Blowup in multidimensional aggregation equations

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Benasque, August 27th 2009

Aggregation Equation

$$\begin{cases} \rho_t + \operatorname{div}(\rho \vec{v}) = 0 \\ \vec{v} = -\nabla K * \rho \end{cases}$$

$\rho(x,t)$: density
$\vec{v}(x,t)$: velocity field
$x \in \mathbb{R}^d$, $t > 0$

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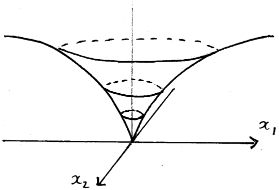
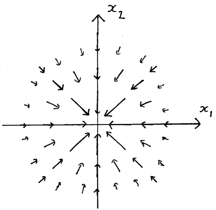
"interaction potential"

$-\nabla K : \mathbb{R}^d \to \mathbb{R}^d$
"attracting field"





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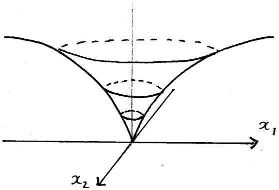
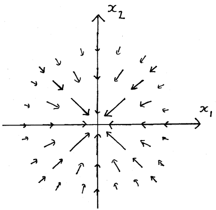


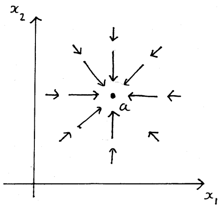


For which interaction potentials do we get finite time blowup?

Aggregation for particles

One particle attracted
by a fixed location $x = a$

$$\dot{X} = -\nabla K(X - a)$$



Multiple particles attracted
by one another

$$\dot{X}_i = -\sum_{j \neq i} m_j \nabla K(X_i - X_j)$$

Continuum model

$\rho(x,t)$ = density of particle at time t

$$\dot{X}_i = -\sum_{j\neq i} \nabla K(X_i - X_j)\, m_j$$

$$\vec{v}(x) = -\int_{\mathbb{R}^d} \nabla K(x-y)\, \rho(y)dy$$

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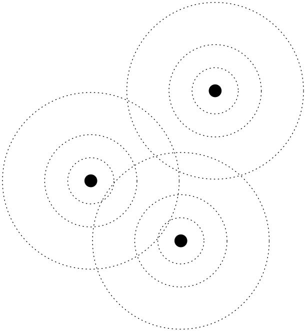
So $\boxed{\vec{v} = -\nabla K * \rho}$

$$\begin{cases} \rho_t + \operatorname{div}(\rho \vec{v}) = 0 \\ \vec{v} = -\nabla K * \rho \end{cases}$$

Model collective motion of cells which are attracted by self-emitted chemical substance (and move with Brownian motion).



$\rho(x,t)$: density of cells
$c(x,t)$: density of chemical substance

- Cells moves toward region with high concentration of chem.

$$
\begin{aligned}
&\rho_t + \operatorname{div}(\rho\ \vec{v}) = \Delta\rho \\
&\vec{v} = \nabla c \\
&c_t - \Delta c = \rho
\end{aligned}
$$

- Cells moves toward region with high concentration of chem.

$$\rho_t + \mathrm{div}\,(\rho\ \vec{v}) = \Delta\rho$$
$$\vec{v} = \nabla c$$
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- Chemical substance "moves" faster than cells

$$\rho_t + \mathrm{div}\,(\rho\ \vec{v}) = \Delta\rho$$
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$$-\Delta c = \rho \qquad (\Rightarrow c = N * \rho)$$

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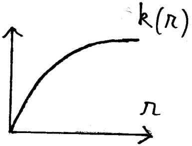
- So we get:

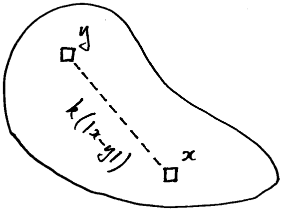
$$\rho_t + \operatorname{div}(\rho\, \vec{v}) = \Delta\rho$$
$$\vec{v} = \nabla N * \rho$$

Interaction energy

$$E_K(\rho) = \frac{1}{2}\iint K(x-y)\;\rho(x)dx\;\rho(y)dy$$
$$= \frac{1}{2}\iint k(|x-y|)\;\rho(x)dx\;\rho(y)dy$$

$$K(x) = k(|x|)$$





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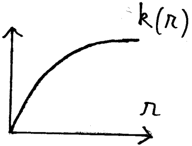

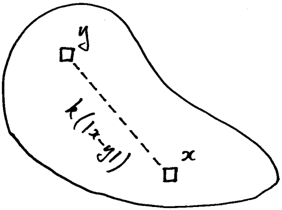

$$0 \leq E_K(\rho) \leq \frac{1}{2} k(\text{diam})$$

$$E_K(\rho) = 0 \ \Leftrightarrow\ \rho = \delta_{x_0}$$

$$\boxed{E_K(\rho) = \frac{1}{2} \iint K(x-y)\,\rho(x)\,\rho(y)\,dxdy}$$

$$\frac{d}{dt} E_K(\rho) = -\int \rho\,|\vec{v}|^2\,dx$$

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$$\boxed{\text{center of mass} = \int \vec{x}\ \ \rho(x,t)\, dx}$$

$$\frac{d}{dt} \int \vec{x}\ \ \rho(x,t)\, dx = 0$$

$$\rho_t + \text{div}\,(\rho \vec{v}) = 0$$
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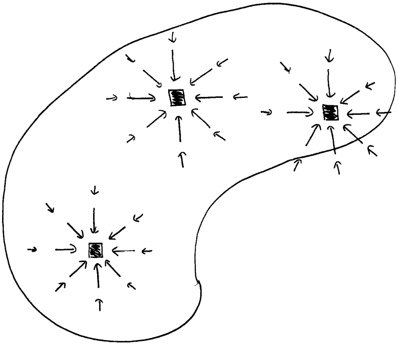
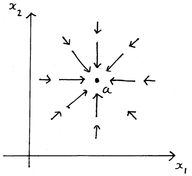


Question: What is the (sharp) condition on the potential in order to have finite time blowup?

$$\dot{X} = -\nabla K(X - a)$$

Question: how long does it takes for a particle to reach the bottom of a fixed potential?



$$\begin{cases} \dot{r} = -k'(r) \\ r(0) = L \end{cases}$$

$$K(x) = k(|x|)$$

Answer: $$T = \int_0^L \frac{dr}{k'(r)}$$

because to move by a distance dr, it takes the particle a time $\frac{dr}{k'(r)}$

Main result

Sharp condition on the interaction potential in order to get blowup

- If $\boxed{\displaystyle\int_0^L \frac{dr}{k'(r)} = +\infty,}$ then we have global existence in

$$C([0,\infty), L^1 \cap L^p) \cap C^1([0,\infty), W^{-1,p}) \qquad \text{for } p > \tfrac{d}{d-1}.$$

$L^1 \cap L^\infty$(Bertozzi, C., Laurent; Nonlinearity (2009))
$L^1 \cap L^p$(Bertozzi, Laurent, Rosado; in preparation)

- If $\boxed{\displaystyle\int_0^L \frac{dr}{k'(r)} < +\infty,}$ then $\rho(t) \to \delta_{X_0}$ in finite time.

(C., Di Francesco, Figalli, Slepcev, Laurent; preprint UAB)

Monotonicity conditions

The Osgood condition is sharp in the class of potential satisfying

- $\exists \delta > 0$ such that $k''(r)$ is monotone in $(0, \delta)$
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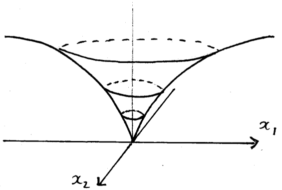
Why do we work with densities in $L^p(\mathbb{R}^d)$ for $p>\frac{d}{d-1}$?



$$\nabla K \in W^{1,q}(\mathbb{R}^d) \text{ for } q<d$$

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$\rho \in L^p$ and $\nabla K \in W^{1,q}$

$$\Rightarrow \quad \boxed{\nabla K * \rho \in C^1}$$

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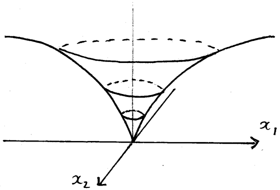

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$\Rightarrow$ Local existence

$\int_0^L \frac{dr}{k'(r)} = +\infty \quad \Rightarrow \quad$ global existence

- We want an apriori bound of $\|\rho(t)\|_{L^p}$ for all time.

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- How to do it?

Show that $\frac{1}{\|\rho(t)\|_{L^p}}$ can not go to 0 in finite time.

$$\frac{d}{dt}\left\{\frac{1}{\|\rho(t)\|_{L^p}^{q/d}}\right\} \geq -c \;\; k'\left(\frac{1}{\|\rho(t)\|_{L^p}^{q/d}}\right)$$

$$\dot{y} = -c \;\; k'(y)$$

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$$\boxed{\frac{1}{\|\rho(t)\|_{L^p}^{q/d}} \geq y(t)}$$

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But $y(t)$ can not go to 0 in finite time.

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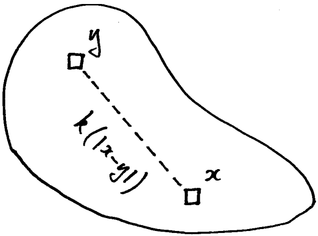
Gradient flow of

$$E_K(\rho) = \frac{1}{2} \iint K(x-y)\, \rho(x)\, \rho(y)\, dxdy$$

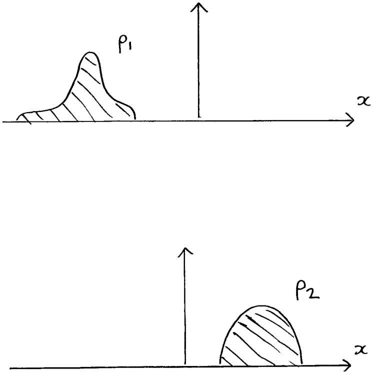
with respect to the Wasserstein distance



What is the Wasserstein distance?

The Wasserstein distance is a distance on the space of probability measure.

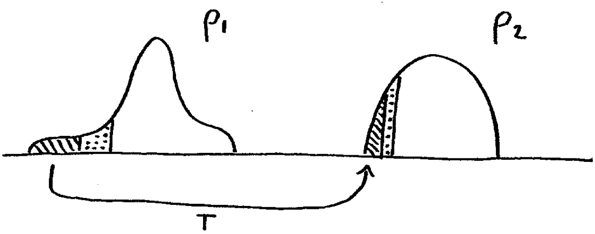
Example: What is the Wasserstein distance between ρ_1 and ρ_2?



Two piles of sand!

Energy needed to transport m kg of sand from $x = a$ to $x = b$:

$$\boxed{\text{energy} = m\,|a - b|^2}$$



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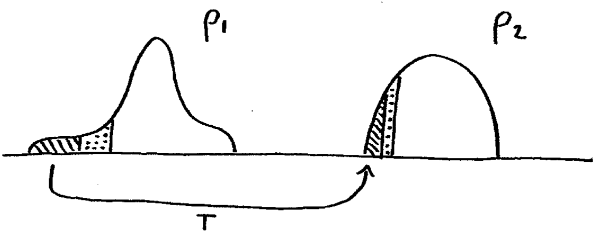

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$$d_W^2(\rho_1, \rho_2) = \int |x - T(x)|^2 \, d\rho_1(x)$$

Definition of the distance[a]

[a] C. Villani, AMS Graduate Texts (2003).

Transporting measures:

Given $T : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ mesurable, we say that $\nu = T\#\mu$, if $\nu[K] := \mu[T^{-1}(K)]$ for all mesurable sets $K \subset \mathbb{R}^d$, equivalently

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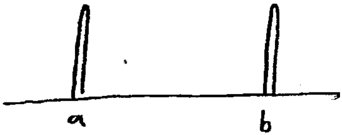
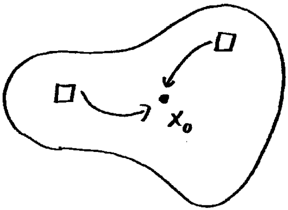
where the transference plan π runs over the set of joint probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals f and $g \in \mathcal{P}_2(\mathbb{R}^d)$.



$$d_W^2(\delta_a, \delta_b) = |a - b|^2$$



$$d_W^2(\rho, \delta_{X_0}) = \int |X_0 - y|^2 \, d\rho(y)$$
$$= \text{Var } (\rho)$$

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JKO scheme (Jordan-Kinderlehrer-Otto)

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$$\rho_{k+1} = \arg \min_{\rho \in \mathcal{P}_2(\mathbb{R}^d)} \left\{ \frac{1}{2\,\Delta t} d_W^2(\rho, \rho_k) + E_K(\rho) \right\}$$

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- As $\Delta t \to 0$ it converges to the solution of a weak form of

$$\begin{cases} \rho_t + \operatorname{div}(\rho \vec{v}) = 0 \\ \vec{v} = -\nabla K * \rho \end{cases}$$

(see "Gradient Flow in Metric Spaces" by Ambrosio, Gigli, Savaré)

Sub-differential Characterization

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Let $\partial K(x)$ be the (possibly multivalued) subdifferential of K at the point x, namely the set

$$\partial K(x) = \left\{ \kappa \in \mathbb{R}^d : K(y) - K(x) \geq \kappa \cdot (y - x) + o(|x - y|) \, \forall y \in \mathbb{R}^d \right\}.$$

Let $\partial^0 K(x)$ be the element of $\partial K(x)$ with minimal norm. Our assumptions, $\partial^0 K(x) = \nabla K(x)$ for all $x \neq 0$ and $\partial^0 K(0) = 0$.

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A vector field $\mathbf{w} \in L^2(d\mu)$ is said to be an element of the subdifferential of E_K, i. e. $\mathbf{w} \in \partial E_K$, if

$$E_K[\nu] - E_K[\mu] \geq \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{w}(x) \cdot (y - x)\, d\gamma(x, y) + o(d_W(\nu, \mu))$$

for all $\gamma \in \Gamma_o(\mu, \nu)$.

Characterization of Sub-differential

Given a locally attractive potential, the vector field

$$\kappa(x) := \int_{y\neq x} \nabla K(x-y)\, d\mu(y) \equiv (\partial^0 K * \mu)(x)$$

is the unique element of the minimal subdifferential of E_K, i.e. $\partial^0 K * \mu = \partial^0 E_K[\mu]$.

Gradient Flow Solution

Concept of Solution

An absolutely continuous curve $\mu : [0, +\infty) \ni t \mapsto \mathcal{P}_2(\mathbb{R}^d)$ is said to be a *weak measure solution* with initial datum $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ if and only if $\partial^0 K * \mu \in L^2(\mu(t))$ a.e. $\tau \in (0, t)$ and

$$\int_0^t \int_{\mathbb{R}^d} \varphi_t(x,\tau)\, d\mu(t)(x) + \int_{\mathbb{R}^d} \phi(x,0)\, d\mu_0(x) =$$
$$\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \nabla\varphi(x,t) \cdot \partial^0 K(x-y)\, d\mu(t)(x)\, d\mu(t)(y),$$

for all test functions $\varphi \in C_c^\infty([0,t) \times \mathbb{R}^d)$.

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An absolutely continuous curve $\mu : [0, +\infty) \ni t \mapsto \mathcal{P}_2(\mathbb{R}^d)$ is said to be a *weak measure solution* with initial datum $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ if and only if $\partial^0 K * \mu \in L^2(\mu(t))$ a.e. $\tau \in (0, t)$ and

$$\int_0^t \int_{\mathbb{R}^d} \varphi_t(x,\tau)\, d\mu(t)(x) + \int_{\mathbb{R}^d} \phi(x,0)\, d\mu_0(x) =$$
$$\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \nabla\varphi(x,t) \cdot \partial^0 K(x-y)\, d\mu(t)(x)\, d\mu(t)(y),$$

for all test functions $\varphi \in C_c^\infty([0,t) \times \mathbb{R}^d)$.

More refined, it is a gradient flow-type solution:

$$v(t) = -\partial^0 E_K[\mu(t)] = -\partial^0 K * \mu(t), \;\; \|v(t)\|_{L^2(\mu(t))} = |\mu'|(t) \; a.e. \; t > 0$$

with $\mu(0) = \mu_0$ and $v(t)$ is the tangent vector to the curve $\mu(t)$ with minimal norm.

Energy equality is satisfied:

$$\int_a^b \int_{\mathbb{R}^d} |v(t)(x)|^2 \, d\mu(t)(x) \, dt + E_K[\mu(a)] = E_K[\mu(b)]$$

holds for all $0 \le a \le b < \infty$.

Well-posedness of Gradient Flow Solutions

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holds for all $0 \leq a \leq b < \infty$.

d_W-Expansion

Given two gradient flow solutions $\mu^1(t)$ and $\mu^2(t)$ in the sense of the theorem above, then

$$d_W(\mu^1(t), \mu^2(t)) \leq e^{-\lambda t} \, d_W(\mu_0^1, \mu_0^2)$$

for all $t \geq 0$. In particular, we have a unique gradient flow solution for any given $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$.

JKO scheme gives measure solutions.
Put particle model and continuum model in the same framework

- If the initial data is

$$\rho_0 = \sum_{i=1}^{N} m_i \, \delta_{X_i}$$

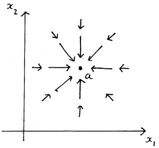
- Then the solution given by the JKO scheme (as $\Delta t \to 0$) is

$$\rho(t) = \sum_{i=1}^{N} m_i \, \delta_{X_i(t)} \qquad \text{where} \qquad \dot{X}_i = -\sum_{j \neq i} m_j \, \nabla K(X_i - X_j)$$

Proof of blowup using the particle model

We want to prove that

$$\boxed{\int_0^L \frac{dr}{k'(r)} < +\infty,} \Longrightarrow \boxed{\rho(t) \to \delta_{x_c} \text{ in finite time}}$$



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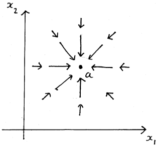

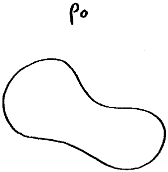

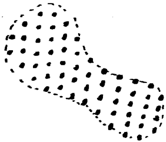

Find a bound (independent of the nb. of particles) for the time it takes for all the particles to arrive at X_0.

$$\dot{X}_i = -\sum_{j\neq i} m_j \, \nabla K(X_i - X_j) = -\sum_{j\neq i} m_j \, \frac{X_i - X_j}{|X_i - X_j|} \, k'(|X_i - X_j|)$$

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$$\frac{d}{dt}R(t)^2 = \frac{d}{dt}|X_i|^2 = 2\,\dot{X}_i\cdot X_i = -2\sum_{j\neq i} m_j \frac{(X_i - X_j)\cdot X_i}{|X_i - X_j|} k'(|X_i - X_j|)$$

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$(X_i - X_j)\cdot X_i \geq 0$ for all j Assume $\frac{k'(r)}{r}$ decreasing

$$\frac{d}{dt}R(t)^2 \leq -2\,\frac{k'(2R(t))}{2R(t)} \sum_{j\neq i} m_j\,(X_i - X_j)\cdot X_i$$

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$$\frac{d}{dt} R(t)^2 \leq -k'(2R(t)) \, R(t)$$

$$\boxed{\frac{d}{dt} R(t) \leq -\frac{1}{2} k'(2R(t))}$$

Summary

- Global existence: estimate of the L^p-norm.

$$\frac{d}{dt}\left\{\frac{1}{\|\rho(t)\|_{L^p}^{q/d}}\right\} \geq -c\ \ k'\left(\frac{1}{\|\rho(t)\|_{L^p}^{q/d}}\right)$$

- Finite time blow-up: estimate of the size of the support.

$$\frac{d}{dt}R(t) \leq -\frac{1}{2}\ \ k'(2R(t))$$

The behavior of $\boxed{\dot{y} = -k'(y)}$ determine whether or not we have finite time blow up

Just for fun!

$$\rho_{k+1} = \arg \min_{\rho \in \mathcal{P}_2(\mathbb{R}^d)} \left\{ \frac{1}{2\,\Delta t} d_W^2(\rho, \rho_k) + E_K(\rho) \right\}$$

What does it happen when you put particles in the JKO scheme?

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What does it happen when you put particles in the JKO scheme?

Theorem: they remain particles if

- $K(x) + \frac{\lambda}{2}|x|^2$ is convex
- $\Delta t < \frac{1}{\lambda}$