this system in a limit of vanishing viscosity?

$$y_t + My_x - \varepsilon y_{xx} = 0 \text{ as } \varepsilon \to 0^+?$$

A motivation

- Boundary control of conservation laws ($k = 1$) or hyperbolic systems of conservation laws ($k \geq 2$)

$$u_t + f(u)_x = 0,\ \ u : [0, T] \times [0, L] \to \mathbb{R}^N,\ \ f : \mathbb{R}^k \to \mathbb{R}^k,$$

  (where for all u, $df(u)$ have real distinct eigenlalues), in particular in the context of (weak) entropy solutions.

- Entropy solutions can be defined as weak solutions obtained by vanishing viscosity:

$$u^\varepsilon \to u \text{ as } \varepsilon \to 0^+ \text{ where } u^\varepsilon_t + f(u^\varepsilon)_x - \varepsilon u^\varepsilon_{xx} = 0.$$

  Cf. Hopf, Oleinik, Lax, Vol'pert, Kruzhkov, Bianchini-Bressan, etc.

- Question. Is it possible to obtain a uniform control for the viscous equation as $\varepsilon \to 0^+$?

The problem of null controllability in the vanishing viscosity limit

- Raised by Coron and Guerrero (2005)
- Consider the control system:

$$\begin{cases} y_t + My_x - \varepsilon y_{xx} = 0 \text{ in } (0,T) \times (0,L), \\ y_{|x=0} = v(t), \ y_{|x=L} = 0 \text{ in } (0,T), \\ y_{|t=0} = y_0 \text{ in } (0,L), \end{cases}$$

Questions

- Standard null-controllability problem. Given $T > 0$, is it possible to drive any y_0 to 0 at time T? (The answer is well-known and positive)
- Uniform controllability problem. Given $T > L/|M|$, is it possible to do so at a bounded cost as $\varepsilon \to 0^+$?
- Is it possible at least for $T \geq CL/|M|$? For which value of C?

Diffusive-dispersive limits

- In the same way, in certain physical situations (e.g. nonlinear elastodynamics with both viscosity and capillarity effects) it is interesting to consider diffusive-dispersive limits:

$$u_t + f(u)_x - \varepsilon u_{xx} + \nu u_{xxx} = 0 \ \text{ as } \ \varepsilon, \nu \to 0^+,$$

  which may converge to a weak solution different to the vanishing viscosity solution or to the same one, according to the situation.

  - Cf. the theory of "nonclassical shock waves", in particular the book of LeFloch.
  - See also Lax-Levermore for the KdV $\to$ Burgers (purely dispersive) limit.

- Consider the control system:

$$\begin{cases} y_t - My_x + \nu y_{xxx} - \varepsilon y_{xx} = 0 \text{ in } Q := (0,T) \times (0,L), \\ y_{|x=0} = v_1(t),\ y_{|x=L} = v_2(t),\ y_{x|x=L} = v_3(t) \text{ in } (0,T), \\ y_{|t=0} = y_0 \text{ in } (0,L), \end{cases} \qquad (1)$$

Questions

- Standard null-controllability problem. Given $T > 0$, is it possible to drive any y_0 to 0 at time T? Is it possible while letting $v_2 = v_3 = 0$? (The answer positive and due to Rosier)

- Uniform controllability problem. Given $T > L/|M|$, is it possible to do so at a bounded cost as $\varepsilon, \nu \to 0^+$?

- Is it possible at least for $T \geq CL/|M|$?

II. Previous studies and results

- The first result, due to Coron and Guerrero, concerns the vanishing viscosity limit.

Theorem (Coron-Guerrero, 2005)

If $M > 0$ and $T > 4.3\,L/M$ or if $M < 0$ and $T > 57.2\,L/|M|$, then the system is uniformly controllable in the sense that there are constants $C, \kappa > 0$, such that for any $y_0 \in L^2(0, L)$ and any $\varepsilon > 0$, one can find a control v driving the system to 0 at time T, at a cost

$$\|v\|_{L^2(0,T)} \leq C \exp(-\frac{\kappa}{\varepsilon})\|y_0\|_{L^2(0,L)}.$$

Theorem (Coron-Guerrero, 2005)

If $M > 0$ and $T < L/M$ or if $M < 0$ and $T < \mathbf{2}L/|M|$, then the system is not uniformly controllable in the sense that there exist $C, \kappa > 0$, such that for any $\varepsilon > 0$, there are initial states $y_0 \in L^2(0, L)$ for which any control v driving the system to 0 at time T satisfies

$$\|v\|_{L^2(0,T)} \geq C \exp(\frac{\kappa}{\varepsilon})\|y_0\|_{L^2(0,L)}.$$

Conjecture. The times L/M if $M > 0$ and $2L/|M|$ if $M < 0$ are optimal, that is, the system is uniformly controllable for times $T > L/M$ if $M > 0$ and $T > 2L/|M|$ if $M < 0$.

The problem is still open!

Other studies on the uniform controllability in the vanishing viscosity limit

- Guerrero-Lebeau: N-D transport equation in the vanishing viscosity limit:

$$y_t + M(t,x).\nabla y - \varepsilon \Delta y = 0.$$

$\rightarrow$ Cost of order $\mathcal{O}(e^{-1/\varepsilon})$ if T is large enough and the characteristics all meet the control zone, of order $\mathcal{O}(e^{1/\varepsilon})$ for T small.

- G.-Guerrero: 1-D Burgers equation in the vanishing viscosity limit:

$$y_t + yy_x - \varepsilon y_{xx} = 0.$$

$\rightarrow$ One can reach a constant state $U \neq 0$ in time $\mathcal{O}(1/|U|)$ at a constant cost, for any initial condition in L^∞.

Diffusive-dispersive limits

Theorem (G.-Guerrero): uniform controllability

There exists a positive constant K_0 such that for any **positive** constant M, there exist $c, C > 0$ such that for any $(\nu, \varepsilon) \in (0,1] \times [0,1]$, any $T \geq K_0 L/M$, any $y_0 \in L^2(0,L)$, there exist a control $v_1 \in L^2(0,T)$ such that the solution of the system with $v_2 = v_3 = 0$ satisfies $y_{|t=T} = 0$ in $(0,L)$ and such that

$$\|v_1\|_{L^2} \leq \frac{C}{\sqrt{\nu}} \exp\left\{-\frac{c}{\max\{\nu^{1/2}, \varepsilon\}}\right\} \|y_0\|_{L^2}.$$

Theorem (G.-Guerrero): non uniform controllability

Consider $M \neq 0$ and $T > 0$ such that $T < \frac{L}{|M|}$. Then there are some constants $c > 0$ and $\ell \in \mathbb{N}$ (independent of $\varepsilon \in [0,1]$ and $\nu \in (0,1]$) and initial states $y_0 \in L^2(0,L)$ such that any control $v_1 \in L^2(0,T)$ driving y_0 to 0 is estimated from below by

$$\|v_1\|_{L^2} \geq c\nu^{\ell} \exp\left\{\frac{c}{\max\{\nu^{1/2}, \varepsilon\}}\right\} \|y_0\|_{L^2}.$$

III. The approach by real analysis

- Initiated by Coron and Guerrero
- We give the example of uniform controllability for diffusive-dispersive systems

The standard duality argument (D. Russell, J.-L. Lions, etc.) shows that if one can prove for the adjoint system

$$\begin{cases} -\varphi_t + M\varphi_x - \nu\varphi_{xxx} - \varepsilon\varphi_{xx} = 0 & \text{in } (0,T)\times(0,L), \\ \varphi(t,0) = \varphi(t,L) = \varphi_x(t,0) = 0 & \text{in } (0,T), \\ \varphi(T,x) = \varphi_T(x) & \text{in } (0,L). \end{cases}$$

the following observability inequality

$$\int_0^L |\varphi(0,x)|^2\,dx \le K(T_0,M,\nu)\int_0^T |\varphi_{xx|x=0}|^2\,dt.$$

then for any y_0, one can find controls v_1, $v_2 = v_3 = 0$ that drive the system to 0, with

$$\|v_1\|^2_{L^2(0,T)} \le \frac{K(T,M,\nu)}{\nu}\|y_0\|^2_{L^2(0,L)}.$$

- With homogeneous boundary conditions, the adjoint equation can be considered as a parabolic equation. One can use typical tools for the control of parabolic equations.
- One proves a Carleman inequality for the system, à la Fursikov-Imanuvilov. In the purely diffusive case ($\nu = 0$), one can use a weight of the form:

$$\exp(-s\alpha) \text{ with } \alpha(t,x) := \frac{\beta(x)}{t(T-t)}, \ s \geq 0,$$

with β a positive increasing concave function.

- In the purely dispersive case ($\varepsilon = 0$), a Carleman inequality was established by Rosier with the previous weight. But optimizing the time dependence (which will be necessary in the sequel), one can use a weight of the form

$$\exp(-s\alpha) \text{ with } \alpha(t,x) := \frac{\beta(x)}{t^{1/2}(T-t)^{1/2}}, \ s \geq 0,$$

In our diffusive-dispersive case, we set

$$\alpha(t,x) = \frac{\beta(x)}{t^\mu (T-t)^\mu},$$

for $\mu \in [1/2, 1]$ and β as previously.

Proposition

There exist a constant $C > 0$ independent of T, $\nu > 0$, $\varepsilon \geq 0$ and $M \in \mathbb{R}$ such that for any $\varphi_T \in L^2(0,L)$, one has

$$s \int_0^T \int_0^L \alpha e^{-2s\alpha} \Big(\nu^2 |\varphi_{xx}|^2 + (\nu^2 s^2 \alpha^2 + \varepsilon^2) |\varphi_x|^2 + (\nu^2 s^4 \alpha^4 + \varepsilon^2 s^2 \alpha^2) |\varphi|^2 \Big) \, dx \, dt$$

$$\leq C\nu \int_0^T (\nu s \alpha_{|x=0} + \varepsilon) e^{-2s\alpha_{|x=0}} |\varphi_{xx|x=0}|^2 \, dt,$$

for any $s \geq CT^\mu(T^\mu + (1 + T^\mu |M|^\mu)/(\nu^{1-\mu}\varepsilon^{2\mu-1}))$, where φ is the corresponding solution of the adjoint system.

- This yields an observability inequality of order
    - $$K \sim \exp\left\{\frac{C}{\nu^{1/2}}\right\},$$
      in the "dispersive regime" where $\nu \gtrsim \varepsilon^2$,
    - $$K \sim \left(\frac{\nu^2}{\varepsilon^2} + \frac{\nu}{\varepsilon}\right) \exp\left\{\frac{C}{\varepsilon}\right\}.$$
      in the "diffusive regime" where $\nu \lesssim \varepsilon^2$.
- The constant are huge. This is normal, since we did not use the transport effect.
- The idea is to use a "dissipation estimate " (here, for the adjoint equation) to compensate the size of these constants.

Exponential dissipation estimates

- A close result was obtained by Danchin for the problem of the vanishing viscosity limit of vortex patches.
- Let us consider some time T_1 and times $0 \leq t_1 < t_2 \leq T_1$.
- One multiplies the adjoint equation with $\exp(r(M(T_1 - t) - x))\varphi$, one integrate with respect to x (where r is a positive parameter).
- It is essential here that the function $(t, x) \mapsto M(T_1 - t) - x$ is a solution of the transport equation.
- After several integration by part, one gets

$$- \frac{d}{dt}\Big(\exp\{-(\nu r^3 + \varepsilon r^2)(T_1 - t)\} \int_0^L \exp\{r(M(T_1 - t) - x)\}|\varphi(t, x)|^2 \, dx\Big) \leq 0.$$

- One integrates with respect to time between t_1 and t_2, and one gets

$$\int_0^L |\varphi(t_1, x)|^2 dx \leq \kappa \int_0^L |\varphi(t_2, x)|^2 dx,$$

with

$$\kappa = \exp\{\nu(t_2 - t_1)r^3 + \varepsilon(t_2 - t_1)r^2 + (L - M(t_2 - t_1))r\}.$$

- One optimizes with respect to r to deduce when $t_2 - t_1 \geq L/M$

$$\int_0^1 |\varphi(t_1, x)|^2 dx \leq \kappa \int_0^1 |\varphi(t_2, x)|^2 dx,$$

with κ satisfying

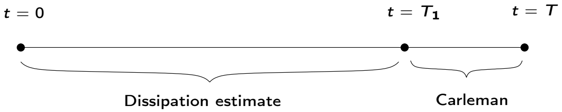
  - if $\varepsilon^2 \gtrsim \nu$:

$$\kappa \leq \exp\left\{-c \frac{(M(t_2 - t_1) - L)^2}{\varepsilon(t_2 - t_1)}\right\},$$

  - if $\varepsilon^2 \lesssim \nu$:

$$\kappa \leq \exp\left\{-c \frac{(M(t_2 - t_1) - L)^{3/2}}{\nu^{1/2}(t_2 - t_1)^{1/2}}\right\}.$$

- If $t_2 - t_1 \geq K_0/M$ for K_0 large enough, this allows to "absorb" the constant coming from the Carleman inequality.



IV. The approach by complex analysis

- One can try to approach Coron and Guerrero's problem (the vanishing viscosity limit for the transport equation), by suitably employing the method of moments, à la Fattorini-Russell.
- This allows to improve the time constants in the Coron-Guerrero theorem.

Theorem (G., 2009)

The control system:

$$\begin{cases} y_t + My_x - \varepsilon y_{xx} = 0 \text{ in } (0,T) \times (0,L), \\ y_{|x=0} = v(t), \ y_{|x=L} = 0 \text{ in } (0,T), \\ y_{|t=0} = y_0 \text{ in } (0,L), \end{cases}$$

is still uniformly controllable if $M > 0$ and $T > 4.2L/M$ or if $M < 0$ and $T > 6.1L/|M|$.

Remark

Coron and Guerrero gave $T > 4.3L/M$ if $M > 0$ and $T > 57.2L/|M|$ if $M < 0$. The main point is that the proof is of completely different nature...

Ideas of proof

- The proof uses the method of moments, cf. Fattorini-Russell (1971).
- It is also connected to the study of the cost of the control of parabolic systems for small times, cf.
  - Seidman, Seidman-Gowda, Seidman-Avdonin-Ivanov,
  - Fernández-Cara-Zuazua,
  - Miller,
  - Tenenbaum-Tucsnak,
  - ...
- Of course, by a time-scaling argument, it is essentially equivalent to control

$$u_t - \Delta u = 0,$$

during the time interval $[0, \varepsilon T]$, and to control

$$u_t - \varepsilon \Delta u = 0,$$

during the time interval $[0, T]$.

- One still wants to prove an observability inequality of the type

$$\|\varphi(0,\cdot)\|_{L^2(0,L)} \leq K \exp\Big(-\frac{\kappa}{\varepsilon}\Big)\|\partial_x \varphi(\cdot,0)\|_{L^2(0,T)},$$

for the adjoint equation

$$\begin{cases} \varphi_t + M\varphi_x + \varepsilon\varphi_{xx} = 0 \text{ in } (0,T)\times(0,L), \\ \varphi = 0 \text{ on } (0,T)\times\{0,L\}, \\ \varphi(T,\cdot) = \varphi_T \text{ in } (0,L). \end{cases}$$

- One can easily diagonalize the operator

$$P := -M\partial_x - \varepsilon\partial^2_{xx},$$

by noticing that

$$\partial^2_{xx}(e^{\frac{Mx}{2\varepsilon}}u) = e^{\frac{Mx}{2\varepsilon}}\left(\partial^2_{xx}u + \frac{M}{\varepsilon}\partial_x u + \frac{M^2}{4\varepsilon^2}u\right),$$

- Hence the operator $-M\partial_x - \varepsilon\partial_{xx}^2$ is diagonalizable in $L^2(0,L)$, with eigenvectors

$$e_k(x) := \sqrt{2}\exp\Big(-\frac{Mx}{2\varepsilon}\Big)\sin\Big(\frac{k\pi x}{L}\Big). \tag{2}$$

for $k \in \mathbb{N}\setminus\{0\}$ and corresponding eigenvalues

$$\lambda_k := \varepsilon\frac{k^2\pi^2}{L^2} + \frac{M^2}{4\varepsilon}, \tag{3}$$

the family $\{e_k,\ k \in \mathbb{N}\setminus\{0\}\}$ being a Hilbert basis of $L^2(0,L)$ for the $L^2((0,L);\exp(\frac{Mx}{\varepsilon})\,dx)$ scalar product.

- Consider a solution φ of the adjoint system, where

$$\varphi_T(x) = \sum_{k=1}^{N} c_k e_k(x).$$

- We deduce easily

$$\partial_x \varphi(t,0) = \sum_{k=1}^{N} c_k \sqrt{2} \frac{k\pi}{L} \exp(-\lambda_k (T-t)).$$

and

$$\varphi(0,x) = \sum_{k=1}^{N} c_k \exp(-\lambda_k T) e_k(x).$$

- Imagine that we have a family ψ_k which is bi-orthogonal to the family $f_k : t \mapsto \exp(-\lambda_k(T-t))$ in $L^2(0,T)$:

$$\langle f_j, \psi_k \rangle_{L^2(0,T)} = \delta_{j,k},$$

then one deduces that

$$\sqrt{2} k \frac{\pi}{L} c_k = \int_0^T (\partial_x \varphi)(t,0)\, \psi_k(t)\, dt.$$

- Then one easily obtains the observability inequality, with a size of the observability constant "essentially" of order

$$\sup_{j,k,l} \exp(-\lambda_j T)\|e_k\|_{L^2(0,L)}\|\psi_l\|_{L^2(0,T)}$$

(This is not completely precise.)

- Should we be able to construct a "nice" bi-orthogonal family ψ_l, we see that this constant will be small provided that T is large enough (remember $\lambda_k = \varepsilon\frac{k^2\pi^2}{L^2} + \frac{M^2}{4\varepsilon} \geq \frac{M^2}{4\varepsilon}$)

- Consequently, the main point is to construct this family and have nice estimates on it.

Construction of the bi-orthogonal family

- Imagine that you are given an entire function $J \in \mathcal{H}(\mathbb{C})$, of exponential type $T/2$: for some constant $C > 0$, one has

$$|J(z)| \leq C \exp(T|z|/2) \text{ for all } z \in \mathbb{C},$$

having simple poles at the points $-i\lambda_k$ and whose restriction to $\mathbb{R}$ is in L^2.

- Then one defines

$$J_k(z) := \frac{J(z)}{J'(-i\lambda_k)(z + i\lambda_k)},$$

which is still an entire function of exponential type $T/2$, is still in L^2 on $\mathbb{R}$, and it satisfies

$$J_k(-i\lambda_j) = \delta_{jk}.$$

- Since J_k is an entire function of exponential type $T/2$ and in $L^2(\mathbb{R})$, by the Paley-Wiener theorem, one can find $\varphi_k \in L^2(\mathbb{R})$, supported in $(-T/2, T/2)$, such that

$$J_k(z) = \widehat{\varphi_k}(z) \text{ for } z \in \mathbb{C}.$$

- The relation $J_k(-i\lambda_j) = \delta_{jk}$ now yields

$$\int_{-T/2}^{T/2} \varphi_k(\tau) \exp(-\lambda_j \tau) \, d\tau = \delta_{jk}.$$

- Translate by $T/2$ and you are done.

- Hence the core of the proof is to construct an entire function J, of exponential type $T/2$, having simple poles at $-i\lambda_k$, whose restriction to $\mathbb{R}$ belongs to L^2, and yielding the best possible estimates.

- An entire function having the k^2, $k \in \mathbb{N} \setminus \{0\}$ as its simple zeros is the following Weierstrass product:

$$\prod_{k=1}^{\infty} \left(1 - \frac{z}{k^2}\right) = \frac{\sin(\pi\sqrt{z})}{\pi\sqrt{z}},$$

  which is an entire function (despite the square roots).

- Now a function having simple zeros exactly at $\{-i\lambda_k,\ k \in \mathbb{N} \setminus \{0\}\}$ by

$$\Phi(z) = \frac{\sin\left(\frac{L}{\sqrt{\varepsilon}}\sqrt{iz - \frac{M^2}{4\varepsilon}}\right)}{\frac{L}{\sqrt{\varepsilon}}\sqrt{iz - \frac{M^2}{4\varepsilon}}}. \tag{4}$$

- It is elementary to see that Φ is of exponential type, and even satisfies

$$|\Phi(z)| \leq C(M,\varepsilon)\exp\left(\frac{L}{\sqrt{2\varepsilon}}\sqrt{|z|}\right) \text{ as } |z| \to +\infty. \tag{5}$$

- But precisely because of this "sub"-exponential estimate, the Phragmen-Lindelöf theorem (or direct computations) proves that this function **cannot** be bounded on the real line.

- Hence, the idea is to find another entire function $F \in \mathcal{H}(\mathbb{C})$, called a multiplier, such that
  - the function $F(z)\Phi(z)$ now suitably behaves on the real line,
  - it is of exponential type $T/2$.

  Such a technique can be traced back to R. Paley and N. Wiener themselves.

The Beurling-Malliavin multiplier

- We use a construction of a multiplier due to Beurling and Malliavin (1961).
- Introduce
$$s(t) = \frac{T}{2\pi} t - \frac{L}{\pi\sqrt{2\varepsilon}}\sqrt{t}.$$
We notice that s is increasing for t larger than
$$A := \frac{1}{2\varepsilon}\left(\frac{L}{T}\right)^2. \tag{6}$$
- Using that
$$\int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| dt^\gamma = |x|^\gamma \pi \cot\frac{\pi\gamma}{2} \text{ for } 0 < \gamma < 2,$$
we see that
$$\int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| ds(t) = -\frac{L}{\sqrt{2\varepsilon}}\sqrt{|x|}.$$

- We introduce
$$B := 4A = \frac{2}{\varepsilon}\left(\frac{L}{T}\right)^2, \tag{7}$$
which satisfies $s(B) = 0$.
- Now one defines ν as the restriction of the measure $ds(t)$ to the interval $[B, +\infty)$. Let us underline that this measure is positive (since $B \geq A$).
- Next we introduce for $z \in \mathbb{C}$:
$$U(z) := \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| \, d\nu(t) = \int_B^\infty \log\left|1 - \frac{z^2}{t^2}\right| \, ds(t), \tag{8}$$
and for $z \in \mathbb{C} \setminus \mathbb{R}$
$$g(z) := \int_0^\infty \log\left(1 - \frac{z^2}{t^2}\right) \, d\nu(t) = \int_B^\infty \log\left(1 - \frac{z^2}{t^2}\right) \, ds(t). \tag{9}$$

- By "atomizing" the measure $d\nu$ in the above integral, we can define

$$\tilde{U}(z) := \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| \, d[\nu(t)], \tag{10}$$

where $[\,\cdot\,]$ denotes the integer part and where

$$\nu(t) = \int_0^t d\nu. \tag{11}$$

In the same way as previously we introduce

$$h(z) := \int_0^\infty \log\left(1 - \frac{z^2}{t^2}\right) \, d[\nu](t). \tag{12}$$

- Of course,

$$U(z) = \mathfrak{Re}(g(z)) \text{ and } \tilde{U}(z) = \mathfrak{Re}(h(z)).$$

Now $\exp(h(z))$ is an entire function. Indeed, calling $\{\mu_k, \ k \in \mathbb{N}\}$ the discrete set in $\mathbb{R}$ consisting of the discontinuities of the function $t \mapsto [\nu(t)]$, we have

$$\exp(h(z)) = \prod_{k \in \mathbb{N}} \left(1 - \frac{z^2}{\mu_k^2}\right). \tag{13}$$

- Finally, the multiplier which we will use is the following:

$$F(z) := \exp(h(z-i)).$$

- The rest of the proof consists in proving that $F(z)\Phi(z)$ is of exponential type $T/2$, and to give estimates on $x \mapsto F(x)\Phi(x)$ on $\mathbb{R}$ and on $F(-i\lambda_k)$, so that we have the correct estimates on

$$J_k(z) = \frac{F(z)\Phi(z)}{F(-i\lambda_k)\Phi'(-i\lambda_k)(z+i\lambda_k)}.$$

- 1. Estimates on the real line.

Lemma

For $x \in \mathbb{R}$, one has

$$U(x) \leq -\frac{L}{\sqrt{2\varepsilon}}\sqrt{|x|} + C_1 aB, \tag{14}$$

where C_1 is the following positive (and finite) constant

$$C_1 := -\min_{x\in\mathbb{R}} \int_0^1 \log\left|1 - \frac{x^2}{t^2}\right| \, d(t - \sqrt{t}) \simeq 2.34 < 2.35. \tag{15}$$

Lemma (Koosis)

We have for $z = x + iy \in \mathbb{C}$:

$$\int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| \, d([\nu](t) - \nu(t)) \leq \log\left(\frac{\max(|x|,|y|)}{2|y|} + \frac{|y|}{2\max(|x|,|y|)}\right). \tag{16}$$

- Using the fact that U is a harmonic function on the upper plane, hence admits an integral representation, one can compare $U(x-i)$ and $U(x)$, and finally get the following estimate on the multiplier:

$$\forall x \in \mathbb{R},\ \tilde{U}(x-i) \leq -\frac{L}{\sqrt{2\varepsilon}}\sqrt{|x|} + aBC_1 + \log^+(|x|) + \frac{T}{2}.$$

- 2. Estimates on the imaginary axis.

Lemma

For all $y \in \mathbb{R}$ one has

$$\int_B^\infty \log\left(1+\frac{y^2}{t^2}\right) d[s] \geq \int_B^\infty \log\left(1+\frac{y^2}{t^2}\right) ds - \log\left(1+\frac{y^2}{B^2}\right). \quad (17)$$

Lemma

One has

$$\int_0^B \log\left|1+\frac{y^2}{t^2}\right| ds = aBG\left(\frac{y}{B}\right). \quad (18)$$

where

$$G(y) := \int_0^1 \log\left|1+\frac{y^2}{t^2}\right| d(t-\sqrt{t})$$

is a bounded function.

- This yields an estimate of the type:

$$\forall y \in \mathbb{R}^-,\ \tilde{U}(iy) \geq \frac{T}{2}|y| - \frac{L}{\sqrt{\varepsilon}}\sqrt{|y|} - \log\left(1 + \frac{y^2}{B^2}\right) - aBG\left(\frac{y}{B}\right).$$

- We can (more easily) obtain an upper bound of the type

$$|\tilde{U}(iy)| \leq \frac{T}{2}|y|,$$

which yields that the multiplier is indeed of exponential type $T/2$.

- Following the constants from line to line, we then deduce the result.

V. Open problems

- The Coron-Guerrero conjecture is still open!
- When dispersion is present, so is the case of negative M...
- Can one estimate the time of uniform controllability for variable M?
- Can one treat the high frequencies and the low frequencies differently? (We are not optimal for the high frequencies; perhaps we could use the Lebeau-Robbiano-Zuazua spectral inequality for the low frequencies?)
- What can be said about nonlinear equations?
- Can one consider the case of systems? (Long horizon quest: control the compressible Navier-Stokes with small viscosity...)