Spectral stability of elliptic operators on variable domains

Pier Domenico Lamberti
based on joint work with Victor I. Burenkov
mailto:lamberti@math.unipd.it

September 4, 2009

Let Ω be an open set in $\mathbb{R}^N$ with continuous boundary.

Let Ω be an open set in $\mathbb{R}^N$ with continuous boundary. We consider the eigenvalue problem

$$-\Delta u = \lambda u, \quad \text{in} \;\; \Omega$$

Let Ω be an open set in $\mathbb{R}^N$ with continuous boundary. We consider the eigenvalue problem

$$-\Delta u = \lambda u, \quad \text{in} \;\; \Omega$$

with Dirichlet boundary conditions

$$u = 0, \quad \text{on} \;\; \partial\Omega$$

Let Ω be an open set in $\mathbb{R}^N$ with continuous boundary. We consider the eigenvalue problem

$$-\Delta u = \lambda u, \quad \text{in} \;\; \Omega$$

with Dirichlet boundary conditions

$$u = 0, \quad \text{on} \;\; \partial\Omega$$

or Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = 0, \quad \text{on} \;\; \partial\Omega.$$

Let Ω be an open set in $\mathbb{R}^N$ with continuous boundary. We consider the eigenvalue problem

$$-\Delta u = \lambda u, \quad \text{in} \;\; \Omega$$

with Dirichlet boundary conditions

$$u = 0, \quad \text{on} \;\; \partial\Omega$$

or Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = 0, \quad \text{on} \;\; \partial\Omega.$$

(These problems have to be interpreted in the weak sense as usual)

The spectrum is discrete

$$\lambda_1[\Omega] \leq \lambda_2[\Omega] \leq \cdots \leq \lambda_n[\Omega] \leq \dots$$

The spectrum is discrete

$$\lambda_1[\Omega] \leq \lambda_2[\Omega] \leq \cdots \leq \lambda_n[\Omega] \leq \dots$$

Consider the function

$$\Omega \mapsto \lambda_n[\Omega]$$

The spectrum is discrete

$$\lambda_1[\Omega] \leq \lambda_2[\Omega] \leq \cdots \leq \lambda_n[\Omega] \leq \ldots$$

Consider the function

$$\Omega \mapsto \lambda_n[\Omega]$$

Can we estimate $|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]|$?

The spectrum is discrete

$$\lambda_1[\Omega] \le \lambda_2[\Omega] \le \cdots \le \lambda_n[\Omega] \le \dots$$

Consider the function

$$\Omega \mapsto \lambda_n[\Omega]$$

Can we estimate $|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]|$?

Yes ...

The spectrum is discrete

$$\lambda_1[\Omega] \leq \lambda_2[\Omega] \leq \cdots \leq \lambda_n[\Omega] \leq \dots$$

Consider the function

$$\Omega \mapsto \lambda_n[\Omega]$$

Can we estimate $|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]|$?

Yes ... if $\Omega_1, \ \Omega_2$ belong to suitable classes of open sets

Open sets with continuous boundaries

Open sets with continuous boundaries

We consider open sets $\Omega \in C(\mathcal{A})$

Open sets with continuous boundaries

We consider open sets $\Omega \in C(\mathcal{A})$, where $\mathcal{A}$ is a fixed atlas

Open sets with continuous boundaries

We consider open sets $\Omega \in C(\mathcal{A})$, where $\mathcal{A}$ is a fixed atlas

$$\mathcal{A} = (\rho, s, s', \{V_j\}_{j=1}^{s}, \{r_j\}_{j=1}^{s})$$

Open sets with continuous boundaries

We consider open sets $\Omega \in C(\mathcal{A})$, where $\mathcal{A}$ is a fixed atlas

$$\mathcal{A} = (\rho, s, s', \{V_j\}_{j=1}^{s}, \{r_j\}_{j=1}^{s})$$

- V_j are cuboids

Open sets with continuous boundaries

We consider open sets $\Omega \in C(\mathcal{A})$, where $\mathcal{A}$ is a fixed atlas

$$\mathcal{A} = (\rho, s, s', \{V_j\}_{j=1}^{s}, \{r_j\}_{j=1}^{s})$$

- V_j are cuboids
- r_j are rotations

Open sets with continuous boundaries

We consider open sets $\Omega \in C(\mathcal{A})$, where $\mathcal{A}$ is a fixed atlas

$$\mathcal{A} = (\rho, s, s', \{V_j\}_{j=1}^{s}, \{r_j\}_{j=1}^{s})$$

- V_j are cuboids
- r_j are rotations
- $\rho > 0$, $s, s' \in \mathbb{N}$ other parameters

Open sets with continuous boundaries

We consider open sets $\Omega \in C(\mathcal{A})$, where $\mathcal{A}$ is a fixed atlas

$$\mathcal{A} = (\rho, s, s', \{V_j\}_{j=1}^{s}, \{r_j\}_{j=1}^{s})$$

- V_j are cuboids
- r_j are rotations
- $\rho > 0$, $s, s' \in \mathbb{N}$ other parameters

$\Omega \in C(\mathcal{A})$ means that $\Omega \cap V_j$ is the subgraph of a continuous function g_j

Open sets with continuous boundaries

We consider open sets $\Omega \in C(\mathcal{A})$, where $\mathcal{A}$ is a fixed atlas

$$\mathcal{A} = (\rho, s, s', \{V_j\}_{j=1}^{s}, \{r_j\}_{j=1}^{s})$$

- V_j are cuboids
- r_j are rotations
- $\rho > 0$, $s, s' \in \mathbb{N}$ other parameters

$\Omega \in C(\mathcal{A})$ means that $\Omega \cap V_j$ is the subgraph of a continuous function g_j

Open sets with continuous boundaries

We consider open sets $\Omega \in C(\mathcal{A})$, where $\mathcal{A}$ is a fixed atlas

$$\mathcal{A} = (\rho, s, s', \{V_j\}_{j=1}^{s}, \{r_j\}_{j=1}^{s})$$

- V_j are cuboids
- r_j are rotations
- $\rho > 0$, $s, s' \in \mathbb{N}$ other parameters

$\Omega \in C(\mathcal{A})$ means that $\Omega \cap V_j$ is the subgraph of a continuous function g_j

Similarly: $C^1(\mathcal{A})$

Open sets with continuous boundaries

We consider open sets $\Omega \in C(\mathcal{A})$, where $\mathcal{A}$ is a fixed atlas

$$\mathcal{A} = (\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s)$$

- V_j are cuboids
- r_j are rotations
- $\rho > 0$, $s, s' \in \mathbb{N}$ other parameters

$\Omega \in C(\mathcal{A})$ means that $\Omega \cap V_j$ is the subgraph of a continuous function g_j

Similarly: $C^1(\mathcal{A})$, $C^2(\mathcal{A})$

Open sets with continuous boundaries

We consider open sets $\Omega \in C(\mathcal{A})$, where $\mathcal{A}$ is a fixed atlas

$$\mathcal{A} = (\rho, s, s', \{V_j\}_{j=1}^{s}, \{r_j\}_{j=1}^{s})$$

- V_j are cuboids
- r_j are rotations
- $\rho > 0$, $s, s' \in \mathbb{N}$ other parameters

$\Omega \in C(\mathcal{A})$ means that $\Omega \cap V_j$ is the subgraph of a continuous function g_j

Similarly: $C^1(\mathcal{A})$, $C^2(\mathcal{A})$, $C^{0,1}(\mathcal{A})$

Open sets with continuous boundaries

We consider open sets $\Omega \in C(\mathcal{A})$, where $\mathcal{A}$ is a fixed atlas

$$\mathcal{A} = (\rho, s, s', \{V_j\}_{j=1}^{s}, \{r_j\}_{j=1}^{s})$$

- V_j are cuboids
- r_j are rotations
- $\rho > 0$, $s, s' \in \mathbb{N}$ other parameters

$\Omega \in C(\mathcal{A})$ means that $\Omega \cap V_j$ is the subgraph of a continuous function g_j

Similarly: $C^1(\mathcal{A})$, $C^2(\mathcal{A})$, $C^{0,1}(\mathcal{A})$, $C^{0,\alpha}(\mathcal{A})$

Open sets with continuous boundaries

We consider open sets $\Omega \in C(\mathcal{A})$, where $\mathcal{A}$ is a fixed atlas

$$\mathcal{A} = (\rho, s, s', \{V_j\}_{j=1}^{s}, \{r_j\}_{j=1}^{s})$$

- V_j are cuboids
- r_j are rotations
- $\rho > 0$, $s, s' \in \mathbb{N}$ other parameters

$\Omega \in C(\mathcal{A})$ means that $\Omega \cap V_j$ is the subgraph of a continuous function g_j

Similarly: $C^1(\mathcal{A})$, $C^2(\mathcal{A})$, $C^{0,1}(\mathcal{A})$, $C^{0,\alpha}(\mathcal{A})$ etc.....

Open sets with continuous boundaries

We consider open sets $\Omega \in C(\mathcal{A})$, where $\mathcal{A}$ is a fixed atlas

$$\mathcal{A} = (\rho, s, s', \{V_j\}_{j=1}^{s}, \{r_j\}_{j=1}^{s})$$

- V_j are cuboids
- r_j are rotations
- $\rho > 0$, $s, s' \in \mathbb{N}$ other parameters

$\Omega \in C(\mathcal{A})$ means that $\Omega \cap V_j$ is the subgraph of a continuous function g_j

Similarly: $C^1(\mathcal{A})$, $C^2(\mathcal{A})$, $C^{0,1}(\mathcal{A})$, $C^{0,\alpha}(\mathcal{A})$ etc..... and, if the

norms are bounded by $M > 0$,

Open sets with continuous boundaries

We consider open sets $\Omega \in C(\mathcal{A})$, where $\mathcal{A}$ is a fixed atlas

$$\mathcal{A} = (\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s)$$

- V_j are cuboids
- r_j are rotations
- $\rho > 0$, $s, s' \in \mathbb{N}$ other parameters

$\Omega \in C(\mathcal{A})$ means that $\Omega \cap V_j$ is the subgraph of a continuous function g_j

Similarly: $C^1(\mathcal{A})$, $C^2(\mathcal{A})$, $C^{0,1}(\mathcal{A})$, $C^{0,\alpha}(\mathcal{A})$ etc..... and, if the norms are bounded by $M > 0$, $C^1_M(\mathcal{A})$

Open sets with continuous boundaries

We consider open sets $\Omega \in C(\mathcal{A})$, where $\mathcal{A}$ is a fixed atlas

$$\mathcal{A} = (\rho, s, s', \{V_j\}_{j=1}^{s}, \{r_j\}_{j=1}^{s})$$

- V_j are cuboids
- r_j are rotations
- $\rho > 0$, $s, s' \in \mathbb{N}$ other parameters

$\Omega \in C(\mathcal{A})$ means that $\Omega \cap V_j$ is the subgraph of a continuous function g_j

Similarly: $C^1(\mathcal{A})$, $C^2(\mathcal{A})$, $C^{0,1}(\mathcal{A})$, $C^{0,\alpha}(\mathcal{A})$ etc..... and, if the norms are bounded by $M > 0$, $C^1_M(\mathcal{A})$, $C^2_M(\mathcal{A})$

Open sets with continuous boundaries

We consider open sets $\Omega \in C(\mathcal{A})$, where $\mathcal{A}$ is a fixed atlas

$$\mathcal{A} = (\rho, s, s', \{V_j\}_{j=1}^{s}, \{r_j\}_{j=1}^{s})$$

- V_j are cuboids
- r_j are rotations
- $\rho > 0$, $s, s' \in \mathbb{N}$ other parameters

$\Omega \in C(\mathcal{A})$ means that $\Omega \cap V_j$ is the subgraph of a continuous function g_j

Similarly: $C^1(\mathcal{A})$, $C^2(\mathcal{A})$, $C^{0,1}(\mathcal{A})$, $C^{0,\alpha}(\mathcal{A})$ etc..... and, if the norms are bounded by $M > 0$, $C^1_M(\mathcal{A})$, $C^2_M(\mathcal{A})$, $C^{0,1}_M(\mathcal{A})$

Open sets with continuous boundaries

We consider open sets $\Omega \in C(\mathcal{A})$, where $\mathcal{A}$ is a fixed atlas

$$\mathcal{A} = (\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s)$$

- V_j are cuboids
- r_j are rotations
- $\rho > 0$, $s, s' \in \mathbb{N}$ other parameters

$\Omega \in C(\mathcal{A})$ means that $\Omega \cap V_j$ is the subgraph of a continuous function g_j

Similarly: $C^1(\mathcal{A})$, $C^2(\mathcal{A})$, $C^{0,1}(\mathcal{A})$, $C^{0,\alpha}(\mathcal{A})$ etc..... and, if the norms are bounded by $M > 0$, $C^1_M(\mathcal{A})$, $C^2_M(\mathcal{A})$, $C^{0,1}_M(\mathcal{A})$, $C^{0,\alpha}_M(\mathcal{A})$

Estimates via Lebesgue measure

Theorem (Dirichlet boundary conditions)

Let $\Omega_1 \in C_M^{0,1}(\mathcal{A})$. *Assume that there exists* $2 < p \leq \infty$ *such that*

$$\left\|\nabla \varphi_n[\Omega_1]\right\|_{L^p(\Omega_1)} < \infty \ \forall \ n \in \mathbb{N}.$$

Then for each $n \in \mathbb{N}$ *there exists* $c_n > 0$ *such that*

$$|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]| \leq c_n |\Omega_1 \setminus \Omega_2|^{1-\frac{2}{p}}$$

for all $\Omega_2 \in C_M^{0,1}(\mathcal{A})$ *with* $\Omega_2 \subset \Omega_1$ *and* $|\Omega_1 \setminus \Omega_2| < c_n^{-1}$.

Estimates via Lebesgue measure

Theorem (Dirichlet boundary conditions)

Let $\Omega_1 \in C^{0,1}_M(\mathcal{A})$. *Assume that there exists* $2 < p \leq \infty$ *such that*

$$\left\|\nabla \varphi_n[\Omega_1]\right\|_{L^p(\Omega_1)} < \infty \ \forall \ n \in \mathbb{N}.$$

Then for each $n \in \mathbb{N}$ *there exists* $c_n > 0$ *such that*

$$|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]| \leq c_n |\Omega_1 \setminus \Omega_2|^{1-\frac{2}{p}}$$

for all $\Omega_2 \in C^{0,1}_M(\mathcal{A})$ *with* $\Omega_2 \subset \Omega_1$ *and* $|\Omega_1 \setminus \Omega_2| < c_n^{-1}$.

THE EXPONENT IS SHARP

Estimates via atlas distance

In $C(\mathcal{A})$ we have a distance:

Estimates via atlas distance

In $C(\mathcal{A})$ we have a distance:

$$d(\Omega_1, \Omega_2) = \max_{j=1,\dots,s} \|g_{j1} - g_{j2}\|_\infty .$$

Estimates via atlas distance

In $C(\mathcal{A})$ we have a distance:

$$d(\Omega_1, \Omega_2) = \max_{j=1,\dots,s} \|g_{j1} - g_{j2}\|_\infty .$$

We call it *Atlas distance*.

Estimates via atlas distance

In $C(\mathcal{A})$ we have a distance:

$$d(\Omega_1, \Omega_2) = \max_{j=1,\dots,s} \|g_{j1} - g_{j2}\|_\infty .$$

We call it *Atlas distance*.

Estimates via atlas distance

In $C(\mathcal{A})$ we have a distance:

$$d(\Omega_1, \Omega_2) = \max_{j=1,\dots,s} \|g_{j1} - g_{j2}\|_\infty .$$

We call it *Atlas distance*.

Theorem

Let $\mathcal{A}$ be fixed. Then for each $n \in \mathbb{N}$ there exists $c_n > 0$ such that

$$|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]| \leq c_n d_{\mathcal{A}}(\Omega_1, \Omega_2),$$

for all $\Omega_1, \Omega_2 \in C(\mathcal{A})$ such that $d_{\mathcal{A}}(\Omega_1, \Omega_2) < c_n^{-1}$.

Estimates via Hausdorff distance

Estimates via Hausdorff distance

Let $A, B \subset \mathbb{R}^N$.

Estimates via Hausdorff distance

Let $A, B \subset \mathbb{R}^N$.

The Hausdorff distance of A and B is

$$d^{\mathcal{H}}(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{x\in B} d(x, A)\right\}.$$

Estimates via Hausdorff distance

Let $A, B \subset \mathbb{R}^N$.

The Hausdorff distance of A and B is

$$d^{\mathcal{H}}(A,B) = \max\left\{\sup_{x\in A} d(x,B),\ \sup_{x\in B} d(x,A)\right\}.$$

The lower Hausdorff deviation of A and B is

$$d_{\mathcal{H}}(A,B) = \min\left\{\sup_{x\in A} d(x,B),\ \sup_{x\in B} d(x,A)\right\}.$$

Estimates via Hausdorff distance

We consider $\Omega \in C^{\omega}_{M}(\mathcal{A})$

Estimates via Hausdorff distance

We consider $\Omega \in C^{\omega}_{M}(\mathcal{A})$ which means that

Estimates via Hausdorff distance

We consider $\Omega \in C_M^{\omega}(\mathcal{A})$ which means that

$$|g_j(x) - g_j(y)| \leq M\omega(|x - y|).$$

Estimates via Hausdorff distance

We consider $\Omega \in C^{\omega}_{M}(\mathcal{A})$ which means that

$$|g_j(x) - g_j(y)| \leq M\omega(|x-y|).$$

Lemma

There exists $K > 0$ such that

$$d_{\mathcal{A}}(\Omega_1, \Omega_2) \leq K\omega(d_{\mathcal{H}}(\partial\Omega_1, \partial\Omega_2)),$$

for all $\Omega_1, \Omega_2 \in C^{\omega}_{M}(\mathcal{A})$.

Estimates via Hausdorff distance

We consider $\Omega \in C^{\omega}_{M}(\mathcal{A})$ which means that

$$|g_j(x) - g_j(y)| \le M\omega(|x-y|).$$

Lemma

There exists $K > 0$ such that

$$d_{\mathcal{A}}(\Omega_1, \Omega_2) \le K\omega(d_{\mathcal{H}}(\partial\Omega_1, \partial\Omega_2)),$$

for all $\Omega_1, \Omega_2 \in C^{\omega}_{M}(\mathcal{A})$.

(The precise statement can be found in the paper [1].)

Estimates via Hausdorff distance

Corollary

Let $\mathcal{A}$, ω, M be fixed. Then for each $n \in \mathbb{N}$ there exists $c_n > 0$ such that

$$|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]| \leq c_n \omega(d_{\mathcal{H}}(\partial\Omega_1, \partial\Omega_2)),$$

for all $\Omega_1, \Omega_2 \in C^{\omega}_{M}(\mathcal{A})$ such that $d_{\mathcal{H}}(\partial\Omega_1, \partial\Omega_2) < c_n^{-1}$.

Using ϵ-neighborhoods

If $\Omega_1, \Omega_2 \in C^{\omega}_{M}(\mathcal{A})$ satisfy

Using ϵ-neighborhoods

If $\Omega_1, \Omega_2 \in C^{\omega}_{M}(\mathcal{A})$ satisfy

$$(\Omega_1)_\epsilon \subset \Omega_2 \subset (\Omega_1)^\epsilon,$$

Using ϵ-neighborhoods

If $\Omega_1, \Omega_2 \in C^\omega_M(\mathcal{A})$ satisfy

$$(\Omega_1)_\epsilon \subset \Omega_2 \subset (\Omega_1)^\epsilon,$$

where

$$\begin{aligned} (\Omega_1)_\epsilon &= \{x \in \Omega_1 : \ d(x, \partial\Omega_1) > \epsilon\} \\ (\Omega_1)^\epsilon &= \{x \in \mathbb{R}^N : \ d(x, \Omega_1) < \epsilon\}, \end{aligned}$$

Using ϵ-neighborhoods

If $\Omega_1, \Omega_2 \in C^{\omega}_{M}(\mathcal{A})$ satisfy

$$(\Omega_1)_\epsilon \subset \Omega_2 \subset (\Omega_1)^\epsilon,$$

where

$$\begin{aligned}(\Omega_1)_\epsilon &= \{x \in \Omega_1 : \ d(x, \partial\Omega_1) > \epsilon\} \\ (\Omega_1)^\epsilon &= \{x \in \mathbb{R}^N : \ d(x, \Omega_1) < \epsilon\},\end{aligned}$$

our estimates can be rewritten as

Using ϵ-neighborhoods

If $\Omega_1, \Omega_2 \in C^{\omega}_{M}(\mathcal{A})$ satisfy

$$(\Omega_1)_\epsilon \subset \Omega_2 \subset (\Omega_1)^\epsilon,$$

where

$$\begin{aligned} (\Omega_1)_\epsilon &= \{x \in \Omega_1 : \ d(x, \partial\Omega_1) > \epsilon\} \\ (\Omega_1)^\epsilon &= \{x \in \mathbb{R}^N : \ d(x, \Omega_1) < \epsilon\}, \end{aligned}$$

our estimates can be rewritten as

$$|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]| \leq c_n \omega(\epsilon).$$

Using ϵ-neighborhoods

If $\Omega_1, \Omega_2 \in C^{\omega}_M(\mathcal{A})$ satisfy

$$(\Omega_1)_\epsilon \subset \Omega_2 \subset (\Omega_1)^\epsilon,$$

where

$$\begin{aligned} (\Omega_1)_\epsilon &= \{x \in \Omega_1 : \ d(x, \partial\Omega_1) > \epsilon\} \\ (\Omega_1)^\epsilon &= \{x \in \mathbb{R}^N : \ d(x, \Omega_1) < \epsilon\}, \end{aligned}$$

our estimates can be rewritten as

$$|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]| \leq c_n \omega(\epsilon).$$

Remark

$(C(\mathcal{A}), d_{\mathcal{A}})$ is a complete metric space

Remark

$(C(\mathcal{A}), d_{\mathcal{A}})$ is a complete metric space

$C^{\omega}_{M}(\mathcal{A})$ is compact in $C(\mathcal{A})$

Remark

$(C(\mathcal{A}), d_{\mathcal{A}})$ is a complete metric space

$C^{\omega}_{M}(\mathcal{A})$ is compact in $C(\mathcal{A})$

$\lambda_n : C(\mathcal{A}) \to \mathbb{R}$ is locally Lipschitz continuous

Remark

$(C(\mathcal{A}), d_{\mathcal{A}})$ is a complete metric space

$C^{\omega}_{M}(\mathcal{A})$ is compact in $C(\mathcal{A})$

$\lambda_n : C(\mathcal{A}) \to \mathbb{R}$ is locally Lipschitz continuous

hence λ_n has maximum and minimum in $C^{\omega}_{M}(\mathcal{A})$

Almost all results presented in this talk can be found in the following papers:

[1] V.I. Burenkov, P.D. Lamberti, Spectral stability of higher order uniformly elliptic operators, in Sobolev spaces in mathematics. **II**, 69–102, *Int. Math. Ser. (N. Y.)*, **9**, Springer, New York, 2009.

[2] V.I. Burenkov, P.D. Lamberti, Spectral stability of Dirichlet second order uniformly elliptic operators, *J. Differential Equations*, **244**, pp. 1712-1740, 2008.

[3] V.I. Burenkov, P.D. Lamberti, Spectral stability of general non-negative self-adjoint operators with applications to Neumann-type operators, *J. Differential Equations*, **233**, 345-379, 2007.