Smoothness of the motion of a rigid body immersed in an incompressible perfect fluid

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The Euler equations in $\mathbb{R}^3$

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = 0 \quad \text{in } (-T, T) \times \mathbb{R}^3 \tag{1}$$

$$\operatorname{div} u = 0 \quad \text{in } (-T, T) \times \mathbb{R}^3 \tag{2}$$

$$u|_{t=0} = u_0 \quad \text{in } \mathbb{R}^3. \tag{3}$$

Hölder spaces

Assume

$$r \in (0, 1).$$

$u \in C^{0,r}(\Omega)$ if $u \in C^0(\Omega)$, u bounded in Ω and

$$|u(x) - u(y)| \leqslant C|x - y|^r.$$

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$$|u(x) - u(y)| \leqslant C|x-y|^r.$$

Assume

$$\lambda \in \mathbb{N}.$$

$u \in C^{\lambda,r}(\Omega)$ if for all $\alpha \in \mathbb{N}^3$, $|\alpha| \leqslant \lambda$,

$$\partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3} u \in C^{0,r}(\Omega).$$

Well-posedness

$$\lambda \in \mathbb{N}, \quad r \in (0,1).$$

Theorem

Assume

$$u_0 \in C^{\lambda+1,r}(\mathbb{R}^3), \quad \operatorname{div} u_0 = 0,$$

Then there exist $T > 0$ *and a unique solution*

$$u \in C_w((-T,T), C^{\lambda+1,r}(\mathbb{R}^3))$$

of (1)–(3).

The flow of the fluid

The flow Φ is defined by

$$\partial_t \Phi(t,x) = u(t,\Phi(t,x)),$$
$$\Phi(0,x) = x$$

for $(t,x) \in (-T,T) \times \mathbb{R}^3$.

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$$\Phi(0,x) = x$$

for $(t,x) \in (-T,T) \times \mathbb{R}^3$.
The material derivative of f is defined by

$$Df = \partial_t f + u \cdot \nabla f.$$

We have

$$\frac{d}{dt} f(\cdot, \Phi(\cdot, x)) = Df(\cdot, \Phi(\cdot, x)).$$

Analyticity of the trajectories

Theorem

Under the hypotheses of Theorem 1, the flow Φ is analytic from $(-T,T)$ to $\mathrm{Id} + C^{\lambda+1,r}(\mathbb{R}^3)$.

More precisely, for all $k \in \mathbb{N}$,

$$\|D^k u\|_{C^{\lambda+1,r}} \leqslant k! L^k \left(\|u\|_{C^{\lambda+1,r}}\right)^{k+1}.$$

Idea of the proof (1/3)

First: (1) can be written as

$$Du + \nabla p = 0.$$

Taking the divergence:

$$-\Delta p = \operatorname{div} Du = \sum_{i,j} \partial_i \partial_j (u_i u_j).$$

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Thus, by setting

$$E(x) = \frac{-2\pi^2}{|x|},$$

we have

$$\nabla p = \sum_{i,j} \int_{\mathbb{R}^3} (\partial_i \partial_j \nabla E)(x-y)(u_i(x) - u_i(y))(u_j(x) - u_j(y)).$$

Idea of the proof (2/3)

Therefore

$$D(\nabla p) = \sum_{i,j} \int_{\mathbb{R}^3} (\partial_i \partial_j \nabla E)(x-y)(Du_i(x) - Du_i(y))(u_j(x)-u_j(y))$$

$$+ \sum_{i,j} \int_{\mathbb{R}^3} (\partial_i \partial_j \nabla E)(x-y)(u_i(x) - u_i(y))(Du_j(x) - Du_j(y))$$

$$\sum_{i,j,k} \int_{\mathbb{R}^3} (\partial_i \partial_j \partial_k \nabla E)(x-y)(u_i(x)-u_i(y))(u_j(x)-u_j(y))(u_k(x) - u_k(y)).$$

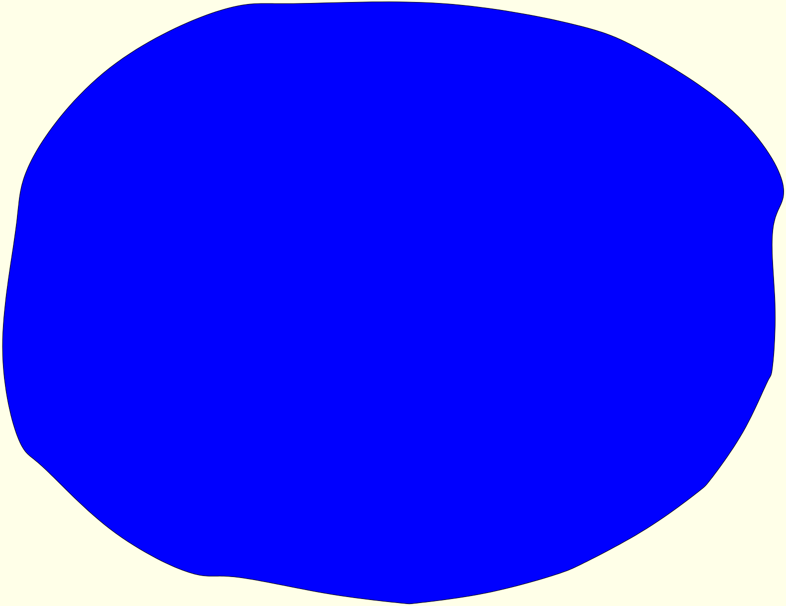
Idea of the proof (3/3)

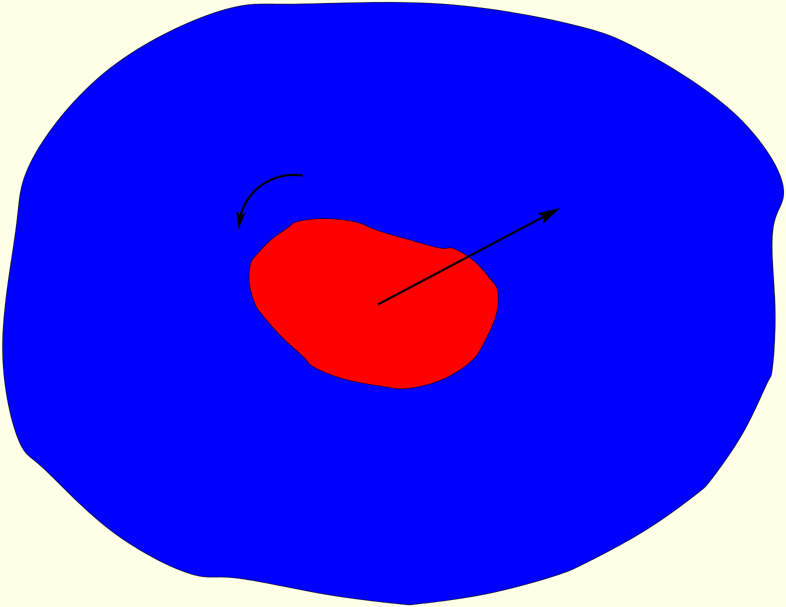
And by induction,

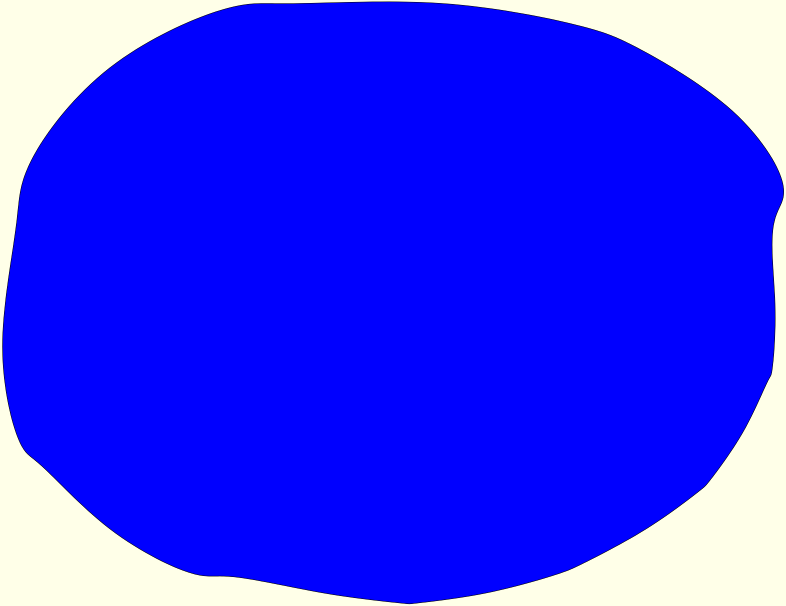
$$D^k(\nabla p) = \sum_{s+|\alpha|=k+2} a^k(s,\alpha) \sum_{\nu_1,\ldots,\nu_s\in\{1,2,3\}} \partial_{\nu_1}\ldots\partial_{\nu_s}\nabla E *_s (D^{\alpha_1}u_{\nu_1},\ldots,D^{\alpha_s}u_{\nu_s}),$$

with

$$|a^k(s,\alpha)| \leqslant \frac{k!}{\alpha!(s+2)!}$$







The Euler equations in a bounded domain

The Euler equations in a bounded domain (without rigid body):

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = 0, \qquad \text{in } (-T,T)\times\Omega, \tag{4}$$

$$\operatorname{div} u = 0 \qquad \text{in } (-T,T)\times\Omega, \tag{5}$$

$$u \cdot n = 0 \qquad \text{on } (-T,T)\times\partial\Omega. \tag{6}$$

Assume

$$\lambda \in \mathbb{N}, \quad r \in (0,1)$$

and

$$u \in C_w((-T,T), C^{\lambda+1,r}(\Omega)).$$

Tool 1: Regularity Lemma (1/2)

Tangential harmonic vector fields:

$$\operatorname{div} u = 0, \quad \operatorname{curl} u = 0, \quad u \cdot n = 0.$$

We denote by H the finite dimensional space of tangential harmonic vector fields.

Tool 1: Regularity Lemma (1/2)

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$$\operatorname{div} u = 0, \quad \operatorname{curl} u = 0, \quad u \cdot n = 0.$$

We denote by H the finite dimensional space of tangential harmonic vector fields.

Notation:

$$|\cdot| := \|\cdot\|_{C^{\lambda,r}(\Omega)} \text{ and } |\cdot|_{\partial\Omega} := \|\cdot\|_{C^{\lambda,r}(\partial\Omega)},$$
$$\|\cdot\| := \|\cdot\|_{C^{\lambda+1,r}(\Omega)} \text{ and } \|\cdot\|_{\partial\Omega} := \|\cdot\|_{C^{\lambda+1,r}(\partial\Omega)}.$$

Tool 1: Regularity Lemma (2/2)

Lemma

For any $u \in C^{\lambda,r}(\Omega)$ such that

$$\operatorname{div} u \in C^{\lambda,r}(\Omega), \quad \operatorname{curl} u \in C^{\lambda,r}(\Omega), \quad u \cdot n \in C^{\lambda+1,r}(\partial\Omega),$$

Tool 1: Regularity Lemma (2/2)

Lemma

For any $u \in C^{\lambda,r}(\Omega)$ *such that*

$$\operatorname{div} u \in C^{\lambda,r}(\Omega), \quad \operatorname{curl} u \in C^{\lambda,r}(\Omega), \quad u \cdot n \in C^{\lambda+1,r}(\partial\Omega),$$

one has $u \in C^{\lambda+1,r}(\Omega)$ *and there exists a constant* c_r *depending only on* Ω *such that*

$$\|u\| \leqslant c_r \left(|\operatorname{div} u| + |\operatorname{curl} u| + \|u \cdot n\| + |\Pi u| \right), \tag{7}$$

where Π *is the orthogonal projection* $L^2(\Omega; \mathbb{R}^3) \to H$.

Tool 2: Commutation rules

$$Df = \partial_t f + u \cdot \nabla f,$$

Commutation rules:

$$D(fg) = (Df)g + f(Dg),$$

$$\nabla Df - D\nabla f = \nabla u \cdot \nabla f,$$

$$\operatorname{div} Df - D \operatorname{div} f = \operatorname{tr}\{\nabla u \cdot \nabla f\},$$

$$\operatorname{curl} Df - D \operatorname{curl} f = \operatorname{as}\{\nabla u \cdot \nabla f\},$$

$$n \cdot Df - D(n \cdot f) = -\nabla^2 \rho\{u, f\}, \quad \text{where } n = \nabla\rho.$$

Tools 1 & 2 applied to the Euler equations

First

$$\operatorname{div} Du = \operatorname{tr}\{\nabla u \cdot \nabla u\},$$

$$\operatorname{curl} Du = -\operatorname{curl} \nabla p = 0,$$

$$n \cdot Du = -\nabla^2 \rho\{u, u\},$$

$$\Pi Du = -\Pi \nabla p = 0.$$

Tools 1 & 2 applied to the Euler equations

First

$$\begin{aligned}
\operatorname{div} Du &= \operatorname{tr}\{\nabla u \cdot \nabla u\}\,, \\
\operatorname{curl} Du &= -\operatorname{curl} \nabla p = 0, \\
n \cdot Du &= -\nabla^2 \rho\{u, u\}, \\
\Pi Du &= -\Pi \nabla p = 0.
\end{aligned}$$

Second

$$Du \in C_w((-T, T), C^{\lambda+1,r}(\Omega))$$

and

$$\begin{aligned}
\|Du\| &\leqslant c_r\left(|\operatorname{div} Du| + |\operatorname{curl} Du| + \|Du \cdot n\|_{\partial\Omega} + |\Pi Du|\right) \\
&\leqslant C\|u\|^2.
\end{aligned}$$

Kato's result

By using a proof by induction, Kato showed that if $\partial\Omega$ is smooth, then for all $k \in \mathbb{N}$,

$$D^k u \in C_w((-T,T), C^{\lambda+1,r}(\Omega))$$

and

$$\|D^k u\| \leqslant C_k \|u\|^{k+1}.$$

To show the analyticity, we need to estimate C_k: we have to count the number of terms appearing when applying D at each time.

Idea of the proof for the Euler equations (1/3)

For $k \in \mathbb{N}^*$, we have in Ω

$$\operatorname{div} D^k u = \operatorname{tr} \left\{ F^k[u] \right\}$$

where

$$F^k[u] := \sum c_k(s,\alpha)\, \nabla D^{\alpha_1} u \cdot \ldots \cdot \nabla D^{\alpha_s} u,$$

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where the sum is over

$$2 \leqslant s \leqslant k+1 \text{ and } \alpha := (\alpha_1, \ldots, \alpha_s) \in \mathbb{N}^s / \ |\alpha| = k+1-s.$$

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Lemma

In the above formula, we have $c_k(s,\alpha) \in \mathbb{Z}$ and

$$|c_k(s,\alpha)| \leqslant \frac{k!}{\alpha!}$$

Idea of the proof for the Euler equations (2/3)

Assume that for all $j \leqslant k-1$,

$$\|D^j u\| \leqslant \frac{j!L^j}{(j+1)^2}\|u\|^{j+1}$$

Then,

$$\left|\operatorname{div} D^k u\right| \quad \leqslant \quad \left|\sum c_k(s,\alpha)\,\nabla D^{\alpha_1}u\cdot\ldots\cdot\nabla D^{\alpha_s}u\right|$$

Idea of the proof for the Euler equations (2/3)

Assume that for all $j \leqslant k-1$,

$$\|D^j u\| \leqslant \frac{j!L^j}{(j+1)^2}\|u\|^{j+1}$$

Then,

$$\begin{aligned} \left|\operatorname{div} D^k u\right| &\leqslant \left|\sum c_k(s,\alpha)\,\nabla D^{\alpha_1}u \cdot \ldots \cdot \nabla D^{\alpha_s}u\right| \\ &\leqslant k!L^k\,\|u\|^{k+1} \sum_{s=2}^{k+1} L^{1-s} \sum_{\alpha/\,|\alpha|=k+1-s} \prod_{i=1}^{s} \frac{1}{(1+\alpha_i)^2} \end{aligned}$$

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&\leqslant k! L^k \|u\|^{k+1} \sum_{s=2}^{k+1} L^{1-s} \frac{20^s}{(k-s+2)^2}
\end{aligned}$$

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&\leqslant k! L^k \|u\|^{k+1} \sum_{s=2}^{k+1} L^{1-s} \frac{20^s}{(k-s+2)^2} \\
&\leqslant \gamma(L) \frac{k! L^k}{(k+1)^2} \|u\|^{k+1},
\end{aligned}$$

with

$$\gamma(L) \to 0 \quad \text{as} \quad L \to \infty.$$

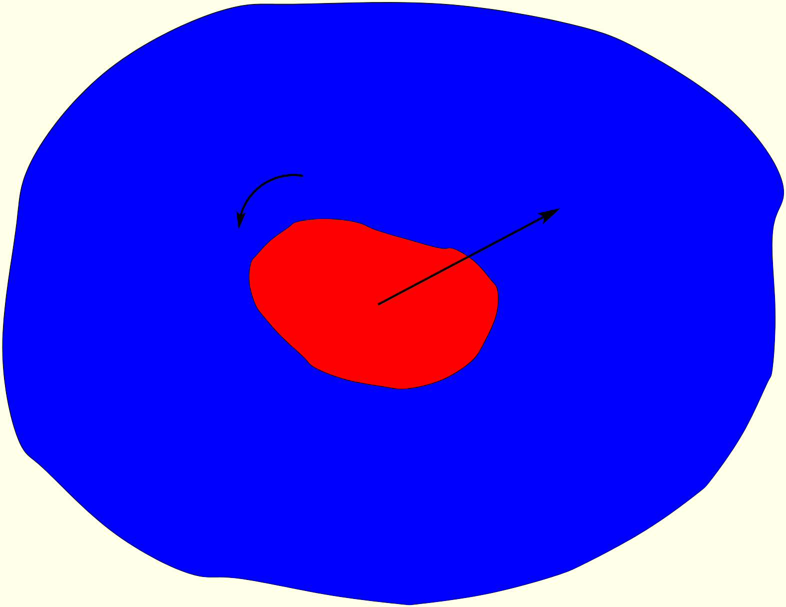
Idea of the proof for the Euler equations (3/3)

Proceeding similarly for

$$\operatorname{curl} D^k u, \quad n \cdot D^k u, \quad \Pi D^k u,$$

and using the Regularity Lemma, we get that for L big enough

$$\|D^k u\| + |\nabla D^{k-1} p| \leqslant \frac{k! L^k}{(k+1)^2} \|u\|^{k+1}.$$



Description of a fluid-structure system

Position of the center of mass of the structure: x_B

Orientation of the structure: Q

$$\mathcal{S}(t) = Q(t)\mathcal{S}(0) + x_B(t).$$

$$x = Q(t)y + x_B(t),$$

Velocity of the rigid body: v

$$v(t,x) = r(t) \wedge (x - x_B(t)) + \ell(t),$$

$$\ell = x_B', \quad Q'y = r \times Qy.$$

Description of a fluid-structure system

Mass of the structure: m

Moment of inertia tensor of the structure: $\mathcal{J}$

Newton's Law:

$$\begin{aligned} m\ell'(t) &= F_{fluid} = \int_{\partial\mathcal{S}(t)} p n \, d\Gamma, \\ (\mathcal{J}r)'(t) &= \tau_{fluid} = \int_{\partial\mathcal{S}(t)} (x - x_B) \wedge p n \, d\Gamma. \end{aligned}$$

$$\mathcal{F}(t) = \Omega \setminus \mathcal{S}(t).$$

Description of a fluid-structure system

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u + \nabla p = 0, \quad x \in \mathcal{F}(t),\ t \in (-T,T), \tag{8}$$

$$\operatorname{div} u = 0 \quad x \in \mathcal{F}(t),\ t \in (-T,T), \tag{9}$$

$$u \cdot n = 0 \quad x \in \partial\Omega,\ t \in (-T,T), \tag{10}$$

$$u \cdot n = v \cdot n \quad x \in \partial\mathcal{S}(t),\ t \in (-T,T), \tag{11}$$

$$m\ell'(t) = \int_{\partial\mathcal{S}(t)} pn \, d\Gamma \quad t \in (-T,T), \tag{12}$$

$$(\mathcal{J}r)'(t) = \int_{\partial\mathcal{S}(t)} (x - x_B) \wedge pn \, d\Gamma \quad t \in (-T,T). \tag{13}$$

Well-posedness studies on the previous system

- D'Alembert, Kelvin, Kirchhoff ($\sim$ 1870, potential case)
- Ortega, Rosier, Takahashi
- Rosier, Rosier
- Houot, Munnier (potential case)
- Houot, Tucsnak

The well-posedness result

Theorem (Houot–Tucsnak)

Assume u_0 *in* $C^{\lambda+1,r}(\mathcal{F}_0)$ *with* λ *in* $\mathbb{N}$ *and* $r \in (0,1)$ *satisfying* $u_0 \cdot n = 0$, *for* $x \in \partial\Omega$ *and* $(u_0 - v_0) \cdot n = 0$ *for* $x \in \partial\mathcal{S}_0$, *with* $v_0 := \ell_0 + r_0 \wedge (x - x_0)$.
Then there exists $T > 0$ *such that the problem (8)–(13) admits a unique solution*

$$(x_B, r, u) \in C^2([-T,T]) \times C^1([-T,T]) \times L^\infty((-T,T), C^{\lambda+1,r}(\mathcal{F}(t))).$$

Moreover $u \in C_w([-T,T]; C^{\lambda+1,r}(\mathcal{F}(t)))$ *and* $u \in C([-T,T]; C^{\lambda+1,r'}(\mathcal{F}(t)))$, *for* $r' \in (0,r)$.

Analyticity of the trajectories

Theorem

Let $\lambda \in \mathbb{N}$. Assume that the boundaries $\partial\Omega$ and $\partial\mathcal{S}_0$ are analytic and that the assumptions of the previous theorem are satisfied. Then the flows $\Phi^{\mathcal{F}}$ and $\Phi^{\mathcal{S}}$ are analytic from $(-T, T)$ to $C^{\lambda+1,r}(\mathcal{F}_0)$ and $C^{\lambda+1,r}(\mathcal{S}_0)$.

Difficulties

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = 0, \quad x \in \mathcal{F}(t),\ t \in (-T,T), \tag{14}$$

$$\operatorname{div} u = 0 \quad x \in \mathcal{F}(t),\ t \in (-T,T), \tag{15}$$

$$u \cdot n = 0 \quad x \in \partial\Omega,\ t \in (-T,T), \tag{16}$$

$$u \cdot n = v \cdot n \quad x \in \partial\mathcal{S}(t),\ t \in (-T,T), \tag{17}$$

$$m\ell'(t) = \int_{\partial\mathcal{S}(t)} pn\ d\Gamma \quad t \in (-T,T), \tag{18}$$

$$(\mathcal{J}r)'(t) = \int_{\partial\mathcal{S}(t)} (x - x_B) \wedge pn\ d\Gamma \quad t \in (-T,T). \tag{19}$$

Tool 1*: Regularity Lemma in moving domain

Lemma

Assume that $\partial\mathcal{F}(0)$ is homeomorph to the 2D sphere. $\exists c, C > 0$ *such that for any $C^{\lambda+1,r}$-diffeomorphism $\eta : \mathcal{F}(0) \to \mathcal{G} := \eta(\mathcal{F}(0))$ satisfying*

$$\|\eta - Id\|_{C^{\lambda+1,r}} < c, \tag{20}$$

one has the following: if

$$\operatorname{div} u \in C^{\lambda,r}(\mathcal{G}), \quad \operatorname{curl} u \in C^{\lambda,r}(\mathcal{G}), \quad u \cdot n \in C^{\lambda+1,r}(\partial\mathcal{G}).$$

then $u \in C^{\lambda+1,r}(\mathcal{G})$ and

$$\|u\|_{C^{\lambda+1,r}(\mathcal{G})} \leqslant C\left(\|\operatorname{div} u\|_{C^{\lambda,r}(\mathcal{G})} + \|\operatorname{curl} u\|_{C^{\lambda,r}(\mathcal{G})} + \|u \cdot n\|_{C^{\lambda+1,r}(\partial\mathcal{G})}\right).$$

Splitting the pressure

We have

$$p = p_1((u,\ell,r),(u,\ell,r)) + p_2\left(\begin{bmatrix}\ell\\ r\end{bmatrix}'\right)$$

so that

$$\mathcal{M}(t)\begin{bmatrix}\ell\\ r\end{bmatrix}' = F((u,\ell,r),(u,\ell,r)).$$

Consequently

$$\left\|\begin{bmatrix}\ell\\ r\end{bmatrix}'\right\| \leqslant C\left(\|r\| + \|\ell\| + \|u\|\right)^2.$$

The Fluid

$$\begin{aligned}
\operatorname{div} Du &= \operatorname{tr}\{\nabla u \cdot \nabla u\} \quad \text{in } \mathcal{F}(t), \\
\operatorname{curl} Du &= -\operatorname{curl} \nabla p = 0 \quad \text{in } \mathcal{F}(t), \\
n \cdot Du &= -\nabla^2 \rho\{u, u\} \quad \text{on } \partial\Omega, \\
n \cdot Du &= n \cdot Dv - \nabla^2 \rho\{u - v, u - v\} \\
&\quad + n \cdot (r \wedge (u - v)) \quad \text{on } \partial\mathcal{S}(t),
\end{aligned}$$

where

$$v(t, x) = r(t) \wedge (x - x_B(t)) + \ell(t)$$

First iteration

We have

$$Dv(t) = r'(t) \wedge (x - x_B(t)) + r(t) \wedge (u(t) - \ell(t)) + \ell'(t).$$

Using the Regularity Lemma (Tool 1*):

$$\|Du\| \leqslant C \left(\|r\| + \|\ell\| + \|u\|\right)^2 + C \left(\|r'\| + \|\ell'\|\right),$$

and from the first step,

$$\|Du\| \leqslant C \left(\|r\| + \|\ell\| + \|u\|\right)^2.$$

Induction

We have proved

$$\|Du\| + \|\ell'\| + \|r'\| \leqslant C\left(\|r\| + \|\ell\| + \|u\|\right)^2 .$$

By induction and counting again the number of terms appearing at each iteration, we obtain for all k

$$\|D^k u\| + \|\ell^{(k)}\| + \|r^{(k)}\| \leqslant \frac{k!L^k}{(k+1)^2}\left(\|r\| + \|\ell\| + \|u\|\right)^{k+1} . \qquad (21)$$

Perspectives

- Is it possible to generalize the result for $\mathcal{F}(t)$ N-torus?
- What happen for non classical solutions?
- Pressure?

$$\|D^k u\| + |\nabla D^{k-1} p| \leqslant \frac{k! L^k}{(k+1)^2} \|u\|^{k+1}.$$

and $D^k u \sim \nabla D^{k-1} p \ldots$

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