Transport with congestion, weak flows and degenerate elliptic PDE's

Guillaume CARLIER, joint work with Lorenzo BRASCO (Pisa and CEREMADE) and Filippo SANTAMBROGIO (CEREMADE).

PDE's, optimal design and numerics, BENASQUE 2009,.

Outline

① A continuous congestion model (joint with FS and C. Jimenez

② Minimal flow formulation

③ Regularity

④ Other formulations, numerical approximation (joint work with FS, F. Benmansour and G. Peyré)

A continuous congestion model

Discrete congested network model: $G = (N, A)$ finite oriented and connected graph, $P \subset N \times N$ (sources/dest.), $\gamma_{s,d} \geq 0$ mass to be sent from s to d, $C_{s,d}$ (nonempty) set of simple paths connecting s to d ($(s, d) \in P$) and C their union. Travelling time functions(congestion), for $a \in A$ $w \mapsto t_a(w)$ ($w \geq 0$ flow on arc a), t_a nonnegative, nondecreasing.

Cost of a path $r \in C$ given the flows $(w_a)_{a \in A}$:

$$T_w(r) := \sum_{a \in r} t_a(w_a)$$

Unknown: arc flows $(w_a)_{a \in A}$ and mass travelling on each road $(h_r)_{r \in C}$, constraints:

$$\gamma_{s,d} = \sum_{r \in C_{s,d}} h_r, \ w_a = \sum_{r \ni a} h_r, \ w_a \geq 0, h_r \geq 0. \tag{1}$$

Pbm: what is a long-term steady state or equilibrium flow-configuration?

Wardrop: used paths have to be shortest paths, given the flow configuration (similar to Nash equilibrium).

Wardrop equilibrium (1952): $(w_a)_{a\in A}$, $(h_r)_{r\in C}$ satisfying (1) such that, $\forall(s,d)\in P$, $\forall r\in C_{s,d}$, if $h_r>0$, then:

$$T_w(r)=\min\{T_w(r'),\ r'\in C_{s,d}\}$$

Beckman, McGuire, Winsten (1956) noticed that $(w_a)_{a\in A}$, $(h_r)_{r\in C}$ is a Wardrop equilibrium iff it minimizes

$$C(w):=\sum_{a\in A}\int_0^{w_a} t_a$$

subject to (1).

Continuous model: Given Ω some bounded open and connected subset of $\mathbb{R}^d$ and probability measures μ_0 and μ_1 on $\overline{\Omega}$ (or a transport plan π that is a joint probability on $\overline{\Omega} \times \overline{\Omega}$) one looks for a probability measure Q on $C([0,1], \overline{\Omega})$ concentrated on absolutely continuous curves such that

$$e_0 \sharp Q = \mu_0,\ e_1 \sharp Q = \mu_1 \text{ or } (e_0, e_1) \sharp Q = \pi,\ \text{with } e_t(\gamma) = \gamma(t)$$

that is an equilibrium i.e. (in a sense to be made precise) such that Q is supported by geodesics for a metric ξ_Q depending on Q itself (congestion).

Intensity of traffic $i_Q \in \mathcal{M}(\overline{\Omega})$, defined by

$$\int \varphi di_Q := \int_{C([0,1],\overline{\Omega})} \Big(\int_0^1 \varphi(\gamma(t))|\dot{\gamma}(t)|dt \Big) dQ(\gamma)$$

for all $\varphi \in C(\overline{\Omega}, \mathbb{R}_+)$. Congestion effect:

$$\xi_Q(x) := g(i_Q(x)), \text{ for } i_Q \ll \mathcal{L}^d \text{ } (+\infty \text{ otherwise}).$$

for a given increasing function $g : \mathbb{R}_+ \to \mathbb{R}_+$. Denote by $\mathcal{Q}(\mu_0, \mu_1)$ (resp.$\mathcal{Q}(\pi)$) the set of probabilities Q such that $(e_0 \sharp Q, e_1 \sharp Q) = (\mu_0, \mu_1)$ (resp. $(e_0, e_1)\sharp Q = \pi$).

Consider then

$$\inf_{Q \in \mathcal{Q}(\mu_0,\mu_1)} \int_\Omega H(i_Q(x))dx \tag{2}$$

where $H' = g$, $H(0) = 0$.

Under the assumptions:

- H is strictly convex and increasing on $\mathbb{R}_+$ with $H(0) = 0$,
- there exists $p > 1$, and positive constants a and b such that $az^p \leq H(z) \leq b(z^p + 1)$ for all $z \in \mathbb{R}_+$,
- the following set

$$\mathcal{Q}^p(\mu_0,\mu_1) := \{Q \in \mathcal{Q}(\mu_0,\mu_1) \ : \ i_Q \in L^p\} \tag{3}$$

  is nonempty,

then (2) has a solution (and the optimal i_Q is unique).

Not easy to check a priori that $\mathcal{Q}^p(\mu_0, \mu_1) \neq \emptyset$, but

- it holds whenever μ_0 and μ_1 are L^p (De Pascale, Pratelli),
- it holds for μ_0 and μ_1 have finite support, $d = 2$ and $p < 2$,
- also when $\overline{\Omega} = [0, 1]^2$ and μ_0 and μ_1 are respectively the one-dimensional Hausdorff measures of the vertical sides of the square.

In dimension 2, $\mathcal{Q}^2(\mu_0, \mu_1) = \emptyset$ as soon as $\mu_0 - \mu_1 \notin H^{1\prime}$.
Indeed associate to every $Q \in \mathcal{Q}(\mu_0, \mu_1)$ the vector-measure σ_Q defined by, $\forall X \in C(\overline{\Omega}, \mathbb{R}^2)$:

$$\int_{\overline{\Omega}} X(x) d\sigma_Q(x) = \int_{C([0,1],\overline{\Omega})} \left(\int_0^1 X(\gamma(t)) \cdot \dot{\gamma}(t) dt \right) dQ(\gamma).$$

It is easy to check:

$$\operatorname{div}(\sigma_Q) = \mu_0 - \mu_1, \text{ and } |\sigma_Q| \leq i_Q.$$

Hence, if $\mu_0 - \mu_1 \notin H^{1\prime}$ there is no L^2, vector-field with divergence $\mu_0 - \mu_1$.

Link with equilibria Further assume that H is differentiable with $H'(z) \leq C(1+z^{p-1})$ and $p < d/(d-1)$ i.e. $q := p' > d$. Geodesic distance : $\xi \in C(\overline{\Omega})$, $\xi \geq 0$, x, y in $\overline{\Omega}^2$:

$$c_\xi(x,y) := \inf_{\gamma\,:\,\gamma(0)=x,\gamma(1)=y} \int_0^1 \xi(\gamma(t))|\dot{\gamma}(t)|dt$$

for ξ only L^q, $\xi \geq 0$:

$$\overline{c}_\xi(x,y) = \sup\left\{c(x,y)\ :\ c \in \mathcal{A}(\xi)\right\},$$

where

$$\mathcal{A}(\xi) = \left\{\lim_n c_{\xi_n} \text{ in } C^0\ :\ (\xi_n)_n \in C^0(\overline{\Omega}),\ \xi_n \geq 0,\ \xi_n \to \xi \text{ in } L^q\right\}.$$

(well defined and Hölder continuous by the Sobolev imbeddings).

Other characterizations of $\overline{c}_\xi$:

$$\overline{c}_\xi = \lim_{\varepsilon} c_{\rho_\varepsilon \star \xi}$$

also $\overline{c}_\xi(x,.)$ is the viscosity solution (i.e. largest a.e. subsolution) of the eikonal equation

$$|\nabla u| = \xi, \ u(x) = 0.$$

For $\xi \in C(\overline{\Omega})$, $\xi \geq 0$ and γ an absolutely continuous curve, set

$$L_\xi(\gamma) := \int_0^1 \xi(\gamma(t))|\dot{\gamma}(t)|dt$$

for $Q \in \mathcal{Q}^p(\mu_0, \mu_1)$, $\xi \in L^q$, $\xi \geq 0$, and $(\xi_n)_n \geq 0$, continuous, $\xi_n \to \xi$ in L^q, then $(L_{\xi_n})_n$ converges strongly in $L^1(C, Q)$ to some limit which is independent of the approximating sequence $(\xi_n)_n$ and which will again be denoted L_ξ.

Theorem 1 *Let* $\overline{Q} \in \mathcal{Q}^p(\mu_0, \mu_1)$ *with* $\overline{Q} := \overline{p} \otimes \overline{\pi}$ *(with* $\overline{\pi} \in \Pi(\mu_0, \mu_1)$*), and set* $\overline{\xi} := H'(i_{\overline{Q}})$*, then* $\overline{Q}$ *solves (2) iff:*

1. $\overline{\pi}$ *solves the Monge-Kantorovich problem:*

$$\inf_{\pi \in \Pi(\mu_0, \mu_1)} \int_{\overline{\Omega} \times \overline{\Omega}} \overline{c}_{\overline{\xi}}(x, y) d\pi(x, y), \tag{4}$$

2. *for* $\overline{Q}$*-a.e.* γ*, one has:*

$$L_{\overline{\xi}}(\gamma) = \overline{c}_{\overline{\xi}}(\gamma(0), \gamma(1)). \tag{5}$$

The second condition is the Wardrop equilibrium condition.

Variant : the transportation plan π is prescribed, then one has a similar variational characterization by considering

$$\inf_{Q \in \mathcal{Q}(\pi)} \int_{\Omega} H(i_Q(x))dx.$$

Minimal flow formulation

For $Q \in \mathcal{Q}^p(\mu_0, \mu_1)$ define as before every the vector-measure σ_Q defined by, $\forall X \in C(\overline{\Omega}, \mathbb{R}^d)$:

$$\int_{\overline{\Omega}} X(x) d\sigma_Q(x) = \int_{C([0,1],\overline{\Omega})} \left(\int_0^1 X(\gamma(t)) \cdot \dot{\gamma}(t) dt \right) dQ(\gamma)$$

which is a kind of vectorial traffic intensity.

It is easy to check:

$$\operatorname{div}(\sigma_Q) = \mu_0 - \mu_1, \ \sigma_Q \cdot n = 0, \ \text{and } |\sigma_Q| \leq i_Q.$$

Since H is increasing, it proves that the value of the scalar problem (2) is larger than that of the minimal flow problem (setting : $\mathcal{H}(\sigma) = H(|\sigma|)$):

$$\inf_{\sigma \in L^p(\Omega, \mathbb{R}^d) \,:\, \mathrm{div}(\sigma) = \mu_0 - \mu_1} \int_\Omega \mathcal{H}(\sigma(x))dx \tag{6}$$

Conversely, if σ is a minimizer of (6) and $Q \in \mathcal{Q}^p(\mu_0, \mu_1)$ is such that $i_Q = |\sigma|$ then Q solves the scalar problem (2) (i.e. is an equilibrium).

Heuristic construction (assuming σ Lipschitz, μ_0, μ_1 Lipschitz densities $\geq c > 0$). Consider (as in Moser, Dacorogna-Moser and more recently Evans and Gangbo) the ODE

$$\dot{X}(t,x) = \frac{\sigma(X(t,x))}{(1-t)\mu_0(X(t,x)) + t\mu_1(X(t,x))}, \; X(0,x) = x.$$

and define $\overline{Q}$ by

$$\overline{Q} = \; \delta_{X(.,x)} \otimes \mu_0$$

Set $\mu_t = (1-t)\mu_0 + t\mu_1$ and

$$v(t,x) = \frac{\sigma(x)}{\mu_t(x)}$$

then by construction μ_t solves the continuity equation:

$$\partial_t \mu_t + \mathrm{div}(\mu_t v) = 0$$

By construction $e_0 \sharp \overline{Q} = \mu_0$ and because of the continuity equation, $X(t,.) \sharp \mu_0 = \mu_t = (1-t)\mu_0 + t\mu_1$. In particular the image of μ_0 by the flow at time 1, $X(1,.)$ is μ_1, which proves that $e_1 \sharp \overline{Q} = \mu_1$ hence $\overline{Q} \in \mathcal{Q}(\mu_0, \mu_1)$. Moreover for every test-function φ:

$$\begin{aligned}
\int_\Omega \varphi \, di_{\overline{Q}} &= \int_\Omega \int_0^1 \varphi(X(t,x))|v(t,X(t,x))| dt d\mu_0(x) \\
&= \int_0^1 \int_\Omega \varphi(x)|v(t,x)|\mu_t(x) dx dt \\
&= \int_\Omega \varphi(x)|\sigma(x)| dx
\end{aligned}$$

so that $i_{\overline{Q}} = |\sigma|$ and then $\overline{Q}$ is optimal.

The previous argument works as soon as $\sigma \in W^{1,\infty}$. By duality, the solution of (6) is $\sigma = \nabla \mathcal{H}^*(\nabla u)$ where $\mathcal{H}^*$ is the Legendre transform of $\mathcal{H}$ and u solves the PDE:

$$\begin{cases} \mathrm{div} \nabla \mathcal{H}^*(\nabla u) = \mu_0 - \mu_1, & \text{in } \Omega, \\ \nabla \mathcal{H}^*(\nabla u) \cdot \nu = 0, & \text{on } \partial\Omega, \end{cases} \tag{7}$$

Let us recall that $H' = g$ where g is the congestion function, natural to have $g(0) > 0$: the metric is positive even if there is no traffic, so that the radial function $\mathcal{H}$ is not differentiable at 0 and then its subdifferential at 0 contains a ball. By duality, this implies $\nabla \mathcal{H}^* = 0$ on this ball which makes (7) very degenerate. A reasonable model of congestion is $g(t) = \lambda + t^{p-1}$ for $t \geq 0$, with $p > 1$ and $\lambda > 0$, so that

$$\mathcal{H}(\sigma) = \frac{1}{p}|\sigma|^p + \lambda|\sigma|, \; \mathcal{H}^*(z) = \frac{1}{q}(|z| - \lambda)_+^q, \text{ with } q = \frac{p}{p-1}. \tag{8}$$

For a general vector field $\mathbf{v}$ under very mild assumptions, the most general meaning that we can give to the flow of $\mathbf{v}$ is in terms of the so-called *superposition principle* (Ambrosio-Crippa), the continuity equation:

$$\partial_t \mu_t + \operatorname{div}(\mathbf{v}\mu_t) = 0, \tag{9}$$

Définition 1 *Let Q be concentrated on the integral curves of $\mathbf{v}$, in the sense that*

$$\int_{C([0,1];\overline{\Omega})} \left| \gamma(t) - \gamma(0) - \int_0^t \mathbf{v}(s, \gamma(s))\, ds \right| \, dQ(\gamma) = 0. \tag{10}$$

If we define the curve of measures μ_t^Q through

$$\int_{\overline{\Omega}} \varphi(x)\, d\mu_t^Q(x) := \int_{C([0,1];\overline{\Omega})} \varphi(\gamma(t))\, dQ(\gamma) \quad \textit{for every } \varphi \in C(\overline{\Omega}), \tag{11}$$

then this curve μ_t^Q is called superposition solution of (9).

Theorem 2 (Superposition principle) *Let μ_t be a positive measure-valued solution of the continuity equation*

$$\frac{\partial}{\partial t}\mu_t + \mathrm{div}(\mathbf{v}\mu_t) = 0,$$

with the vector field $\mathbf{v}$ *satisfying the following condition*

$$\int_0^1 \int_{\overline{\Omega}} \frac{|\mathbf{v}(t,x)|}{1+|x|}\, d\mu_t(x)\, dt < +\infty, \tag{12}$$

then μ_t is a superposition solution.

One can still relate (6) and (2) under quite weak assumptions thanks to the superposition principle (Ambrosio-Crippa), assume that μ_0 and μ_1 have L^p densities bounded from below by a positive constant, define σ, μ_t as before and $\widehat{\sigma} = \sigma/\mu_t$. Since

$$\frac{\partial}{\partial t}\mu_t + \mathrm{div}(\widehat{\sigma}\mu_t) = 0,$$

with initial datum μ_0. By the superposition principle, μ_t is a superposition solution: $\mu_t = \mu_t^Q$ with $Q \in \mathcal{Q}^p(\mu_0, \mu_1)$ and $i_Q = |\sigma|$ so that Q solves (2). In particular the values of (6) and (2) coincide.

To sum up, we have seen how to construct an optimal Q for

$$\inf_{Q \in \mathcal{Q}(\mu_0, \mu_1)} \int_\Omega H(i_Q(x))dx$$

using the flow of the ODE

$$\dot{\gamma}(t) = \widehat{\sigma}(t, \gamma(t)),\ \widehat{\sigma}(t, x) = \frac{\sigma(t, x)}{(1-t)\mu_0(x) + t\mu_1(x)}$$

and $\sigma = \nabla \mathcal{H}^*(\nabla u)$ with

$$\mathrm{div} \nabla \mathcal{H}^*(\nabla u) = \mu_0 - \mu_1,\ \text{in } \Omega,\ \nabla \mathcal{H}^*(\nabla u) \cdot \nu = 0,\ \text{on } \partial\Omega.$$

- Cauchy Lipschitz case : requires σ to be Lipschitz, not realistic in traffic congestion models,
- in the general case, using supeposition solutions of the continuity equation: not really satisfactory, the regularity of the curves charged by Q is quite poor, no flow, no group property...

Assume Ω Lipschiz, μ_0, μ_1 have Lipschitz densities $\geq c > 0$. Intermediate approach : DiPerna-Lions theory. Requires $\widehat{\sigma}$ to have Sobolev regularity and an L^∞ bound on

$$\operatorname{div}(\widehat{\sigma}) = \frac{\operatorname{div}(\sigma)}{\mu_t} - \frac{1}{\mu_t^2}\nabla\mu_t \cdot \sigma = \frac{\mu_0 - \mu_1}{\mu_t} - \frac{1}{\mu_t^2}\nabla\mu_t \cdot \sigma.$$

The issue then becomes proving Sobolev regularity and an L^∞ bound on σ.

Regularity

Aim: prove Sobolev and L^∞ estimates for the optimizer σ of (6) under the following assumptions:

(i) $\mu_i = f_i \mathcal{L}^d$, with $f_i \in \mathrm{Lip}\,(\Omega)$ and $f_i \geq c > 0$, for $i = 0, 1$;

(ii) Ω open connected bounded subset of $\mathbb{R}^d$ having Lipschitz boundary.

in the case where the congestion takes the form

$$\mathcal{H}(\sigma) = \frac{1}{p}|\sigma|^p + |\sigma|,\ \mathcal{H}^*(z) = \frac{1}{q}(|z|-1)_+^q,\ \text{with } q = \frac{p}{p-1} \quad (13)$$

with $q \geq 2$.

so that the optimal σ is

$$\sigma = \Big(|\nabla u| - 1\Big)_+^{q-1} \frac{\nabla u}{|\nabla u|}.$$

where u solves the very degenerate PDE:

$$\operatorname{div}\left(\Big(|\nabla u| - 1\Big)_+^{q-1} \frac{\nabla u}{|\nabla u|}\right) = f = f_0 - f_1, \tag{14}$$

with Neumann boundary condition

$$\Big(|\nabla u| - 1\Big)_+^{q-1} \frac{\nabla u}{|\nabla u|} \cdot \nu = 0.$$

Note that there is no uniqueness for u but there is for σ.

Setting

$$G(z) = |\nabla \mathcal{H}^*(z)|^{\frac{p}{2}} \frac{z}{|z|} = (|z|-1)_+^{\frac{q}{2}} \frac{z}{|z|}, \;\; z \in \mathbb{R}^d$$

using

$$(\nabla \mathcal{H}^*(z) - \nabla \mathcal{H}^*(w)) \cdot (z-w) \geq \frac{4}{q^2} |G(z) - G(w)|^2 \,,$$

and

$$\begin{aligned} & |\nabla \mathcal{H}^*(z) - \nabla \mathcal{H}^*(w)| \\ & \leq (q-1) \left(|G(z)|^{\frac{q-2}{q}} + |G(w)|^{\frac{q-2}{q}} \right) |G(z) - G(w)| \end{aligned}$$

together with arguments originally due to Bojarski and Iwaniec for the p-laplacian, we first get:

Theorem 3 *$\mathcal{G} \in W^{1,2}(\Omega)$, where the function $\mathcal{G}$ is defined by*

$$\mathcal{G}(x) := G(\nabla u(x)) = (|\nabla u(x)| - 1)_+^{\frac{q}{2}} \frac{\nabla u(x)}{|\nabla u(x)|}, \;\; x \in \Omega. \tag{15}$$

Corollary 1

$$\sigma = \nabla \mathcal{H}^*(\nabla u) = |\mathcal{G}|^{\frac{q-2}{q}} \mathcal{G} \in W^{1,r}(\Omega), \tag{16}$$

for suitable exponents $r = r(d,q)$ given by

$$r(d,q) = \begin{cases} 2, & \text{if } d = q = 2, \\ \text{any value } < 2, & \text{if } d = 2,\; q > 2, \\ \frac{dq}{(d-1)q+2-d}, & \text{if } d > 2. \end{cases}$$

Regularizing (14) and using the fact that convex transforms of derivatives of the solution are subsolutions (in fact we use $(\partial_1 u - 2)_+^r$ of an elliptic PDE and using a bootstrap argument, we can prove the following:

Theorem 4 *If u solves (14), then u is globally Lipschitz on Ω.*

This enables us to define a flow à la DiPerna-Lions for the ODE related to the traffic congestion problem.

Other formulations, numerical approximation

Here we consider the case where the transport plan γ is fixed (so that the equivalence with the minimal flow problem does not hold any more). Recall that our study of equilibria relies on the following convex optimization problem:

$$(\mathcal{P})\inf\left\{\int_{\Omega}H(x,i_Q(x))dx\,:\,Q\in\mathcal{Q}(\gamma)\right\} \tag{17}$$

We will also assume here that $d=2$ and $q>2$ i.e. $p<2$.

For every $x \in \Omega$ and $\xi \geq 0$, let us define

$$H^*(x,\xi) := \sup\{\xi i - H(x,i),\ i \geq 0\},\ \xi_0(x) := g(x,0).$$

Let us now define the functional

$$J(\xi) = \int_\Omega H^*(x,\xi(x))dx - \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_\xi(x,y)d\gamma(x,y) \tag{18}$$

and consider:

$$(\mathcal{P}^*) \sup\{-J(\xi) : \xi \in L^q, \xi \geq \xi_0\} \tag{19}$$

Theorem 5 *If the domain of* $(\mathcal{P})$ *is nonempty, then*

$$\min(\mathcal{P}) = \max(\mathcal{P}^*) \tag{20}$$

and $\xi \in L^q$ *solves* $(\mathcal{P}^*)$ *if and only if* $\xi = \xi_Q$ *for some* $Q \in \mathcal{Q}(\gamma)$ *solving* $(\mathcal{P})$*.*

In the sequel, we will numerically approximate the unique equilibrium metric ξ_Q by a descent method on $(\mathcal{P}^*)$. One can recover the corresponding equilibrium intensity i_Q by inverting the relation $\xi(x) = g(x, i_Q(x))$.

Discretization

Start with the dual formulation

$$\inf_{\xi \in L^q,\ \xi \geq \xi_0 = g(.,0)} J(\xi) = \int_\Omega H^*(x, \xi(x))dx - W(\xi)$$

to compute the optimal metric $\xi = \partial_i H(x, i_Q(x))$. Case of a fixed (discrete) transport plan $\gamma = \sum \gamma_{\alpha\beta}\delta_{(S_\alpha, T_\beta)}$:

$$W(\xi) := \sum \gamma_{\alpha\beta} c_\xi(S_\alpha, T_\beta).$$

Where $c_\xi(S, .)$ is the *viscosity* solution (or largest $W^{1,q}$ a.e. subsolution) of the Eikonal equation

$$\|\nabla \mathcal{U}\| = \xi; \quad \mathcal{U}_\xi(S) = 0 \tag{21}$$

(and we assume that $q > 2$ so that the domain of the primal is nonempty).

Space discretization, mesh size h, consistent (Souganidis, Barles-Souganidis, Rouy-Tourin) discretization of the Eikonal equation:

$$\left(\frac{\max\{(\mathcal{U}_{i,j}-\mathcal{U}_{i-1,j}),(\mathcal{U}_{i,j}-\mathcal{U}_{i+1,j}),0\}}{h_x}\right)^2 + \left(\frac{\max\{(\mathcal{U}_{i,j}-\mathcal{U}_{i,j-1}),(\mathcal{U}_{i,j}-\mathcal{U}_{i,j+1}),0\}}{h_y}\right)^2 = (\xi_{i,j})^2.$$

can be solved efficiently by Sethian's Fast Marching Method.
Notation : $c_\xi^h(S,T)$, discrete functional

$$J^h(\xi) = h^2\sum_{i,j} H^*(i,j;\xi_{i,j}) - \sum_{r,s} c_\xi^h(S_\alpha,T_\beta)\gamma_{\alpha,\beta},$$

Note that each J^h is convex.

Γ-convergence:

Theorem 6 *The sequence of functionals J^h $\Gamma-$converges with respect to the weak L^q convergence to the limit functional J. Moreover, as the sequence $(J^h)_h$ is equi-coercive and every functional, J included, is strictly convex, (strong) convergence of the unique minimizers and of the values of the minima is guaranteed.*

Solving the discrete problem by a subgradient descent method, J^h involves a differentiable part and a convex homogenous one. Problem : compute at each iteration a subgradient of the second part. Not straightforward but possible recursively by a method that uses the same recursivity as the FMM. We call this method the Fast Subgradient Marching Method, it enables to compute efficiently $(N^2 \log(N))$ a supergradient of the (discrete) geodesic distance with respect to the values of the metric on a grid. See the problem as an optimization problem over metrics.

There are several other applications of this strategy to compute by FMM a supergradient of distances with respect to metrics: inverse problems in travel-time tomography for instance.
Optimal design of obstacles to prevent mass transfer or the invasion of an army (Buttazzo):

$$\max_{\xi} \sum \alpha_i \bar{c}_{\xi}(x_i, y_i)$$

subject to $\underline{\xi} \leq \xi \leq \overline{\xi}$ and

$$\int \xi = \lambda.$$