# Hilbert Uniqueness Method and Regularity: Applications to the order of convergence of discrete controls for the wave equation

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# Outline of the talk

1. Introduction: The Hilbert Uniqueness Method
2. An alternate HUM type method
3. Application: the order of convergence of discrete controls

# An abstract control problem

Let $\mathcal{A}$ be a skew-adjoint operator defined on a Hilbert space $\mathfrak{X}$.

Consider the following model:

$$y'(t) = \mathcal{A}\,y(t) + \mathcal{B}v(t), \qquad y(0) = y^0 \in \mathfrak{X},$$

where $\mathcal{B} \in \mathfrak{L}(\mathcal{Y}, \mathcal{D}(\mathcal{A})^*)$ and $v \in L^2(0, T; \mathcal{Y})$.

## Assumption

For all $v \in L^2(0, T; \mathcal{Y})$, solutions can be defined in the sense of transposition in $C^0([0, T]; \mathfrak{X})$.

## Goal : Exact controllability

Fix a time $T > 0$ and $y^0 \in \mathfrak{X}$. Can we find $v \in L^2(0, T; \mathcal{Y})$ such that $y(T) = 0$ ?

# Hypotheses

- $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathfrak{X}$ is a skew-adjoint operator.
  $\Longrightarrow$ The energy $\|z(t)\|_{\mathfrak{X}}^2$ of solutions is constant.
- $\mathcal{A}$ has compact resolvent.
  $\Longrightarrow$ Its spectrum is discrete.

$\rightsquigarrow$ Spectrum of $\mathcal{A}$:

$$\sigma(\mathcal{A}) = \{i\mu^j : \ j \in \mathbb{N}\},$$

where $(\mu^j)_{j\in\mathbb{N}}$ is an increasing sequence of real numbers, corresponding to an orthonormal basis $(\Psi^j)_{j\in\mathbb{N}}$

$$\mathcal{A}\Psi^j = i\mu^j\Psi^j.$$

# Examples

- Wave equation in a *bounded domain*+ BC with distributed control

$$\begin{cases} u'' - \Delta u = \chi_\omega v, & (t,x) \in \mathbb{R} \times \Omega, \\ u_{|\partial\Omega} = 0, \\ (u(0), \dot{u}(0)) = (u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega), \end{cases}$$

$$\mathcal{A} = \begin{pmatrix} 0 & Id \\ \Delta & 0 \end{pmatrix}, \quad \mathfrak{X} = H_0^1(\Omega) \times L^2(\Omega),$$

$$\mathcal{B} = \begin{pmatrix} 0 \\ \chi_\omega \end{pmatrix}, \quad \mathcal{Y} = L^2(\omega).$$

- Wave equation in a *bounded domain*+ BC with boundary control
- Schrödinger equation $\mathcal{A} = -i\Delta$ + BC, Linearized KdV $\mathcal{A} = \partial_{xxx}$ + BC, Maxwell equation,...

# Duality

Use the adjoint system to characterize the controls !

For all $z$ solution of

$$z' = \mathcal{A} z, \qquad z(0) = z^0 \in \mathfrak{X},$$

we have

$$\langle y(T), z(T) \rangle_{\mathfrak{X}} - \langle y^0, z^0 \rangle_{\mathfrak{X}} = \int_0^T \langle v(t), \mathcal{B}^* z(t) \rangle_{\mathcal{Y}} \, dt.$$

In particular, $v$ is a control if and only if $\forall z^0 \in \mathfrak{X}$

$$0 = \int_0^T \langle v(t), \mathcal{B}^* z(t) \rangle_{\mathcal{Y}} \, dt + \langle y^0, z^0 \rangle_{\mathfrak{X}}.$$

# Fundamental hypotheses

- $\mathcal{B}^* : \mathcal{D}(\mathcal{A}) \to \mathcal{Y}$, $\mathcal{B}^* \in \mathfrak{L}(\mathcal{D}(\mathcal{A}), \mathcal{Y})$.

**Definition**

$\mathcal{B}^*$ is admissible if $\forall T > 0, \exists K_T > 0$,

$$\int_0^T \|\mathcal{B}^* z(t)\|_{\mathcal{Y}}^2 \, dt \leq K_T \left\| z^0 \right\|_{\mathfrak{X}}^2, \qquad \forall\, z^0 \in \mathcal{D}(\mathcal{A}).$$

**Definition**

$\mathcal{B}^*$ is exactly observable at time $T^* > 0$ if $\exists k_* > 0$,

$$k_* \left\| z^0 \right\|_{\mathfrak{X}}^2 \leq \int_0^{T^*} \|\mathcal{B}^* z(t)\|_{\mathcal{Y}}^2 \, dt, \qquad \forall\, z^0 \in \mathfrak{X}.$$

# The Hilbert Uniqueness Method (Lions '86)

Let $T \geq T^*$.
Define, for $z^0 \in \mathfrak{X}$,

$$J(z^0) = \frac{1}{2} \int_0^T \|\mathcal{B}^* z(t)\|_{\mathcal{Y}}^2 \, dt + \langle y^0, z^0 \rangle,$$

where $z$ satisfies $z' = \mathcal{A}z$, $z(0) = z^0$.
Observability $\Rightarrow$ Existence and Uniqueness of a minimizer $Z^0$.
Then $v = \mathcal{B}^* Z$ is such that the solution $y$ of

$$y' = \mathcal{A}y + \mathcal{B}v, \qquad y(0) = y^0,$$

satisfies $y(T) = 0$.
Besides, $v$ is the control of minimal $L^2(0, T; Y)$-norm.

# A regularity problem

## On the regularity

If $y^0 \in \mathcal{D}(\mathcal{A})$,

- Does the function $Z^0$ computed that way belongs to $\mathcal{D}(\mathcal{A})$ ?
- Is the controlled solution $(y, v)$ a strong solution ?
  i.e. $y \in C^1([0, T]; \mathfrak{X})$

General Answer : NO !

Consider the wave equation

$$\begin{cases} w_{tt} - w_{xx} = 0, & 0 < x < 1,\ 0 < t < T, \\ w(0,t) = 0,\ \ w(1,t) = v(t), & 0 < t < T, \\ (w(x,0), w_t(x,0)) = (w^0(x), w^1(x)) & \in L^2(0,1) \times H^{-1}(0,1). \end{cases}$$

The adjoint problem is

$$q_{tt} - q_{xx} = 0,\ \ q(0,t) = q(1,t) = 0,\ \ (q^0, q^1) \in H_0^1(0,1) \times L^2(0,1),$$

and the solutions write

$$q = \sqrt{2} \sum_{k \geq 1} \left( \hat{q}_k^0 \cos(k\pi t) + \frac{\hat{q}_k^1}{k\pi} \sin(k\pi t) \right) \sin(k\pi x),$$

Controllability in time $T = 4$ :

If $\left(w^0(x),\ w^1(x)\right) = \sqrt{2} \sum_{k \geq 1} \left(\hat{w}_k^0,\ \hat{w}_k^1\right) \sin(k\pi x)$,

$$\hat{Q}_k^0 = \frac{\hat{w}_k^1}{4k^2\pi^2}, \quad \hat{Q}_k^1 = -\frac{\hat{w}_k^0}{4}.$$

In particular, the HUM control can be computed explicitly

$$\begin{aligned} v(t) &= Q_x(1,t) \\ &= \frac{1}{4}\sum_{k\geq 1}(-1)^k k\pi\left(\frac{\hat{w}_k^1}{k^2\pi^2}\cos(k\pi t) - \frac{\hat{w}_k^0}{k\pi}\sin(k\pi t)\right). \end{aligned}$$

$$\Longrightarrow v(0) = \frac{1}{4}\sum_{k\geq 1}(-1)^k \frac{\hat{w}_k^1}{k\pi} \neq 0\ !$$

$\Longrightarrow$ If $w^0 \in H_0^1(0,1)$, the controlled solution is not a strong solution in general because of the failure of the compatibility conditions $w^0(1) = v(0) = 0$.

# Main question

## Main question

How to construct a control method which respects the regularity of the solutions ?

If $y^0 \in \mathcal{D}(\mathcal{A})$, we want

- $Z^0 \in \mathcal{D}(\mathcal{A})$
- the controlled equation $y' = \mathcal{A}y + \mathcal{B}v$ is satisfied in the strong sense.

Related result - Dehman Lebeau 2009:
The wave equation with distributed control $\mathcal{B} = \chi_\omega$ where $\chi_\omega$ is smooth, and where the HUM operator is modified by a function $\eta(t)$ vanishing at $t \in \{0, T\}$.

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# The modified HUM method

Let $y^0 \in \mathfrak{X}$, and $\delta > 0$ such that $T - 2\delta \geq T^*$, where $T^*$ is the time of observability. Define, for $z^0 \in \mathfrak{X}$,

$$J(z^0) = \frac{1}{2} \int_0^T \eta(t) \, \|\mathcal{B}^* z(t)\|_{\mathcal{Y}}^2 \, dt + \langle y^0, z^0 \rangle,$$

where $z$ satisfies $z' = \mathcal{A} z$, $z(0) = z^0$ and

$$\eta \in C^\infty(\mathbb{R}), \quad \eta = \begin{cases} 0 & \text{on } (-\infty, 0] \cup [T, \infty) \\ 1 & \text{on } [\delta, T - \delta] \end{cases} \qquad \eta \geq 0.$$

Observability $\Rightarrow$ Existence and Uniqueness of a minimizer $Z^0$.
Then $v = \eta \mathcal{B}^* Z$ is such that the solution $y$ of

$$y' = \mathcal{A} y + \mathcal{B} v, \qquad y(0) = y^0,$$

satisfies $y(T) = 0$.
Besides, $v$ is the control of minimal $L^2((0, T), dt/\eta; Y)$-norm.

# Main result

**Theorem (SE Zuazua)**

Assume that admissibility and observability property hold.
If $y^0 \in \mathcal{D}(\mathcal{A})$, then the minimizer $Z^0$ computed by the above method and the control function $v = \eta B^* Z$ are more regular:

- $Z^0 \in \mathcal{D}(\mathcal{A})$,
- $v \in H_0^1(0, T; \mathcal{Y})$.

In particular, the controlled solution $y$ with control $v$ is a strong solution of the controlled equation.
Moreover, there exists a constant $C = C(\eta)$ such that

$$\left\| Z^0 \right\|_{\mathcal{D}(\mathcal{A})} + \|v\|_{H_0^1(0,T;\mathcal{Y})} \leq C \left\| y^0 \right\|_{\mathcal{D}(\mathcal{A})}.$$

# Before the proof

First remark that, due to the classical observability property,

$$\left\| Z^0 \right\|_{\mathfrak{X}} + \|v\|_{L^2(0,T;\mathcal{Y})} \leq C \left\| y^0 \right\|_{\mathfrak{X}}.$$

Also remark that admissibility and observability properties yield

$$k \left\| z^0 \right\|_{\mathcal{D}(\mathcal{A})} \leq \int_0^T \eta(t) \left\| \mathcal{B}^* z'(t) \right\|_{\mathcal{Y}}^2 \, dt \leq K \left\| z^0 \right\|_{\mathcal{D}(\mathcal{A})}.$$

$\longrightarrow$ It is sufficient to prove that

$$\int_0^T \eta(t) \left\| \mathcal{B}^* Z'(t) \right\|_{\mathcal{Y}}^2 \, dt < \infty.$$

Indeed, this implies $Z^0 \in \mathcal{D}(\mathcal{A})$ and $v \in H_0^1(0,T;\mathcal{Y})$.

# Idea of the proof

Write the characterization of the control $v = \eta \mathcal{B}^* Z$:

$$0 = \int_0^T \eta(t) \langle \mathcal{B}^* Z(t), \mathcal{B}^* z(t) \rangle_{\mathcal{Y}} \, dt + \langle y^0, z^0 \rangle_{\mathfrak{X}},$$

for all $z$ solution of $z' = \mathcal{A} z, \quad z(0) = z^0$.
Then take formally $z = Z'' = \mathcal{A}^2 Z$:

$$\int_0^T \eta(t) \left\| \mathcal{B}^* Z'(t) \right\|_{\mathcal{Y}}^2 \, dt = -\langle \mathcal{A} y^0, \mathcal{A} Z^0 \rangle_{\mathfrak{X}} - \int_0^T \eta'(t) \langle \mathcal{B}^* Z'(t), \mathcal{B}^* Z(t) \rangle_{\mathcal{Y}} \, dt.$$

Using observability,

$$\int_0^T \eta(t) \left\| \mathcal{B}^* Z'(t) \right\|_{\mathcal{Y}}^2 \, dt \leq C(\|\eta'\|_\infty) \left\| y^0 \right\|_{\mathcal{D}\mathcal{A})}^2 .$$

# The 1d wave equation

$$\begin{cases} w_{tt} - w_{xx} = 0, & 0 < x < 1,\ 0 < t < T, \\ w(0,t) = 0,\ w(1,t) = v(t), & 0 < t < T, \\ (w(x,0), w_t(x,0)) = (w^0(x), w^1(x)) & \in L^2(0,1) \times H^{-1}(0,1). \end{cases}$$

The adjoint problem is

$$q_{tt} - q_{xx} = 0,\ \ q(0,t) = q(1,t) = 0,\ \ (q^0, q^1) \in H_0^1(0,1) \times L^2(0,1),$$

Controllability is OK for $T \geq T^* = 2$.

# Computation of the control

Hilbert Uniqueness Method, cf J.-L. Lions.

*Assume $T > T^* = 2$ and $\eta$ vanishing at $t = 0, T$.*

Initial data to be controlled: $(w^0, w^1) \in H^{-1}(\Omega) \times L^2(\Omega)$.

Minimize the functional

$$J(q^0, q^1) = \frac{1}{2} \int_0^T \eta |\partial_x q(1,t)|^2 \, dt + \langle w^1, q^0 \rangle_{H^{-1} \times H_0^1} - \int_\Omega w^0 q^1.$$

over $(q^0, q^1) \in H_0^1(\Omega) \times L^2(\Omega)$, $q$ solution of the adjoint problem. Minimizer $= (Q^0, Q^1)$.

Then $v = \eta \partial_x Q(1,t)$ is the control of minimal $L^2((0,T), dt/\eta)$ -norm.

# Our result

## Theorem

If $(w^0, w^1) \in H_0^1(0,1) \times L^2(0,1)$, then $(Q^0, Q^1) \in H^2 \cap H_0^1(0,1) \times H_0^1(0,1)$ and $v \in H_0^1(0,T)$ .
Besides, there exists a constant $C$ independant of $(w^0, w^1)$ such that

$$\begin{aligned} \left\| (Q^0, Q^1) \right\|_{H^2 \cap H_0^1(0,1) \times H_0^1(0,1)} &\leq C \left\| (w^0, w^1) \right\|_{H_0^1(0,1) \times L^2(0,1)} , \\ \| v \|_{H_0^1(0,T)} &\leq C \left\| (w^0, w^1) \right\|_{H_0^1(0,1) \times L^2(0,1)} . \end{aligned}$$

# The 1-d discrete case

Space semi-discretization (finite difference, $h = \frac{1}{N+1}$)

$$\begin{cases} w_j'' - \dfrac{1}{h^2}(w_{j-1} + w_{j+1} - 2w_j) = 0, & j \in \{1, \cdots, N\}, t \geq 0, \\ w_0(t) = 0, \quad w_{N+1}(t) = v(t), & t \geq 0. \end{cases}$$

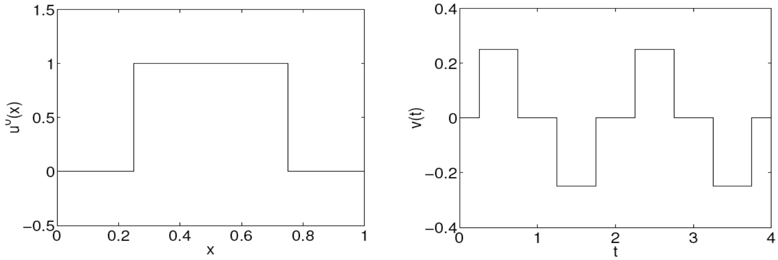


Figure: Left, the initial data $u(0)$. Right, the HUM control for the continuous system for initial data $(u(0), 0)$.

# Numerical experiments

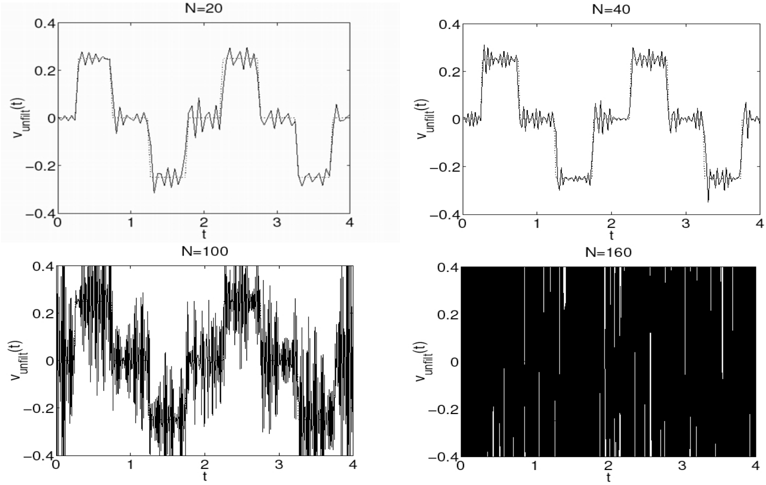


Figure: Discrete controls for different values of $N$.

# Spectral explanation

Discrete schemes are not uniformly observable

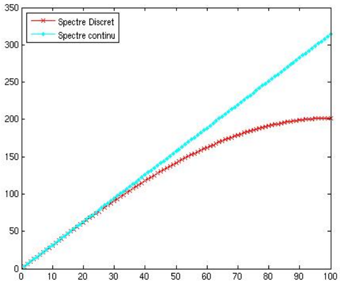


Figure: Discrete Spectrum vs Continuous Spectrum.

$\rightsquigarrow$ Filtering techniques are needed.

# Results (Infante Zuazua 99)

Spectrum of the discrete Laplace operator:

$$-\Delta_h \varphi = \lambda \varphi, \quad \varphi_0 = \varphi_{N+1} = 0$$

is given by the sequence $(\varphi^k, \lambda^k(h))$ $(k \in \{1, \cdots, N\})$:

$$\varphi_j^k = \sqrt{2}\, \sin(k\pi jh), \quad j \in \{1, \cdots, N\}, \quad \lambda^k(h) = \frac{4}{h^2} \sin^2\left(\frac{k\pi h}{2}\right).$$

Define, for $\gamma \in (0, 4)$,

$$\mathcal{C}_h(\gamma) = \mathrm{Span}\left\{\varphi_k, \;\; \lambda^k(h) \leq \frac{\gamma}{h^2}\right\}$$

and the orthogonal projection $\pi_\gamma^h$ over $\mathcal{C}_h(\gamma)$.

# Theorem (Infante Zuazua 99), slightly revisited

Let $\gamma \in (0,4)$ and $T > 2/(1-\gamma/4)$. Consider a sequence

$$(w_h^0, w_h^1) \underset{h\to 0}{\longrightarrow} (w^0, w^1) \quad \text{in } L^2(0,1) \times H^{-1}(0,1).$$

Define the functionals

$$J_h(q_h^0, q_h^1) = \frac{1}{2} \int_0^T \eta(t) \left| \frac{q_N}{h} \right|^2 \, dt + \langle w_h^1, q_h^0 \rangle_{H_h^{-1} \times H_h^1} - \int_\Omega w_h^0 q_h^1,$$

where $q$ is the solution of

$$\begin{cases} q_j'' - \dfrac{1}{h^2}(q_{j-1} + q_{j+1} - 2q_j) = 0, \quad j \in \{1, \cdots, N\}, t \geq 0, \\ q_0(t) = 0, \quad q_{N+1}(t) = 0, \quad t \geq 0. \\ (q_j(0), q_j'(0)) = (q_j^0, q_j^1). \end{cases}$$

# Theorem (Infante Zuazua 99), slightly revisited

The functionals

$$J_h(q_h^0, q_h^1) = \frac{1}{2} \int_0^T \eta(t) \left| \frac{q_N}{h} \right|^2 dt + \langle w_h^1, q_h^0 \rangle_{H_h^{-1} \times H_h^1} - \int_\Omega w_h^0 q_h^1,$$

have a unique minimizer $(Q_h^0, Q_h^1)$ on $\mathcal{C}_h(\gamma)^2$. The functions

$$v_h(t) = -\eta(t) \frac{Q_N(t)}{h}$$

are such that the solution $y_h$ of the discrete wave equation with initial data $(y_h^0, y_h^1)$ and control function $v_h$ satisfies

$$\pi_\gamma^h(y_h(T), y_h'(T)) = (0, 0).$$

Moreover, $(v_h) \longrightarrow v$ strongly in $L^2(0, T; dt/\eta)$, where $v$ is the HUM control of the continuous wave equation for $(w^0, w^1)$.

# Order of convergence

## Approximation of smooth data

$\exists C$ independent of $h > 0$ such that $\forall (w^0, w^1) \in H_0^1(0,1) \times L^2(0,1)$, there exists a sequence $(w_h^0, w_h^1)$ of discrete data such that $\forall h > 0$,

$$\begin{aligned} \left\| (w_h^0, w_h^1) \right\|_{H_0^1 \times L^2} &\leq C \left\| (w^0, w^1) \right\|_{H_0^1 \times L^2} \\ \left\| (w_h^0, w_h^1) - (w^0, w^1) \right\|_{L^2 \times H^{-1}} &\leq Ch \left\| (w^0, w^1) \right\|_{H_0^1 \times L^2}. \end{aligned}$$

# Order of convergence

## Theorem (SE & Zuazua)

$\exists C$ independent of $h > 0$ such that for all $(w^0, w^1) \in H_0^1(0,1) \times L^2(0,1)$, the discrete controls $v_h$ computed for the discrete data $(w_h^0, w_h^1)$ given above satisfy:

$$\|v_h - v\|_{L^2(0,T;dt/\eta)} \leq C h^{2/3} \left\|(w^0, w^1)\right\|_{H_0^1 \times L^2}$$

First result on the order of convergence of discrete controls.

# Idea of the proof-I

★ For $(w^0, w^1) \in H_0^1(0,1) \times L^2(0,1)$, the control is $v = \eta(t)\partial_x Q(1,t)$ for a solution $Q$ of the adjoint wave equation, with initial data $(Q^0, Q^1) \in (H^2 \cap H_0^1(0,1) \times H_0^1(0,1)) \cap \mathcal{C}_h(\gamma)^2$ .

★ One can approximate $(Q^0, Q^1)$ and $Q$ by discrete data $(\tilde{Q}_h^0, \tilde{Q}_h^1)$ such that

$$\begin{aligned} \left\| (\tilde{Q}_h^0, \tilde{Q}_h^1) \right\|_{H^2 \cap H_0^1 \times H_0^1} &\leq C \left\| (w^0, w^1) \right\|_{H_0^1 \times L^2} \\ \left\| \frac{\tilde{Q}_{N,h}}{h} + \partial_x Q(1,t) \right\|_{L^2(0,T)} &\leq C h^{2/3} \left\| (w^0, w^1) \right\|_{H_0^1 \times L^2} . \end{aligned}$$

Set $\tilde{v}_h = \eta(t) \dfrac{\tilde{Q}_{N,h}}{h}$:

$$\| \tilde{v}_h - v \|_{L^2(0,T; dt/\eta)} \leq C h^{2/3} \left\| (w^0, w^1) \right\|_{H_0^1 \times L^2} .$$

# Idea of the proof-II

★ The control $\tilde{v}_h = \eta(t)\frac{\tilde{Q}_{N,h}}{h}$ is an approximate control for the discrete equations: if $\tilde{w}_h$ denotes the solution of the discrete equation with control $\tilde{v}_h$, we have

$$\left\|(\tilde{w}_h(T), \tilde{w}_h'(T))\right\|_{L^2\times H^{-1}} \leq Ch^{2/3}\left\|(w^0, w^1)\right\|_{H_0^1\times L^2}.$$

★ Compute the control $\hat{v}_h$ of minimal $L^2(0,T;dt/\eta)$ norm such that

$$\begin{cases} p_j'' - \frac{1}{h^2}(p_{j-1}+p_{j+1}-2p_j) = 0, \quad j\in\{1,\cdots,N\}, t\geq 0, \\ p_0(t) = 0, \quad p_{N+1}(t) = \hat{v}_h(t), \quad t\geq 0. \\ (p_h(0), p_h'(0)) = (0,0), \quad (p_h(T), p_h'(T)) = -(\tilde{w}_h(T), \tilde{w}_h'(T)) \end{cases}$$

$\Rightarrow \hat{v}_h = -\eta(t)\frac{\hat{Q}_N}{h}$, $\hat{Q}_h$ solution of the discrete adjoint system:

$$\left\|\hat{v}_h\right\|_{L^2(0,T;dt/\eta)} \leq Ch^{2/3}\left\|(w^0, w^1)\right\|_{H_0^1\times L^2}.$$

# Idea of the proof-III

★ The function $\tilde{v}_h + \hat{v}_h$ is a discrete exact control which can be written as

$$\tilde{v}_h + \hat{v}_h = -\eta \frac{Q_{N,h}}{h},$$

where $Q_h$ is a solution of the discrete adjoint system in $\mathcal{C}_h(\gamma)$.
Uniqueness of such exact controls $\longrightarrow$ $v_h = \tilde{v}_h + \hat{v}_h$

$$\begin{aligned} \|v_h - v\|_{L^2(0,T;dt/\eta)} &\leq \|\tilde{v}_h - v\|_{L^2(0,T;dt/\eta)} + \|\hat{v}_h\|_{L^2(0,T;dt/\eta)} \\ &\leq Ch^{2/3} \left\|(w^0, w^1)\right\|_{H_0^1 \times L^2}. \end{aligned}$$

# Comments

## About $h^{2/3}$

- Remark that $\sqrt{\lambda^k(h)} = \frac{2}{h}\sin\left(\frac{k\pi h}{2}\right) \simeq k\pi$ for $k = o(h^{-2/3})$
  $\Rightarrow$ Convergence of the eigenvalues OK at scale $h^{-2/3}$.
- See also Baker SIAM JNA '76 and Rauch SIAM JNA '85: Distance between the continuous and semi-discrete semi-groups is exactly $h^{2/3}$.

- Optimality of this rate of convergence ?
- Applications to other situations:
  - Different numerical methods:
    $\star$ finite element (Infante Zuazua '99, SE '09),
    $\star$ mixed finite elements (Castro Micu '06, SE'09),
    $\star$ bi-grid techniques (Negreanu Zuazua '04)
  - Higher dimensions

$\longrightarrow$ See Zuazua's Survey '05 for extensive references)

*Thank you for your attention !*