The Sharp Hardy Uncertainty Principle for Schrödinger Evolutions

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- Joint work with C. Kenig, G. Ponce and L. Vega.
- $$\widehat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx.$$
- *Theorem (Hardy's Uncertainty Principle)*

  Assume $f(x) = O(e^{-x^2/\beta^2})$ *and* $\widehat{f}(\xi) = O(e^{-4\xi^2/\alpha^2})$. *If* $\frac{1}{\alpha\beta} > \frac{1}{4}$, $f \equiv 0$. *If* $\frac{1}{\alpha\beta} = \frac{1}{4}$, f *is a constant multiple of* e^{-x^2/β^2}.

- $$\begin{cases} \partial_t u = i\triangle u, \ x \in \mathbb{R}^n, \ t \in \mathbb{R}, \\ u(0) = u_0. \end{cases}$$

- $$\int_{\mathbb{R}^n} \frac{e^{i|x-y|^2/4t}}{(4\pi it)^{n/2}}\, u_0(y)\, dy = \frac{e^{i|x|^2/4t}}{(4\pi it)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot y/2t} e^{i|y|^2/4t} u_0(y)\, dy$$

- $$(4\pi it)^{\frac{n}{2}} e^{-i|x|^2/4t}\, u(x,t) = (e^{i|\cdot|^2/4t} u_0)^{\wedge}\left(\frac{x}{2t}\right)$$

- If $\frac{T}{\alpha\beta} > \frac{1}{4}$,

$$u(0) = O(e^{-|x|^2/\beta^2}) \text{ and } u(T) = O(e^{-|x|^2/\alpha^2})$$

then, $u \equiv 0$ in $\mathbb{R}^n \times [0,T]$. If $\frac{T}{\alpha\beta} = \frac{1}{4}$, u has initial data

$$\omega e^{-\left(\frac{1}{\beta^2} + \frac{i}{4T}\right)|x|^2}, \ \omega \in \mathbb{C}.$$

(Cowling M. and Price J. F.) If p, $q \in [1, \infty]$ with at least one of them finite, $\frac{1}{\alpha\beta} \geq \frac{1}{4}$, $e^{|x|^2/\beta^2} f \in L^p(\mathbb{R})$ and $e^{4|\xi|^2/\alpha^2} \widehat{f} \in L^q(\mathbb{R})$, then $f \equiv 0$.
(Beurling) If f is in $L^1(\mathbb{R})$ and

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |f(x)||\widehat{f}(\xi)| e^{|x\,\xi|}\, dx\, d\xi < \infty,$$

then $f \equiv 0$.

- If $e^{\frac{x^2}{\beta^2}} u_0 \in L^p(\mathbb{R})$, $e^{\frac{x^2}{\alpha^2}} e^{iT\partial_x^2} u_0 \in L^q(\mathbb{R})$, p, $q \in [1, \infty]$ with at least one of them finite and $\frac{T}{\alpha\beta} \geq \frac{1}{4}$, then $u_0 \equiv 0$.
- If $u_0 \in L^1(\mathbb{R})$ and

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |u_0(x)||e^{iT\partial_x^2} u_0(y)| e^{|xy|/2T}\, dx\, dy < \infty,$$

then $u_0 \equiv 0$.

General Framework

- u verifies $\partial_t u = (A + iB)\left(\triangle u + V(x,t)u\right)$, $f = e^{\varphi(x,t)}u$.
- f verifies $|\partial_t f - \mathcal{S}f - \mathcal{A}f| \leq M|f|$, $\mathcal{S}$ is symmetric and $\mathcal{A}$ is skew-symmetric.

$$\mathcal{S} = A\left(\triangle + |\nabla\varphi|^2\right) - iB\left(2\nabla\varphi\cdot\nabla + \triangle\varphi\right) + \partial_t\varphi,$$

$$\mathcal{A} = iB\left(\triangle + |\nabla\varphi|^2\right) - A\left(2\nabla\varphi\cdot\nabla + \triangle\varphi\right),$$

$$\mathcal{S}_t + [\mathcal{S},\mathcal{A}] = \partial_t^2\varphi + 4A\nabla\varphi\cdot\nabla\partial_t\varphi - 2iB\left(2\nabla\partial_t\varphi\cdot\nabla + \triangle\partial_t\varphi\right)$$
$$-\left(A^2+B^2\right)\left[4\nabla\cdot\left(D^2\varphi\nabla\ \ \right) - 4D^2\varphi\nabla\varphi\cdot\nabla\varphi + \triangle^2\varphi\right].$$

- When $f = e^{\mu|x|^2}u$,

$$\mathcal{S}_t + [\mathcal{S},\mathcal{A}] = -\mu\left(A^2+B^2\right)\left[8\triangle - 32\mu^2|x|^2\right],$$

$$\left(\mathcal{S}_t f + [\mathcal{S},\mathcal{A}]f, f\right) = \left(A^2+B^2\right)\int_{\mathbb{R}^n} 8\mu|\nabla f|^2 + 32\mu^3|x|^2|f|^2\,dx.$$

General Framework

- May need to get control on $H(t) = \|f(t)\|^2 = \|e^{\varphi} u\|$, for $t \geq 0$.

$$\partial_t H(t) = 2\Re(\partial_t f, f) = 2\Re(\partial_t f - \mathcal{S}f - \mathcal{A}f, f) + 2(\mathcal{S}f, f)$$

Is $\mathcal{S}$ a negative operator?

$$\partial_t^2 H = 2\partial_t Re\left(\partial_t f - \mathcal{S}f - \mathcal{A}f, f\right) + 2\left(\mathcal{S}_t f + [\mathcal{S}, \mathcal{A}]\, f, f\right) + \|\partial_t f - \mathcal{A}f + \mathcal{S}f\|^2 - \|\partial_t f - \mathcal{A}f - \mathcal{S}f\|^2$$

Is $\mathcal{S}_t + [\mathcal{S}, \mathcal{A}]$ a non-negative operator?

General Framework

- To obtain logarithmic convexity for $H(t)$

$$\partial_t \log H(t) = 2\Re(\partial_t f, f) = 2\Re(\partial_t f - \mathcal{S}f - \mathcal{A}f, f)/H + N(t).$$

- $N(t) = 2(\mathcal{S}f, f)/H$ and

$$\begin{aligned}\partial_t N(t) &= 2\left(\mathcal{S}_t f + [\mathcal{S}, \mathcal{A}]\, f, f\right)/H \\ &+ \left[\|\partial_t f - \mathcal{A}f + \mathcal{S}f\|^2\|f\|^2 - \left(\Re\left(\partial_t f - \mathcal{A}f + \mathcal{S}f, f\right)\right)^2\right]/H^2 \\ &+ \left[\left(Re\left(\partial_t f - \mathcal{A}f - \mathcal{S}f, f\right)\right)^2 - \|\partial_t f - \mathcal{A}f - \mathcal{S}f\|^2\|f\|^2\right]/H^2\,.\end{aligned}$$

Is $\mathcal{S}_t f + [\mathcal{S}, \mathcal{A}]$ a non-negative operator?

• To prove an $L^2 - L^2$ Carleman inequality

$$\|e^{\varphi} u\|_{L^2_{x,t}} \lesssim \|e^{\varphi} (\partial_t - (A+iB)\triangle) u\|_{L^2_{x,t}},$$

the standard argument is to write $f = e^{\varphi} u$,

$$e^{\varphi} (\partial_t - (A+iB)\triangle) u = \partial_t f - \mathcal{S} f - \mathcal{A} f.$$

$$\|\partial_t f - \mathcal{S} f - \mathcal{A} f\|^2_{L^2_{x,t}} = \|\mathcal{S} f\|^2_{L^2_{x,t}} + \|\partial_t f - \mathcal{A} f\|^2_{L^2_{x,t}} - 2\mathrm{Re} \iint \mathcal{S} f \overline{\partial_t f - \mathcal{A} f}\, dxdt,$$

$$-2\mathrm{Re} \iint \mathcal{S} f \overline{\partial_t f - \mathcal{A} f}\, dxdt = \int (\mathcal{S}_t f + [\mathcal{S}, \mathcal{A}] f, f)\, dt.$$

• Relation between Carleman inequalities for evolutions, convexity and logarithmic convexity properties of solutions.

- $\partial_t\left(\gamma(t)\partial_t \log H(t)\right) \ge \frac{2}{H}\left(\gamma\, \mathcal{S}_t f + \gamma\,[\mathcal{S},\mathcal{A}]f + \dot\gamma\,\mathcal{S}f, f\right).$
- When $f(x,t) = e^{a(t)|x+b(t)\xi|^2}u(x,t)$, $\xi\in\mathbb{R}^n$, $H(t)=\|f(t)\|^2$ and u is a free wave

$$
\begin{aligned}
&\left(\gamma\, \mathcal{S}_t f + \gamma\,[\mathcal{S},\mathcal{A}]f + \dot\gamma\,\mathcal{S}f, f\right)\\
&\quad \ge 8a\gamma \int_{\mathbb{R}^n} \left| -i\nabla f + \frac{\dot b}{2}\,\xi\, f + \left(\frac{\dot a}{2a} + \frac{\dot\gamma}{4\gamma}\right)(x+b\,\xi)\, f\right|^2 dx\\
&\qquad + F(a,\gamma)\int_{\mathbb{R}^n} \left|x + b\,\xi + \frac{a\gamma \ddot b}{F(a,\gamma)}\,\xi\right|^2 |f|^2\, dx\\
&\qquad\qquad - \frac{a^2\gamma^2\ddot b^2}{F(a,\gamma)}\,|\xi|^2 \int_{\mathbb{R}^n} |f|^2\, dx,
\end{aligned}
$$

- $$F(a,\gamma) = \gamma\left[\ddot a + 32a^3 - \frac{3\dot a^2}{2a} - \frac{a}{2}\left(\frac{\dot a}{a} + \frac{\dot\gamma}{\gamma}\right)^2\right].$$

- $$u_R(x,t) = R^{-\frac{n}{2}} \left(t - \frac{i}{R}\right)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4i(t-\frac{i}{R})}}$$
$$= (Rt - i)^{-\frac{n}{2}} e^{-\frac{(R-iR^2t)}{4(1+R^2t^2)}|x|^2}, \ R > 0.$$

- $$u_R(x, t + t_0), \ R > 0, a \in \mathbb{R}.$$

- Choosing R and t_0 can find free waves such that

$$|u(-1)| \approx e^{-\frac{|x|^2}{\beta^2}}, \ |u(1)| \approx e^{-\frac{|x|^2}{\alpha^2}},$$

when

$$\frac{2}{\alpha\beta} \leq \frac{1}{4}.$$

- When $f = e^{a(t)|x|^2}u$, f verifies $\partial_t f = \mathcal{S}f + \mathcal{A}f$, with

$$\mathcal{S} = -4ia\left(x\partial_x + \tfrac{1}{2}\right) + \dot{a}x^2 \quad , \quad \mathcal{A} = i\left(\partial_x^2 + 4a^2x^2\right).$$

$$\tfrac{1}{a}\,\mathcal{S}_t + \tfrac{1}{a}\,[\mathcal{S},\mathcal{A}] - \tfrac{\dot{a}}{a^2}\,\mathcal{S} = -8\partial_x^2 + F\left(a,\tfrac{1}{a}\right)x^2,$$

$$F\left(a,\tfrac{1}{a}\right) = \tfrac{1}{a}\left(\ddot{a} - \tfrac{3\dot{a}^2}{2a} + 32a^3\right).$$

- If a is a positive and even solution of

$$F\left(a,\tfrac{1}{a}\right) \geq 0,\ \text{in } [-1,1],$$

the formal calculations show that $H_a(t) = \|e^{a(t)x^2}u(t)\|^2$ verifies

$$\partial_t\left(\frac{1}{a}\partial_t \log H_a(t)\right) \geq 0,\ \text{in } [-1,1]$$

and

$$H_a(0) \leq H_a(-1)^{\frac{1}{2}}H_a(1)^{\frac{1}{2}}.$$

- The solutions of $F(a, \frac{1}{a}) = 0$ are the functions

$$Ra(Rt + t_0), R > 0, t_0 \in \mathbb{R}, \text{with } a(t) = \frac{1}{4\,(1 + t^2)}.$$

- If formal calculations are correct for H_{a_R},

$$a_R(t) = Ra(Rt) = \frac{R}{4\,(1 + R^2 t^2)}$$

and some free wave u, we get

$$\|e^{\frac{Rx^2}{4}} u(0)\|^2 \le \|e^{\frac{Rx^2}{4(1+R^2)}} u(-1)\| \|e^{\frac{Rx^2}{4(1+R^2)}} u(1)\|$$

and $u \equiv 0$, letting $R \to +\infty$.

- $$u(x, t) = (t - i)^{-\frac{1}{2}}\, e^{-\frac{|x|^2}{4i(t-i)}} = (t - i)^{-\frac{1}{2}}\, e^{-\frac{(1-it)}{4(1+t^2)}|x|^2}$$

contradicts this.

- (C.E. Kenig, G. Ponce, L. Vega)

Theorem

Assume $u \in C([0,T], L^2(\mathbb{R}^n))$ verifies

$$\partial_t u = i\left(\triangle u + F(x,t)\right), \; \textit{in } \mathbb{R}^n \times [0,T],$$

then

$$\sup_{[0,T]} \|e^{\lambda\cdot x}u(t)\| \leq \|e^{\lambda\cdot x}u(0)\| + \|e^{\lambda\cdot x}u(T)\| + \|e^{\lambda\cdot x}F\|_{L^1([0,T],L^2(\mathbb{R}^n))}.$$

- $f = e^{\lambda\cdot x}u$ verifies

$$\begin{aligned}
&\partial_t f - i\triangle f - i|\lambda|^2 f + 2i\lambda\cdot\nabla f = -iF,\\
&\partial_t \widehat{f}(t) + \left(i|\xi|^2 - i|\lambda|^2 - 2\lambda\cdot\xi\right)\widehat{f}(t) = -i\widehat{F}(t).
\end{aligned}$$

- There is $\epsilon > 0$ such that if $u \in C([0,T], L^2(\mathbb{R}^n))$ verifies

$$\partial_t u = i\left(\triangle u + V(x,t)\right), \text{ in } \mathbb{R}^n \times [0,T], \;\; V \in L^\infty(\mathbb{R}^n \times [0,T]),$$

and

$$\lim_{R \to +\infty} \int_0^1 \|V(t)\|_{L^\infty(\mathbb{R}^n \setminus B_R)}\, dt = 0,$$

then

$$\sup_{[0,T]} \|e^{\lambda \cdot x} u(t)\| \le \|e^{\lambda \cdot x} u(0)\| + \|e^{\lambda \cdot x} u(T)\| + e^{|\lambda| \|V\|_\infty} \sup_{[0,T]} \|u(t)\|.$$

- The identity, $\int_{\mathbb{R}^n} e^{2\sqrt{\mu}\lambda \cdot x - \frac{|\lambda|^2}{2}}\, d\lambda = (2\pi)^{n/2} e^{2\mu |x|^2}$, gives

$$\sup_{[0,T]} \|e^{\mu |x|^2} u(t)\| \le \|e^{\mu |x|^2} u(0)\| + \|e^{\mu |x|^2} u(T)\| + e^{\mu \|V\|_\infty^2} \sup_{[0,T]} \|u(t)\|.$$

Theorem

Assume $u \in C([0,1], H^2(\mathbb{R}^n))$ *is a strong solution to*

$$\partial_t u = i\left(\triangle u + V(x,t)u\right),$$

$$V : \mathbb{R}^n \times [0,1] \to \mathbb{C},\ \ V, \nabla V \in L^\infty(\mathbb{R}^n \times [0,1]),$$

and

$$\lim_{R \to +\infty} \int_0^1 \|V(t)\|_{L^\infty(\mathbb{R}^n \setminus B_R)}\, dt = 0.$$

Then, there is $c = c(n, \|u\|_{L_t^\infty H_x^2}, \|V\|_{L_{t,x}^\infty}, \|\nabla V\|_{L_t^1 L_x^\infty})$ *such that if* $u(0)$ *and* $u(1)$ *are in* $H^1(e^{\mu|x|^2}dx)$ *and* $\mu \geq c$, *then* $u \equiv 0$.

Theorem

$\varphi : [0,1] \longrightarrow \mathbb{R}$ *is a smooth function. Then, there is*

$$N = N(\|\dot{\varphi}\|_\infty + \|\ddot{\varphi}\|_\infty) > 0$$

such that the inequality

$$\alpha^{3/2}|\xi|\|e^{\alpha|x+\varphi(t)\xi|^2} f\|_{L^2(dxdt)} \leq N\|e^{\alpha|x+\varphi(t)\xi|^2}(\partial_t - i\triangle)f\|_{L^2(dxdt)}$$

holds, when $\alpha \geq N$, $\xi \in \mathbb{R}^n$ *and* $f \in C_0^\infty(\mathbb{R}^{n+1})$ *is supported in the set*

$$\{(x,t) : |x + \varphi(t)\xi| \geq |\xi|\} \ .$$

- $u_1,\ u_2 \in C([0,1], H^k(\mathbb{R}^n)),\ k > n/2+1$, verify

$$i\partial_t u + \triangle u + F(u,\overline{u}) = 0,$$

$$F : \mathbb{C}^2 \to \mathbb{C},\quad F \in C^k,\quad F(0) = \partial_u F(0) = \partial_{\bar{u}} F(0) = 0.$$

There is $c = c(n, \|u_1\|_{L_t^\infty H_x^2}, \|u_2\|_{L_t^\infty H_x^2}, \|F\|_{C^k})$ such that if $u_1(0) - u_2(0)$ and $u_1(1) - u_2(1)$ are in $H^1(e^{\mu|x|^2}dx)$ and $\mu \geq c$, then $u_1 \equiv u_2$.

For u_0 in $\mathbb{S}'(\mathbb{R}^n)$ the following statements are equivalent:

(*i*) There are two different real numbers t_1 and t_2, such that $e^{a_j|x|^2}e^{it_j\triangle}u_0$ is in $L^2(\mathbb{R}^n)$, for some $a_j > 0$, $j = 1, 2$.

(*ii*) $e^{b_1|x|^2}u_0$ and $e^{b_2|x|^2}\widehat{u}_0$ are in $L^2(\mathbb{R}^n)$, for some $b_j > 0$, $j = 1, 2$.

(*iii*) There is $\nu : [0, +\infty) \longrightarrow (0, +\infty)$ such that $e^{\nu(t)|x|^2}e^{it\triangle}u_0$ is in $L^2(\mathbb{R}^n)$, for all $t \geq 0$.

(*iv*) $g(x) \equiv e^{i\tau|x|^2}u_0(x)$, $\tau \in \mathbb{R}$, verifies (*ii*) with possibly different constants.

(*v*) $u_0(x + iy)$ is an entire function and

$$|u_0(x + iy)| \leq Ne^{-a|x|^2+b|y|^2}, \text{ for some } N, a, b > 0.$$

(*vi*) $\widehat{u}_0(\xi + i\eta)$ verifies (*v*) with possibly different constants.

(*vii*) There are δ and $\epsilon > 0$ and h in $L^2(e^{\epsilon|x|^2}dx)$ such that $u_0(x) = e^{\delta\Delta}h$.

Assume α, β, T are positive and λ is in $\mathbb{R}^n$. Then,

- $$\|e^{\frac{\lambda\cdot x}{\alpha t+\beta}} u(t)\| \le \|e^{\frac{\lambda\cdot x}{\beta}} u(0)\|^{\frac{\beta(T-t)}{T(\alpha t+\beta)}} \|e^{\frac{\lambda\cdot x}{\alpha T+\beta}} u(T)\|^{\frac{(\alpha T+\beta)t}{(\alpha t+\beta)T}},$$

  when $0 \le t \le T$.

- $$\|e^{\frac{\lambda\cdot x}{\alpha t+\beta}} u(t)\| \le \|e^{\frac{\lambda\cdot x}{\beta}} u(0)\|^{\frac{\beta}{\alpha t+\beta}} \|e^{\frac{2\lambda\cdot \xi}{\alpha}} \widehat{u(0)}\|^{\frac{\alpha t}{\alpha t+\beta}},$$

  when $t \ge 0$.

- $f(x,t) = e^{\frac{\lambda\cdot x}{\alpha t+\beta}} u(x,t),\ \lambda \in \mathbb{R}^n.$
- $u(0)$ extends to $\mathbb{C}^n$ as an analytic function and

$$|u(x+iy,0)| \le N e^{-a|x|^2+b|y|^2},\ \text{for all } x,y \in \mathbb{R}^n.$$

- $\sup_{0\le t\le T} \|e^{a|x|^2} u(t)\| < +\infty.$
- $\partial_t f = \mathcal{S}f + \mathcal{A}f,\ H(t) = \|f(t)\|^2.$
- $(\alpha t+\beta)^2 \mathcal{S}_t + (\alpha t+\beta)^2 [\mathcal{S},\mathcal{A}] + 2\alpha(\alpha t+\beta)\mathcal{S} \ge 0.$
- $\partial_t \left((\alpha t+\beta)^2 \log H(t) \right) \ge 0.$
- $T^{\frac{n}{2}} |u(xT,T)| \to 2^{-\frac{n}{2}} |\widehat{u}(\xi/2,0)|$ and

$$\|e^{\frac{\lambda\cdot x}{\alpha T+\beta}} u(T)\|^{\frac{(\alpha T+\beta)t}{(\alpha t+\beta)T}} = \|e^{\frac{\lambda\cdot xT}{\alpha T+\beta}} T^{\frac{n}{2}} u(Tx,T)\|^{\frac{(\alpha T+\beta)t}{(\alpha t+\beta)T}},$$

and converges to

$$\|e^{\frac{2\lambda\cdot\xi}{\alpha}} \widehat{u(0)}\|^{\frac{\alpha t}{\alpha t+\beta}},$$

when $T \to +\infty$.

- $$\|e^{\frac{|x|^2}{(\alpha t+\beta)^2}} u(t)\| \le \|e^{\frac{|x|^2}{\beta^2}} u(0)\|^{\frac{\beta(T-t)}{T(\alpha t+\beta)}} \|e^{\frac{|x|^2}{(\alpha T+\beta)^2}} u(T)\|^{\frac{(\alpha T+\beta)t}{(\alpha t+\beta)T}},$$

  when $0 \le t \le T$.

- $$\|e^{\frac{|x|^2}{(\alpha t+\beta)^2}} u(t)\| \le \|e^{\frac{|x|^2}{\beta^2}} u(0)\|^{\frac{\beta}{\alpha t+\beta}} \|e^{\frac{4|\xi|^2}{\alpha^2}} \widehat{u(0)}\|^{\frac{\alpha t}{\alpha t+\beta}},$$

  when $t \ge 0$.

- $$\int_{\mathbb{R}^n} e^{\lambda\cdot\frac{2\sqrt{\mu}x}{\alpha t+\beta}-\frac{|\lambda|^2}{2}}\, d\lambda = (2\pi)^{n/2} e^{\frac{2\mu|x|^2}{(\alpha t+\beta)^2}}.$$

Other Convex Weights

Given $\vec{\mu} = (\mu_1, \ldots, \mu_n)$ in $[0, \infty)^n$, the following holds:

- $$\|e^{\frac{\mu_j x_j^2}{(\alpha t+\beta)^2}} u(t)\| \le \|e^{\frac{\mu_j x_j^2}{\beta^2}} u(0)\|^{\frac{\beta(T-t)}{T(\alpha t+\beta)}} \, \|e^{\frac{\mu_j x_j^2}{(\alpha T+\beta)^2}} u(T)\|^{\frac{(\alpha T+\beta)t}{(\alpha t+\beta)T}},$$

  when $0 \le t \le T$.
- $$\|e^{\frac{\mu_j x_j^2}{(\alpha t+\beta)^2}} u(t)\| \le \|e^{\frac{\mu_j x_j^2}{\beta^2}} u(0)\|^{\frac{\beta}{\alpha t+\beta}} \, \|e^{\frac{4\mu_j \xi_j^2}{\alpha^2}} \widehat{u(0)}\|^{\frac{\alpha t}{\alpha t+\beta}},$$

  when $t \ge 0$.

Given $p \in (1,2]$, there is $c = c(p,n) > 0$ such that

- $$\|e^{\left|\frac{x}{\alpha t+\beta}\right|^p} u(t)\| \le c\, \|e^{\left|\frac{x}{\beta}\right|^p} u(0)\|^{\frac{\beta(T-t)}{T(\alpha t+\beta)}}\, \|e^{\left|\frac{x}{\alpha T+\beta}\right|^p} u(T)\|^{\frac{(\alpha T+\beta)t}{(\alpha t+\beta)T}},$$

  when $0 \le t \le T$.

- $$\|e^{\left|\frac{x}{\alpha t+\beta}\right|^p} u(t)\| \le c\, \|e^{\left|\frac{x}{\beta}\right|^p} u(0)\|^{\frac{\beta}{\alpha t+\beta}}\, \|e^{\left|\frac{2\xi}{\alpha}\right|^p} \widehat{u(0)}\|^{\frac{\alpha t}{\alpha t+\beta}},$$

  when $t \ge 0$.

- There is $c = c(n,p)$ such that

  $$c^{-1} e^{\frac{|x|^p}{p}} \le \int_{\mathbb{R}^n} e^{\lambda\cdot x - \frac{|\lambda|^{p'}}{p'}} |\lambda|^{\frac{n(p'-2)}{2}}\, d\lambda \le c\, e^{\frac{|x|^p}{p}},\ \text{when } x \in \mathbb{R}^n.$$

Theorem

- *Assume that u in $C([0,T]), L^2(\mathbb{R}^n))$ verifies*

$$\partial_t u = i\left(\triangle u + V(x,t)u\right), \ \text{in } \mathbb{R}^n \times [0,T],$$

$$\|e^{\frac{|x|^2}{\beta^2}} u(0)\| + \|e^{\frac{|x|^2}{\alpha^2}} u(T)\| < +\infty,$$

the potential $V(x,t)$ verifies one of the conditions in the next slides

$$\frac{T}{\alpha\beta} > \frac{1}{4}\ .$$

Then, $u \equiv 0$ in $\mathbb{R}^n \times [0,1]$.

- When

$$\|e^{\mu|x|^2} u(0)\| + \|e^{\mu|x|^2} u(T)\| < +\infty,$$

the same holds provided that

$$\mu > \frac{1}{4T}.$$

Conditions on the Potential

(i) $V(x,t) = V_1(x)$ is real-valued and $\|V_1\|_{L^\infty(\mathbb{R}^n)}$ is finite.

(ii) $V(x,t) = V_1(x) + V_2(x,t)$, V_1 as above and

$$\sup_{[0,1]} \|e^{\frac{|x|^2}{(\alpha t+\beta(1-t))^2}} V_2(t)\|_{L^\infty(\mathbb{R}^n)} \text{ is finite.}$$

(iii) $\|V\|_{L^\infty(\mathbb{R}^n\times[0,1])}$ is finite and

$$\lim_{R\to+\infty} \int_0^1 \|V(t)\|_{L^\infty(\mathbb{R}^n\setminus B_R)}\, dt = 0.$$

(i) $V(x,t) = V_1(x)$ is real-valued and $\|V_1\|_\infty$ is finite.

(ii) $V(x,t) = V_1(x) + V_2(x,t)$, V_1 as above and

$$\sup_{[0,1]} \|e^{\mu|x|^2} V_2(t)\|_\infty \text{ is finite.}$$

(iii) $\|V\|_\infty$ is finite and

$$\lim_{R\to+\infty} \int_0^1 \|V(t)\|_{L^\infty(\mathbb{R}^n\setminus B_R)}\, dt = 0.$$

Theorem

Assume $T/\alpha\beta = 1/4$. *Then, there is a smooth complex-valued potential* V *verifying*

$$|V(x,t)| \lesssim \frac{1}{1+|x|^2}, \ \text{in } \mathbb{R}^n \times [0,T]$$

and u *in* $C^\infty([0,T], \mathcal{S}(\mathbb{R}^n))$ *such that*

$$\partial_t u = i\left(\triangle u + V(x,t)u\right), \ \text{in } \mathbb{R}^n \times [0,T]$$

and

$$\|e^{\frac{|x|^2}{\beta^2}} u(0)\|_{L^2(\mathbb{R}^n)} + \|e^{\frac{|x|^2}{\alpha^2}} u(T)\|_{L^2(\mathbb{R}^n)}$$

is finite.

- Reduction to case $\alpha = \beta$ using the Appell (conformal) transform:

$$w(x,t) = t^{-\frac{n}{2}} u(x/t, 1/t) e^{\frac{|x|^2}{4(A+iB)t}}$$

verifies

$$\partial_t w = -(A+Bi)\left(\triangle w + t^{-\frac{n}{2}-2} F(x/t, 1/t) e^{\frac{|x|^2}{4(A+iB)t}}\right),$$

when

$$\partial_s u = (A+Bi)\left(\triangle u + F(y,s)\right).$$

-

$$\widetilde{u}(x,t) = \left(\frac{\sqrt{\alpha\beta}}{\alpha(1-t)+\beta t}\right)^{\frac{n}{2}} u\left(\frac{\sqrt{\alpha\beta}\, x}{\alpha(1-t)+\beta t}, \frac{\beta t}{\alpha(1-t)+\beta t}\right) e^{\frac{(\alpha-\beta)|x|^2}{4i(\alpha(1-t)+\beta t)}},$$

verifies

$$\partial_t \widetilde{u} = i\left(\triangle \widetilde{u} + \widetilde{V}(x,t)\widetilde{u}\right), \text{ in } \mathbb{R}^n \times [0,1],$$

$$\widetilde{V}(x,t) = \frac{\alpha\beta}{(\alpha(1-t)+\beta t)^2}\, V\left(\frac{\sqrt{\alpha\beta}\,x}{\alpha(1-t)+\beta t}, \frac{\beta t}{\alpha(1-t)+\beta t}\right)$$

and

$$\|\widetilde{u}(t)e^{\mu|x|^2}\| = \|u(s)e^{\frac{|y|^2}{(\alpha s+\beta(1-s))^2}}\|\ , \text{ when } \mu = \frac{1}{\alpha\beta}\ , \ s = \frac{\beta t}{\alpha(1-t)+\beta t}\ .$$

- $$\|\widetilde{V}\|_{L^\infty(\mathbb{R}^n\times[0,1])} \le \max\left\{\frac{\alpha}{\beta}, \frac{\beta}{\alpha}\right\}\|V\|_{L^\infty(\mathbb{R}^n\times[0,1])}.$$
- $$\int_0^1 \|\widetilde{V}(t)\|_{L^\infty(\mathbb{R}^n)}\, dt = \int_0^1 \|V(s)\|_{L^\infty(\mathbb{R}^n)}\, ds.$$

- $$u(x,t) = (1+it)^{-2k-\frac{n}{2}} \left(1+|x|^2\right)^{-k} e^{-\frac{(1-it)}{4(1+t^2)}|x|^2},$$
for some $k > \frac{n}{2}$.

- $$\|e^{|x|^2/8} u(\pm 1)\| < +\infty \quad , \quad \partial_t u = i\left(\triangle u + V(x,t)u\right), \text{ in } \mathbb{R}^{n+1}$$

and
$$|V(x,t)| \leq \frac{1}{1+|x|^2}, \text{ in } \mathbb{R}^n \times [-1,1].$$

Variable Coefficients and the same Gaussian decay

Assume $\mu > 0$ and that u in $C([0,1]), L^2(\mathbb{R}^n))$ verifies

$$\partial_t u = i\left(\triangle u + V(x,t)u\right), \text{ in } \mathbb{R}^n \times [0,1].$$

Then, there is a universal constant $N = N(\mu)$ such that

$$sup_{[0,1]}\|e^{\mu|x|^2}u(t)\| + \|\sqrt{t(1-t)}e^{\mu|x|^2}\nabla u\|_{L^2(\mathbb{R}^n\times[0,1])} \le e^{N\left(1+\|V\|_\infty^2\right)}\left[\|e^{\mu|x|^2}u(0)\| + \|e^{\mu|x|^2}u(1)\|\right].$$

Moreover, $\|e^{\mu|x|^2}u(t)\|$ is "logarithmically" convex in $[0,1]$:

$$\|e^{\mu|x|^2}u(t)\| \le e^{N\left(1+\|V\|_\infty^2\right)}\|e^{\mu|x|^2}u(0)\|^{1-t}\|e^{\mu|x|^2}u(1)\|^t .$$

Theorem

Assume $\mu > 0$, u *in* $C([0,1]), L^2(\mathbb{R}^n))$ *verifies*

$$\partial_t u = i\left(\triangle u + V_1(x)u\right), \text{ in } \mathbb{R}^n \times [0,1],$$

with V_1 *real and bounded. Then, there is a universal constant* $N = N(\mu)$ *such that*

$$sup_{[0,1]}\|e^{\mu|x|^2}u(t)\| + \|\sqrt{t(1-t)}e^{\mu|x|^2}\nabla u\|_{L^2(\mathbb{R}^n\times[0,1])} \leq e^{N\left(1+\|V_1\|_\infty^2\right)}\left[\|e^{\mu|x|^2}u(0)\| + \|e^{\mu|x|^2}u(1)\|\right].$$

Gaussian decay for diffusions

Theorem

Assume that u *in* $L^\infty([0,1], L^2(\mathbb{R}^n)) \cap L^2([0,1]), H^1(\mathbb{R}^n))$ *satisfies*

$$\partial_t u = (A + iB)\left(\triangle u + V(x,t)u + F(x,t)\right), \text{ in } \mathbb{R}^n \times (0,1],$$

$A > 0$ *and* $B \in \mathbb{R}$. *Then,*

$$e^{-M_T} \| e^{\frac{\mu A |x|^2}{A + 4\mu(A^2+B^2)T}} u(T) \|$$
$$\leq \| e^{\mu |x|^2} u(0) \| + \sqrt{A^2 + B^2} \| e^{\frac{\mu A |x|^2}{A + 4\mu(A^2+B^2)t}} F(t) \|_{L^1([0,T], L^2(\mathbb{R}^n))},$$

for all $T > 0$ *and with*

$$M_T = \| A \mathrm{Re}\, V - B \mathrm{Im}\, V \|_{L^1([0,T], L^\infty(\mathbb{R}^n))}.$$

Convexity for Diffusions

- u in $L^\infty([0,1]), L^2(\mathbb{R}^n)) \cap L^2([0,1], H^1(\mathbb{R}^n))$ verifies

$$\partial_t u = (A + iB)\left(\triangle u + V(x,t)u\right), \text{ in } \mathbb{R}^n \times [0,1],$$

$A > 0$, $B \in \mathbb{R}$. Then,

$$\|e^{\mu|x|^2} u(t)\|$$

is logarithmically convex in $[0,1]$ and there is N such that

$$\|e^{\mu|x|^2} u(t)\| \ \le e^{N\left(1+(A^2+B^2)\|V\|_\infty^2\right)} \|e^{\mu|x|^2} u(0)\|^{1-t} \|e^{\mu|x|^2} u(1)\|^t,$$

when $0 \le t \le 1$.

Convexity for Diffusions

Moreover,

$$\|\sqrt{t(1-t)}e^{\mu|x|^2}\nabla u\|_{L^2(\mathbb{R}^n\times[0,1])} \leq e^{N\left(1+(A^2+B^2)\|V\|_\infty^2\right)}\left(\|e^{\mu|x|^2}u(0)\|+\|e^{\mu|x|^2}u(1)\|\right).$$

- When $u(t)=e^{itH}u_0$, $H=\triangle+V_1(x)$,

$$\begin{cases}\partial_t u=i\left(\triangle u+V_1(x)\right), \text{in } \mathbb{R}^n\times[0,1],\\ u(0)=u_0.\end{cases}$$

- $u_\epsilon(t) = e^{(\epsilon+i)tH}u_0$ solves

$$\begin{cases} \partial_t u = (\epsilon + i)\left(\triangle u + V_1(x)\right), \text{in } \mathbb{R}^n \times [0,1], \\ u(0) = u_0. \end{cases}$$

- $u_\epsilon(1) = e^{(\epsilon+i)H}u_0 = e^{\epsilon H}e^{iH}u_0 = e^{\epsilon H}u(1)$, and

$$\|e^{\mu_\epsilon |x|^2}e^{\epsilon H}u(1)\| \le e^{\epsilon \|V_1\|_\infty}\|e^{\mu |x|^2}u(1)\|, \text{ with } \mu_\epsilon = \frac{\mu}{1+4\mu\epsilon}.$$

- $\|e^{\mu_\epsilon |x|^2}u_\epsilon(t)\|$ is logarithmically convex and

$$\sup_{[0,1]} \|e^{\mu_\epsilon |x|^2}u_\epsilon(t)\| + \|\sqrt{t(1-t)}e^{\mu_\epsilon |x|^2}\nabla u_\epsilon\|_{L^2(\mathbb{R}^n\times[0,1])}$$
$$\le e^{N\left(1+\|V_1\|_\infty^2\right)}\left(\|e^{\mu_\epsilon |x|^2}u_\epsilon(0)\| + \|e^{\mu_\epsilon |x|^2}u_\epsilon(1)\|\right).$$

Then, let ϵ tend to zero.

A Carleman inequality

- The inequality

$$|\xi| \| e^{\mu|x+(1-t^2)\xi|^2-(1+\epsilon)\frac{|\xi|^2(1-t^2)}{16\mu}} f \|_{L^2(\mathbb{R}^{n+1})} \leq$$

$$N_{\mu,\epsilon} \| e^{\mu|x+(1-t^2)\xi|^2-(1+\epsilon)\frac{|\xi|^2(1-t^2)}{16\mu}} (\partial_t - i\triangle) f \|_{L^2(\mathbb{R}^{n+1})}$$

holds, when $\mu > 0$, $\epsilon > 0$, $\xi \in \mathbb{R}^n$ and $f \in C_0^\infty(\mathbb{R}^{n+1} \times (-1,1))$.

- Related to the logarithmic convexity of

$$H(t) = \| e^{\mu|x+(1-t^2)\xi|^2-\frac{|\xi|^2(1-t^2)}{16\mu}} u(t)\|^2, \quad \mu > 0,\ \xi \in \mathbb{R}^n,$$

when u is a free wave in $\mathbb{R}^n \times [-1,1]$.

- $H(t) \leq H(-1)^{\frac{1}{2}} H(1)^{\frac{1}{2}}$, when $-1 \leq t \leq 1$.

$$H(-1)^{\frac{1}{2}} H(1)^{\frac{1}{2}} = \|e^{\mu|x|^2} u(-1)\| \|e^{\mu|x|^2} u(1)\| \leq 1,$$

$$H(t) = \|e^{\mu|x|^2 - (1-t^2)|\xi|^2 \frac{1-16\mu^2(1-t^2)}{16\mu} + 2\mu(1-t^2)x\cdot\xi} u(t)\|^2 \leq 1.$$

proves theorem, when $\mu \geq \frac{1}{4}$, in $\mathbb{R}^n \times [-1, 1]$. Otherwise,

$$\|e^{\mu|x|^2 - (1-t^2)|\xi|^2 \frac{1+\epsilon-16\mu^2(1-t^2)}{16\mu} + 2\mu(1-t^2)x\cdot\xi} u(t)\|^2 \leq e^{-\frac{2\epsilon(1-t^2)}{16\mu}|\xi|^2}$$

and integrate $d\xi$ to get

- $$\sup_{[-1,1]} \|e^{(a_2(t)-\epsilon)\,|x|^2} u(t)\|^2 < +\infty, \text{ for all } \epsilon > 0,$$

where
$$a_2(t) = \frac{\mu}{1 - 16\mu^2(1-t^2)} > \mu, \text{ in } (-1,1).$$

- Also,
$$F(a_2, \tfrac{1}{a_2}) > 0, \text{ in } [-1,1]$$

and the latter justifies the log-convexity calculations and implies that
$$\sup_{[-1,1]} \|e^{a_2(t)|x|^2} u(t)\|^2 \le 1.$$

Theorem

$\mu \le 1/8$. Then, there is $N = N(\mu)$ such that

$$\sup_{[-1,1]} \|e^{\frac{R}{4(1+R^2t^2)}|x|^2} u(t)\| + \|\sqrt{1-t^2}\,\nabla\left[e^{\frac{(R-iR^2t)}{4(1+R^2t^2)}|x|^2} u(t)\right]\|_{L^2(\mathbb{R}^n\times[-1,1])} \le e^{N\left(1+\|V\|_\infty^2\right)}\left[\|e^{\mu|x|^2}u(-1)\| + \|e^{\mu|x|^2}u(1)\|\right],$$

where R is the smallest root of the equation

$$\mu = \frac{R}{4\left(1+R^2\right)}.$$

- $$u_R(x,t) = R^{-\frac{n}{2}}\left(t - \frac{i}{R}\right)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4i(t-\frac{i}{R})}}$$
$$= (Rt - i)^{-\frac{n}{2}} e^{-\frac{(R-iR^2t)}{4(1+R^2t^2)}|x|^2}, \; R > 0.$$

- u in $C([-1,1], L^2(\mathbb{R}^n))$ is a solution of

$$\partial_t u - i\triangle u = 0, \text{ in } \mathbb{R}^n \times [-1,1]$$

and

$$\|e^{\mu|x|^2} u(-1)\| + \|e^{\mu|x|^2} u(1)\| \leq 1, \text{ for some } \mu > 0.$$

- Show that either $u \equiv 0$ or

$$\sup_{[-1,1]} \|e^{\frac{R|x|^2}{4(1+R^2t^2)}} u(t)\| \leq 1,$$

where R is the smallest root of the equation

$$\mu = \frac{R}{4(1+R^2)}.$$

- $$\sup_{[-1,1]} \|e^{\mu|x|^2}u(t)\| + \epsilon_\mu \|\sqrt{1-t^2}\, e^{\mu|x|^2}\nabla u\|_{L^2(\mathbb{R}^n\times[-1,1])} \le 1,$$

- kth step: k smooth functions $a_i : [-1,1] \longrightarrow (0,+\infty)$ have been constructed:

$$a_1 \equiv \mu < a_2 < \cdots < a_k, \text{ in } (-1,1),$$

$$F(a_i, \tfrac{1}{a_i}) > 0, \text{ in } [-1,1], \; a_i(-1) = a_i(1) = \mu, \; i = 1, \dots, k,$$

where

$$F(a, \tfrac{1}{a}) = \tfrac{1}{a}\left(\ddot{a} - \tfrac{3\dot{a}^2}{2a} + 32a^3\right)$$

- $$\sup_{[-1,1]} \|e^{a_i(t)|x|^2} u(t)\| + \epsilon_{a_i} \|e^{a_i(t)|x|^2} \nabla u\|_{L^2(\mathbb{R}^n \times [-1,1])} \leq 1. \qquad (1)$$

- $$\partial_t \left(\frac{1}{a} \partial_t \log H_b \right) \geq -\frac{2\ddot{b}^2 |\xi|^2}{F(a)}, \qquad (2)$$

$$\partial_t \left(\frac{1}{a} \partial_t H \right) \geq \epsilon_a \int_{\mathbb{R}^n} e^{a|x|^2} \left(|\nabla u|^2 + |x|^2 |u|^2 \right) dx,$$

when $F(a, \frac{1}{a}) > 0$, in $[-1, 1]$ and

$$H_b(t) = \|e^{a(t)|x+b(t)\xi|^2} u(t)\|^2_{L^2(\mathbb{R}^n)}, \quad H(t) = \|e^{a(t)|x|^2} u(t)\|^2_{L^2(\mathbb{R}^n)}.$$

- Take $a = a_k$ in (2) with a suitable choice of a certain $b = b_k$ in $(-1, 1)$, with

$$b(-1) = b(1) = 0,$$

gives:

$$\int_{\mathbb{R}^n} e^{2a_k(t)|x|^2 - 2|\xi|^2\left(T_k(t) - a_k(t)b_k(t)^2\right) + 4a_k(t)b_k(t)x\cdot\xi} |u(t)|^2 \, dx \leq 1,$$

for all $\xi \in \mathbb{R}^n$.

- $$\partial_t\left(\frac{1}{a_k}\,\partial_t T_k\right) = -\frac{\ddot{b}_k^2}{F(a_k, \frac{1}{a_k})}, \text{ in } [-1, 1], \quad T_k(-1) = T_k(1) = 0.$$

- Is $T_k(t) - a_k(t)b_k(t)^2 \leq 0$, somewhere in $(-1, 1)$?.

- A positive answer implies $u \equiv 0$.

- Otherwise

$$\int_{\mathbb{R}^n} e^{2a_k(t)|x|^2-2|\xi|^2\left((1+\epsilon)T_k(t)-a_k(t)b_k(t)^2\right)+4a_k(t)b_k(t)x\cdot\xi}|u(t)|^2\,dx \\ \le e^{-2\epsilon T_k(t)|\xi|^2}$$

and integrate $d\xi$ to find that

$$\sup_{[-1,1]} \|e^{(a_{k+1}(t)-\epsilon)|x|^2}u(t)\| < +\infty, \text{ for all } \epsilon > 0$$

with

$$a_{k+1} = \frac{a_k T_k}{T_k - a_k b_k^2},$$

$$a_1 \equiv \mu < a_2 < \cdots < a_k < a_{k+1}, \text{ in } (-1,1),$$

$$F(a_{k+1}, \tfrac{1}{a_{k+1}}) > 0, \text{ in } [-1,1], \; a_{k+1}(-1) = a_{k+1}(1) = \mu.$$

- When $\lim_{k\to+\infty} a_k(0) < +\infty$, the sequence a_k converges to an even function a verifying

$$\begin{cases} F(a, \frac{1}{a}) = \frac{1}{a}\left(\ddot{a} - \frac{3\dot{a}^2}{2a} + 32a^3\right) = 0, \text{ in } [-1,1], \\ a(1) = \mu. \end{cases}$$

-

$$\frac{R}{4\,(1+R^2t^2)}\,, \quad R \in \mathbb{R},$$

are all the even solutions of this equation and a must be one of them:

$$\mu = \frac{R}{4\,(1+R^2)}\,,$$

for some $R > 0$. In particular, $u \equiv 0$, when $\mu > 1/8$.

Parabolic analog

Theorem

Assume that

$$|\triangle u - \partial_t u| \leq M\left(|u| + |\nabla u|\right)\ ,\ |u(x,t)| \leq Me^{M|x|^2}$$

in $\mathbb{R}^n \times [0, T]$ *and*

$$|u(x,T)| \leq C_k e^{-k|x|^2},\ \textit{in}\ \mathbb{R}^n,\ \textit{for all}\ k \geq 1.$$

Then, $u \equiv 0$ *in* $\mathbb{R}^n \times [0,1]$.

- If $e^{\frac{|x|^2}{\beta^2}} e^{T\triangle} u_0 \in L^2(\mathbb{R}^n)$, $u_0 \in L^2(\mathbb{R}^n)$ and $\frac{\sqrt{T}}{\beta} \geq \frac{1}{2}$. Then,

$$u_0 \equiv 0.$$

Parabolic analog

Theorem

Let u in $L^\infty([0,T], L^2(\mathbb{R}^n))$ *verify*

$$\partial_t u = \triangle u + V(x,t)u \ , \ \ in \ \mathbb{R}^n \times [0,T],$$

for some bounded potential V,

$$\|u(0)\| + \|e^{\frac{|x|^2}{\beta^2}} u(T)\| < +\infty$$

and assume $\frac{\sqrt{T}}{\beta} \geq 1$. *Then,* $u \equiv 0$.

- Log-convexity in $[0,1]$ of

$$H(t) = \int_{\mathbb{R}^n} e^{2\mu|x+\xi t(1-t)e_1|^2 + \frac{|\xi|^2 t(1-t)(1-2t)}{3} - \frac{|\xi|^2 t(1-t)}{8\mu}} |\widetilde{u}(t)|^2 \, dx,$$

when $\xi \in \mathbb{R}^n$,

$$\|e^{\mu|x|^2}\widetilde{u}(0)\| + \|e^{\mu|x|^2}\widetilde{u}(1)\| < +\infty$$

and $\mu = \frac{1}{2\beta} \geq \frac{1}{2}$, implies $\widetilde{u} \equiv 0$.

- $$u_R(x,t) = R^{-\frac{n}{2}} \left(t - \tfrac{i}{R}\right)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4(t-\frac{i}{R})}} = (Rt-1)^{-\frac{n}{2}} e^{-\frac{\left(R^2 t + iR\right)|x|^2}{4(1+R^2t^2)}}.$$

- $$a_R(t) = \frac{R^2 t}{4(1+R^2t^2)}, \ \mu = \frac{R^2}{4(1+R^2)}.$$

- $$F(a) = e^{8A}\left(\ddot{a} + 24a\dot{a} + 64a^3\right) = \frac{\dddot{e^{8A}}}{8}, \text{ where } \dot{A} = a.$$
- $$\partial_t\left(e^{8A}\partial_t \log H_b\right) \geq -e^{8A}|\xi|^2 \frac{\left(\ddot{b} + 2\left(\dot{a} + 8a^2\right)\dot{b}\right)^2}{\ddot{a} + 24a\dot{a} + 64a^3},$$
- $$H_b(t) = \|e^{a(t)|x+b(t)\xi|^2}u(t)\|^2_{L^2(\mathbb{R}^n)}.$$
- What is the choice of b so that if

$$\begin{cases} \partial_t\left(e^{8A}\partial_t T\right) = -e^{8A}\frac{\left(\ddot{b}+2\left(\dot{a}+8a^2\right)\dot{b}\right)^2}{\ddot{a}+24a\dot{a}+64a^3}, \\ T(0) = T(1) = 0, \end{cases} \qquad \widetilde{a} = \frac{aT}{T - ab^2},$$

$\widetilde{a}$ verifies

$$F(\widetilde{a}) > 0, \text{ in } [0,1],$$

when $F(a) > 0$ in $[0,1]$ and $T - ab^2 > 0$ in $(0,1)$?

Finding Gaussian decaying solutions

- Gaussian decaying solutions for $\partial_t = i\left(\triangle + V_1(x)\right)$:
- (R. Killip)

When $e^{\mu|x|^2}u_0$ is in $L^2(\mathbb{R}^n)$ and $u(t) = e^{itH}\left(e^H u_0\right)$,

$$u(t) = e^{\left(\frac{1}{t}+i\right)tH}u_0$$

and

$$\|e^{\frac{\mu|x|^2}{1+4\mu\left(1+t^2\right)}}u(t)\| \le e^{\|V_1\|_\infty}\|e^{\mu|x|^2}u_0\|\ , \text{ when } t \ge 0.$$