Topology and Shape Optimization for Smart Materials

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The problem:cartoon

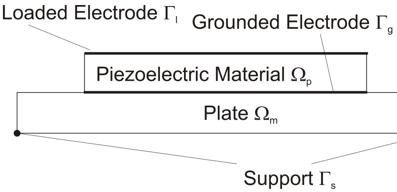


Figure: The active device

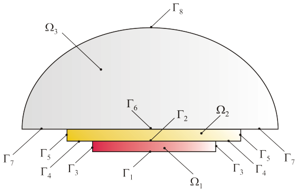


Figure: FEM realization

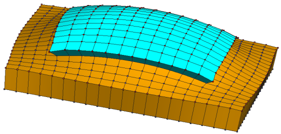


Figure: FEM realization

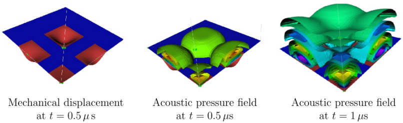


Figure: FEM realization of coupled system

The time harmonic model

Piezoelectricity

The material law describing the piezoelectric effect is given by

$$\begin{cases} S(w) &= A\varepsilon(w)\ , \\ \sigma(u,q) &= C\varepsilon(u) - Pe(q)\ , \\ \psi(u,q) &= P^T\varepsilon(u) + De(q)\ , \end{cases}$$

$$\begin{cases} \sigma n &= 0 \\ q &= 0 \end{cases} \text{on } S_0 \quad \begin{cases} \varphi \cdot n &= 0 \\ u &= 0 \end{cases} \text{on } S_1\ ,$$

We consider the following transmission conditions

$$\begin{cases} [\![\sigma]\!] n &= 0 \\ [\![u]\!] &= 0 \end{cases} \quad \text{and} \quad \begin{cases} [\![\varphi]\!] \cdot n &= 0 \\ [\![q]\!] &= 0 \end{cases}\ ,$$

where, for any $x \in \Gamma_i$, $i = 1, 2, ..., m$, the symbol $[\![(\cdot)]\!]$ is used to denote the jump between quantities evaluated on the boundary Γ_i of each pair Ω_{i-1} and Ω_i, that is

$$[\![(\cdot)]\!] = (\cdot)^{(i)} - (\cdot)^{(i-1)}\ ,$$

and $n = n^{(i)} = -n^{(i-1)}$ is the unit normal vector pointing toward the exterior of Ω_i.

The weak system

The weak formulation of the piezoelectric problem reads:

$$\left\{ \begin{array}{rcl} \langle -\omega^2 u, \eta \rangle_\Omega + a_\Omega^{MM}(u,\eta) + a_\Omega^{EM}(q,\eta) & = & 0 \quad \forall \eta \in \mathcal{W}_M(\Omega) \\ a_\Omega^{EE}(q,\xi) - a_\Omega^{ME}(u,\xi) & = & 0 \quad \forall \xi \in \mathcal{W}_E(\Omega) \end{array} \right. ,$$

$$a_\Omega^{MM}(u,\eta) = \int_\Omega C \nabla^s u \cdot \nabla^s \eta \quad \text{and} \quad a_\Omega^{EM}(q,\eta) = \int_\Omega P \nabla q \cdot \nabla^s \eta \, ,$$

$$a_\Omega^{EE}(q,\xi) = \int_\Omega D \nabla q \cdot \nabla \xi \quad \text{and} \quad a_\Omega^{ME}(u,\xi) = \int_\Omega P^T \nabla^s u \cdot \nabla \xi \, ,$$

with $a_\Omega^{EM}(q,u) = a_\Omega^{ME}(u,q)$ and $\nabla := \partial / \partial x$ and

$$\begin{array}{rcl} \mathcal{W}_M(\Omega) & = & \{ u \in [H^1(\Omega)]^3 : u|_{S_1} = 0, \ [\![u]\!]|_{\Gamma_i} = 0, \ i = 1,2,...,m \} \, , \\ \mathcal{W}_E(\Omega) & = & \{ q \in H^1(\Omega) : q|_{S_0} = 0, \ [\![q]\!]|_{\Gamma_i} = 0, \ i = 1,2,...,m \} \, . \end{array}$$

Shape sensitivity: Novotny, Perla-Menzala, G.L., Sokolowski 2009 also for time-dependent problems and acoustic, mechanic, piezo-electric coupling

Adjoint system

In order to simplify the further calculation, let us introduce the adjoint displacement $v \in \mathcal{W}_M(\Omega)$ and the adjoint electrical potential $p \in \mathcal{W}_E(\Omega)$, such that

$$\left\{ \begin{array}{rcll} \langle -\omega^2 v, \eta\rangle_\Omega + a_\Omega^{MM}(v,\eta) - a_\Omega^{EM}(p,\eta) & = & -\langle D_u(J_\Omega(u,q)),\eta\rangle & \forall \eta \in \mathcal{W}_M(\Omega) \\ a_\Omega^{EE}(p,\xi) + a_\Omega^{ME}(v,\xi) & = & -\langle D_q(J_\Omega(u,q)),\xi\rangle & \forall \xi \in \mathcal{W}_E(\Omega) \end{array} \right. .$$

From the above system, we can define the adjoint stress tensor σ_a and the adjoint electrical displacement φ_a as following

$$\left\{ \begin{array}{rcl} \sigma_a(v,p) & = & C\nabla^s v - P\nabla p \, , \\ \varphi_a(v,p) & = & -P^T\nabla^s v - D\nabla p \, . \end{array} \right.$$

Adjoint equation

- The perturbed domain:

$$\Omega_\tau = \{x_\tau \in \mathbb{R}^3 : x_\tau = x + \tau V,\ x \in \Omega,\ \tau \geq 0\} ,$$

where V is a smooth vector field defined in Ω that represents the shape change velocity.

- The perturbed shape functional:

$$J_{\Omega_\tau}(u_\tau, q_\tau) ,$$

where the pair $u_\tau = u_\tau(x_\tau, t)$ and $q_\tau = q_\tau(x_\tau, t)$ solve

- The perturbed variational problem:

$$\left\{ \begin{array}{rcl} \langle -\omega^2 u_\tau, \eta\rangle_{\Omega_\tau} + a^{MM}_{\Omega_\tau}(u_\tau, \eta) + a^{EM}_{\Omega_\tau}(q_\tau, \eta) & = & 0 \quad \forall \eta \in \mathcal{W}_M(\Omega_\tau) \\ a^{EE}_{\Omega_\tau}(q_\tau, \xi) - a^{ME}_{\Omega_\tau}(u_\tau, \xi) & = & 0 \quad \forall \xi \in \mathcal{W}_E(\Omega_\tau) \end{array} \right. ,$$

- The perturbed forms:

$$a^{MM}_{\Omega_\tau}(u_\tau, \eta) = \int_{\Omega_\tau} C\nabla^s u_\tau \cdot \nabla^s \eta \quad \text{and} \quad a^{EM}_{\Omega_\tau}(q_\tau, \eta) = \int_{\Omega_\tau} P\nabla q_\tau \cdot \nabla^s \eta ,$$

$$a^{EE}_{\Omega_\tau}(q_\tau, \xi) = \int_{\Omega_\tau} D\nabla q_\tau \cdot \nabla \xi \quad \text{and} \quad a^{ME}_{\Omega_\tau}(u_\tau, \xi) = \int_{\Omega_\tau} P^T\nabla^s u_\tau \cdot \nabla \xi ,$$

with $a^{EM}_{\Omega_\tau}(q_\tau, u_\tau) = a^{ME}_{\Omega_\tau}(u_\tau, q_\tau)$ and $\nabla := \partial/\partial x_\tau$.

Shape sensitivity

Derivatives....

The shape derivative of the functional $J_\Omega(u,q)$ is given by

$$\dot{J}_\Omega(u,q) = \langle D_\Omega(J_\Omega(u,q)), V\rangle + \langle D_u(J_\Omega(u,q)), \dot{u}\rangle + \langle D_q(J_\Omega(u,q)), \dot{q}\rangle \ .$$

$$\dot{a}_\Omega^{MM}(u,\eta) = a_\Omega^{MM}(\dot{u},\eta) + \int_\Omega (C\nabla^s u \cdot \nabla^s \eta)\mathrm{div}V - \int_\Omega (\nabla u^T (C\nabla^s \eta) + \nabla \eta^T (C\nabla^s u) \cdot \nabla V \ ,$$

$$\dot{a}_\Omega^{EM}(q,\eta) = a_\Omega^{EM}(\dot{q},\eta) + \int_\Omega (P\nabla q \cdot \nabla^s \eta)\mathrm{div}V - \int_\Omega (\nabla q \otimes P^T \nabla^s \eta + \nabla \eta^T P \nabla q) \cdot \nabla V \ ,$$

$$\dot{a}_\Omega^{EE}(q,\xi) = a_\Omega^{EE}(\dot{q},\xi) + \int_\Omega (D\nabla q \cdot \nabla \xi)\mathrm{div}V - \int_\Omega (\nabla q \otimes D\nabla \xi + \nabla \xi \otimes D\nabla q) \cdot \nabla V \ ,$$

$$\dot{a}_\Omega^{ME}(u,\xi) = a_\Omega^{ME}(\dot{u},\xi) + \int_\Omega (P^T \nabla^s u \cdot \nabla \xi)\mathrm{div}V - \int_\Omega (\nabla u^T P \nabla \xi + \nabla \xi \otimes P^T \nabla^s u) \cdot \nabla V \ ,$$

Shape derivative

$$
\begin{aligned}
\langle D_u(J_\Omega(u,q)),\dot{u}\rangle &= \int_\Omega (C\nabla^s u\cdot\nabla^s v + P\nabla q\cdot\nabla^s v)\mathrm{div}V + a_\Omega^{EM}(\dot{q},v) + a_\Omega^{EM}(p,\dot{u}) \\
&- \int_\Omega (\nabla u^T(C\nabla^s v) + \nabla v^T(C\nabla^s u) + \nabla q\otimes P^T\nabla^s v + \nabla v^T P\nabla q)\cdot\nabla V \\
&+ \omega^2\int_\Omega (\nabla u^T v + \nabla v^T u)\cdot V - \omega^2\int_{\partial\Omega}(u\cdot v)n\cdot V\ ,
\end{aligned}
$$

$$
\begin{aligned}
\langle D_q(J_\Omega(u,q)),\dot{q}\rangle &= \int_\Omega (D\nabla q\cdot\nabla p - P^T\nabla^s u\cdot\nabla p)\mathrm{div}V - a_\Omega^{ME}(\dot{u},p) - a_\Omega^{ME}(v,\dot{q}) \\
&- \int_\Omega (\nabla q\otimes D\nabla p + \nabla p\otimes D\nabla q - \nabla u^T P\nabla p - \nabla p\otimes P^T\nabla^s u)\cdot\nabla V\ .
\end{aligned}
$$

we have

$$
\langle D_u(J_\Omega(u,q)),\dot{u}\rangle + \langle D_q(J_\Omega(u,q)),\dot{q}\rangle = \int_\Omega S\cdot\nabla V - \omega^2\int_{\partial\Omega}(u\cdot v)n\cdot V + \omega^2\int_\Omega(\nabla u^T v + \nabla v^T u)\cdot V\ ,
$$

Shape derivative: final form

The tensor S

The tensor S reads

$$S = (\sigma \cdot \nabla^s v - \varphi \cdot \nabla p) I - (\nabla u^T \sigma_a + \nabla v^T \sigma - \nabla q \otimes \varphi_a - \nabla p \otimes \varphi) \; ,$$

In addition, we observe that

$$\int_\Omega S \cdot \nabla V = \int_{\partial\Omega} Sn \cdot V + \sum_{i=1}^{m} \int_{\Gamma_i} [\![S]\!] n \cdot V - \int_\Omega \mathrm{div} S \cdot V \; .$$

Shape gradient

We obtain the shape derivative of the functional $J_\Omega(u, q)$ independent of $\dot{u}$ and $\dot{q}$, namely

$$\begin{aligned} \dot{J}_\Omega(u, q) &= \langle D_\Omega(J_\Omega(u, q)), V \rangle + \int_{\partial\Omega} (-\omega^2 u \cdot v) n \cdot V \\ &+ \int_{\partial\Omega} Sn \cdot V + \sum_{i=1}^{m} \int_{\Gamma_i} [\![S]\!] n \cdot V - \int_\Omega \mathrm{div}\ S \cdot V \\ &+ \omega^2 \int_{\partial\Omega} (\nabla u^T v + \nabla v^T u) \cdot V \; , \end{aligned}$$

Topological gradient

Topological gradient

We dig a hole ω_h of radius h into Ω centered at the origin and denote the new domain $\Omega(h) = \Omega \setminus \bar{\omega}_h$. The system then reads

$$\begin{array}{rcll} D(-\nabla_x)^T A(x) D(\nabla_x) u^h(x) & = & f(x), & x \in \Omega(h) \\ D(n(x)^h)^T A(x) D(\nabla_x) u^h(x) & = & g(x), & x \in \Gamma_N \\ D(n(x)^h)^T A(x) D(\nabla_x) u^h(x) & = & 0, & x \in \partial\omega_h \\ u^h(x) & = & 0, & x \in \Gamma_D \end{array}$$

$$u^h(x) = u(x) + \chi(x)(hw(x/h) + h^2 w^2(x/h) + \ldots)$$

The shape functional is of the form:

$$\mathcal{J}(u;\Omega) = \int_\Omega J(u(x);x)dx$$

with some canonical regularity conditions and growth bounds

Topological gradient: Cardone, Nazarov, Sokolowski 2009

Formula

We obtain the following formula for the topological gradient:

$$\begin{aligned}\mathcal{J}(u^h;\Omega(h)) &= \mathcal{J}(u;\Omega) + h^3 J(u(0);0) + h^3(-P(0)^T f(0)\text{mes}(\omega) - D(\nabla_x)P(0)^T M\epsilon^0) \\ &\quad + O(h^{3+\cdots})\end{aligned}$$

where P solves the adjoint problem

$$\begin{aligned} D(-\nabla_x)^T A(x) D(\nabla_x) P(x) &= J'(u(x),x), & x \in \Omega \\ D(n(x))^T A(x) D(\nabla_x) P(x) &= 0, & x \in \Gamma_N \\ P(x) &= 0, & x \in \Gamma_D \end{aligned}$$

and M is the so-called polarization matrix which has to be computed numerically, as no analytical formula is yet available.

SIMP-Results (Wein,Schury,G.L., Bänsch, Kaltenbacher 2009)

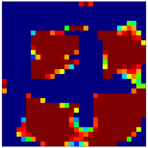


Figure: topology at 1850/950 Hz

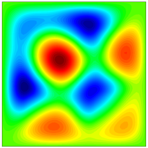
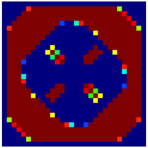
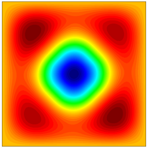


Figure: displacement at 1850/950 Hz





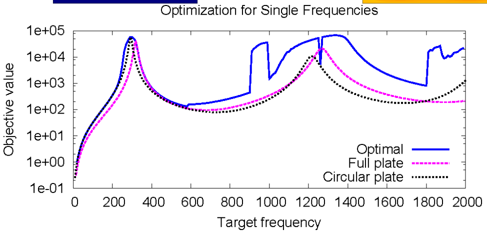


Figure: Sweep over frequency range

robust SIMP

- Robust design for SIMP using MMA as in the previous case
- Perform this with shape optimization
- combine with topological gradient method
- transient problem in progress!!

Thank you for your attention ...