Generic properties of the Laplace-Dirichlet and Schrödinger operators, with applications to quantum control

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Partial differential equations, optimal design and numerics
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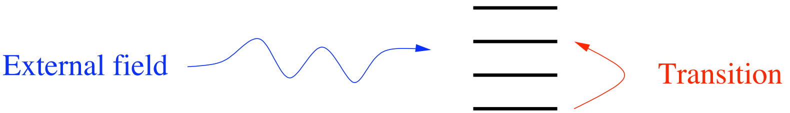
Main motivation: quantum control

Many technologies require the ability to induce a transition from a state to another of a quantum mechanical system.

- **Photochemistry** (to induce certain chemical reactions)
- **Magnetic Resonance** (in order to exploit spontaneous emission)
- **Realization of Quantum Computers** (to stock information)



Population transfer Problem: Design external fields

- Lasers
- X-Rays
- Magnetic Fields

to drive a quantum mechanical system from one state to another

Bilinear Schrödinger equation

$$i\dot{\psi} = -\Delta\psi + V\psi + uW\psi$$

$\Omega \subset \mathbf{R}^d$
$\psi(t,x)$ wave function, $\psi(t,\cdot) \in L^2(\Omega)$, $\|\psi(t,\cdot)\|_2 = 1$
$-\Delta + V$ Schrödinger operator
$V : \Omega \to \mathbf{R}$ uncontrolled potential
$u = u(t)$ real-valued control
$W : \Omega \to \mathbf{R}$ controlled potential

Most relevant cases:

- $\Omega \subset \mathbf{R}^d$ bounded domain and $\psi|_{\partial\Omega} \equiv 0$
- Ω compact connected manifold, Δ Laplace-Beltrami operator
- $\Omega = \mathbf{R}^d$

Controllability results

Negative results

- non-exact controllability in the unit sphere of $L^2(\Omega)$ (Turinici [2000]);
- non-controllability for the harmonic oscillator: $\Omega = \mathbf{R}$, $V(x) = x^2$, $W(x) = x$ (Mirrahimi-Rouchon [2004]).

Positive results

- exact controllability in $H^{5+\varepsilon}(\Omega)$ with $\Omega = (-1/2, 1/2)$, $V = 0$, $W(x) = x$ (Beauchard [2005], Beauchard-Coron [2006]);
- L^2-approximate controllability by Lyapunov methods (Mirrahimi [2006], Nersesyan [2009], Ito-Kunisch [2009]);
- L^2-approximate controllability by finite-dimensional techniques (Chambrion-Mason-S-Boscain [2009]).

More than one control (Eberly–Law-like systems): Adami-Boscain [2005], Bloch-Brockett-Rangan [2006], Ervedoza-Puel [2009].

Discrete spectrum

If Ω is a bounded domain or a compact manifold and $V \in L^\infty(\Omega)$, then $-\Delta + V$ has discrete spectrum.

Theorem (Reed-Simon)

Let $\Omega = \mathbf{R}^d$, $V \in L^\infty_{\mathrm{loc}}(\mathbf{R}^d, \mathbf{R})$ be such that $\lim_{|x| \to \infty} V(x) = +\infty$. Then $-\Delta + V$, defined as a sum of quadratic forms, is a self-adjoint operator with compact resolvent. In particular $\sigma(-\Delta + V)$ is discrete and admits a family of eigenfunctions in $H^2(\mathbf{R}^d, \mathbf{R})$ which forms an orthonormal basis of $L^2(\mathbf{R}^d, \mathbf{C})$. For every eigenfunction ϕ of $-\Delta + V$ and every $a > 0$, $x \mapsto e^{a\|x\|}\phi(x)$ is in $L^2(\mathbf{R}^d, \mathbf{C})$.

Conditions ensuring approximate controllability for $-\Delta + V$ with discrete spectrum

$(\lambda_n(V,\Omega), \phi_n(V,\Omega))_{n\in\mathbf{N}}$ eigenpairs of $-\Delta + V$ on Ω

Theorem (U. Boscain, T. Chambrion, P. Mason, M. S.)

Let $V, W \in L^\infty_{\mathrm{loc}}(\mathbf{R}^d)$ and $U = [0,\delta]$. Assume that $\lim_{|x|\to\infty} V(x) = +\infty$ and that W has at most exponential growth at infinity. If $(\lambda_{k+1}(V,\Omega) - \lambda_k(V,\Omega))_{k\in\mathbf{N}}$ are $\mathbf{Q}$-linearly independent and if

$$\int_\Omega W\phi_k(V,\Omega)\phi_{k+1}(V,\Omega)dx \neq 0$$

for every $j \in \mathbf{N}$, then the Schrödinger equation corresponding to (Ω, V, W) is approximately controllable in $L^2(\Omega)$.

Advantages:

- Controllability extends to density matrices and tracking
- W unbounded is allowed
- bounded (arbitrarily small) control

Genericity

Genericity is a measure of frequency and robustness.

X complete metric space $\Longrightarrow$ X Baire space, ie, $\cap_{n \in \mathbf{N}} O_n$ is dense if each $O_n \subset X$ is open and dense

Residual set: intersection of countably many open and dense subsets of a Baire space

A boolean function $P : X \to \{0, 1\}$ on a Baire space X is called a generic property if there exists a residual subset Y of X such that every x in Y satisfies property P, that is, $P(x) = 1$.

The sufficient conditions for controllability are in the form of a countable family of non-vanishing relations. The idea is then to associate to each of them a set O_n.

Baire spaces and topologies

We consider the cases Ω bounded domain and $\Omega = \mathbf{R}^d$. In the first case we can consider genericity w.r.t. (Ω, V, W).

$$\Omega \;\rightarrow\; \Sigma_m = \{\Omega \mid \Omega \text{ bounded domain with } \mathcal{C}^m \text{ boundary}\}, m \in \mathbf{N}$$

$$V \;\rightarrow\; \mathcal{V}(\Omega) = \begin{cases} L^\infty(\Omega) & \Omega \in \Sigma_m \\ \{V \in L^\infty_{loc} \mid \lim_{x\to\infty} V(x) = +\infty\} & \Omega = \mathbf{R}^d \end{cases}$$

$$W \;\rightarrow\; \mathcal{W}(\Omega) = \begin{cases} L^\infty(\Omega) & \Omega \in \Sigma_m \\ \{W \in L^\infty_{loc} \mid \limsup_{x\to\infty} \frac{\log(|W(x)|+1)}{\|x\|} < \infty\} & \Omega = \mathbf{R}^d \end{cases}$$

$$(V, W) \;\rightarrow\; \mathcal{Z}(\Omega) = \{(V, W) \in \mathcal{V} \times \mathcal{W} \mid V + uW \in \mathcal{V} \quad \forall u \in U\}$$

We endow these spaces with the $\mathcal{C}^m$, L^∞ and $L^\infty \times L^\infty$ topology

Analytic propagation of non-vanishing conditions and the role of the Laplace–Dirichelet operator when Ω is bounded

If Ω and V satisfy the non-resonance condition

$$(\lambda_k(V,\Omega))_{k\in\mathbf{N}} \text{ are } \mathbf{Q}\text{-linearly independent}$$

then it is clear that generically w.r.t. W the system is approximately controllable, since every condition

$$\int_\Omega W\phi_k(V,\Omega)\phi_{k+1}(V,\Omega)dx \neq 0$$

defines an open dense subset of $\mathcal{W}$.

If the non-resonance condition is true for $\lambda_k(0,\Omega)$, then, by analytic perturbation, it is true for $\lambda_k(\mu V,\Omega)$ for a generic $\mu \in \mathbf{R}$

Similarly, if $\phi_k(0,\Omega)^2$ are linearly independent, then, thanks to

$$\frac{d}{d\mu}|_{\mu=0}\lambda_k(\mu V,\Omega) = \int_\Omega V\phi_k(0,\Omega)^2$$

generically with respect to V the sequence $\frac{d}{d\mu}|_{\mu=0}\lambda_k(\mu V,\Omega)$ is non-resonant. This would imply that generically w.r.t. μ the same is true for $\lambda_k(\mu V,\Omega)$

Generic approximate controllability

Hence, we are left to prove that, generically with respect to $\Omega \in \Sigma_m$, either $\lambda_k(0,\Omega)$ is non-resonant or $\phi_k(0,\Omega)^2$ is free.

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Theorem (Y. Privat, M. S.)

Generically with respect to $\Omega \in \Sigma_m$, $\lambda_k(0,\Omega)$ is non-resonant and $\phi_k(0,\Omega)^2$ is free.

Corollary

Generically with respect to $\{(\Omega, V, W) \mid \Omega \in \Sigma_m,\ (V,W) \in \mathcal{Z}(\Omega)\}$ the Schrödinger equation

$$i\dot{\psi} = -\Delta\psi + V\psi + uW\psi, \quad \psi|_{\partial\Omega} = 0, \quad u \in [0,\delta]$$

is approximately controllable for every $\delta > 0$.

Techniques

The openness of the sets O_n follows from standard continuity results. The hard point is their density.

Local step

Use local perturbations to get a domain Ω satisfying a desired property (eg, smooth perturbation of a rectangle to obtain a Lipschitz domain for which a prescribed linear combinations of eigenvalues does not vanish and approximate it by a $\mathcal{C}^m$ domain)

Global step

Consider an analytic path of domains starting from Ω in order to *propagate* the good property. The property will be true for all but countably many points of the path.

Tricky point of the global perturbation analysis: intersection of eigenvalues

If λ_2 and λ_3 cross λ_4 along the analytic perturbation, then the condition $\lambda_3 - \lambda_2 \neq 0$ becomes $\lambda_4 - \lambda_3 \neq 0$.

The best would be to propagate a domain satisfying all the required properties.

Two strategies to avoid the bad effect of eigenvalue rearrangement along the path:

- Intersections are meagre: the eigevalues of

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

  are double if $a = c$ and $b = 0$, two conditions on three parameters! (Von Neumann-Wigner [1929], Lupo-Micheletti [1995], Lamberti-Lanza de Cristoforis [2006]). The idea is that by small perturbation of the analytic path we avoid intersections (Arnold, Colin de Verdière, Teytel [1999])
- Limit situations (converge to an example –possibly non-admissible– that satisfies all rearranged conditions)

Generic analytic properties of $-\Delta$ for topological balls

Let $F_n : \mathbf{R}^{n(n+1)} \longrightarrow \mathbf{R}$, $n \in \mathbf{N}$, be a sequence of analytic functions. Ω satisfies property P_n if the first n eigenvalues $\lambda_1, \ldots, \lambda_n$ of the Laplace-Dirichlet operator on Ω are simple and if $\exists x_1, \ldots, x_n \in \Omega$ and a choice $\phi_1, \ldots, \phi_n$ of corresponding eigenfunctions such that

$$F_n(\phi_1(x_1), \ldots, \phi_n(x_1), \ldots, \phi_1(x_n), \ldots, \phi_n(x_n), \lambda_1, \ldots, \lambda_n) \neq 0.$$

Assume that, for every $n \in \mathbf{N}$, there exists a topological ball $\mathcal{R}_n$ with Lipschitz boundary satisfying property P_n. Then a generic $\Omega \in \Sigma_m$ that is a topological ball satisfies P_n for every $n \in \mathbf{N}$.

Key steps of the proof:

- approximation by smooth domains (L^∞ convergence of eigenfunctions – Arendt-Daners, 2007)
- analytic propagation by deformations of the domain
- non-crossing of eigenvalues: given two smooth topological balls Ω_0 and Ω_1, there exists an analytic path $\eta \mapsto \Omega_\eta$ joining them such that the first n eigenvalues of Ω_η are simple for $\eta \in (0,1)$ (Teytel, 1999)

Generic analytic properties of $-\Delta$ for richer topologies

For every $n \in \mathbf{N}$ let $J_n \subset \mathbf{N}^n$ be made of all n-uples whose entries are pairwise distinct. Given $j = (j_1, \ldots, j_n)$ in J_n, we say that Ω satisfies property $\hat{P}_j$ if $\lambda_{j_1}, \ldots, \lambda_{j_n}$ are simple and if $\exists x_1, \ldots, x_n \in \Omega$ and a choice of $\phi_1, \ldots, \phi_{j_n}$ such that

$$F_n(\phi_{j_1}(x_1), \ldots, \phi_{j_n}(x_1), \ldots, \phi_{j_1}(x_n), \ldots, \phi_{j_n}(x_n), \lambda_{j_1}, \ldots, \lambda_{j_n}) \neq 0.$$

Assume that, for every $n \in \mathbf{N}$ and $j \in J_n$, there exists a Lipschitz topological ball $\hat{R}_j$ satisfying property $\hat{P}_j$. Then a generic $\Omega \in \Sigma_m$ satisfies $\hat{P}_j$ for every $j \in \cup_{n \in \mathbf{N}} J_n$.