Operational interpretation of quantum discord

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Quantum Discord (QD)

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• *was introduced* [*Zurek (00), Ollivier& Zurek (01)*] *to quantify all quantum correlations.*

• *Since its definition, it has received LOTS of attention. (36 arXiv titles over the last two years.)*

• *Interpretations in terms of the gain (in work extraction) a Maxwell´s demon obtains when operating quantumly with respect to classically have been provided. See Aharon´s talk!!!*

• *However, astonishingly, up to now QD lacked an (information-theoretic) operational interpretation.*

• *Quantum information community not happy about this* :-(

• *Curiosity: discord is an asymmetric correlation quantifier...*

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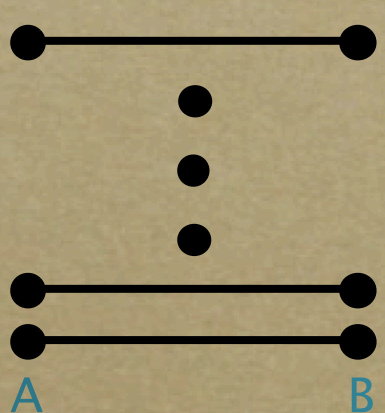
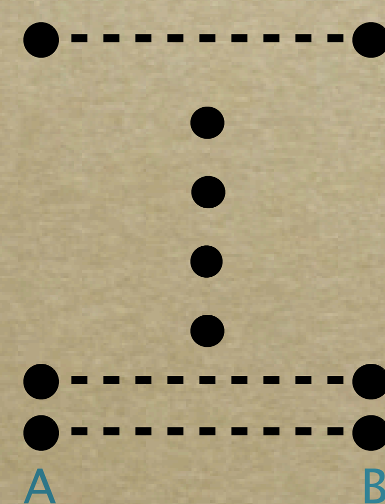
- *Paradigmatic examples:*



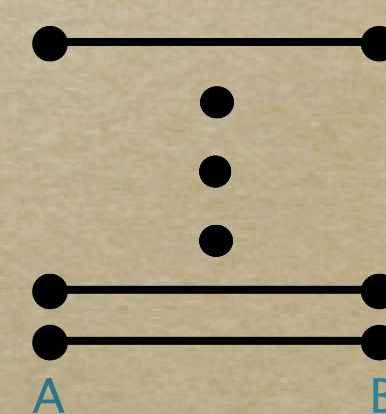
$|\Psi^+_{AB}\rangle^{\otimes M}$

$$E_C = \min_{LOCC} \frac{M}{N}$$



$\varrho_{AB}^{\otimes N}$

$$E_D = \max_{LOCC} \frac{M'}{N}$$



$|\Psi^+_{AB}\rangle^{\otimes M'}$

QD has a clear operational interpretation: it quantifies the total singlet consumption in state merging!!!!

The intrinsic asymmetry in QD plays a natural role in this scenario!!!

Discord imbalances quantify the efficiencies in different strategies of state merging and dense coding!!!

[*D. Cavalcanti, L. Aolita, S. Boixo, K. Modi, M. Piani, & A. Winter, arXiv:*1008.3205]

See also [*V. Madhok & A. Datta, arXiv:*1008.4135] *for related results!!!*

Outline of the talk

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- *Conditional entropy and coherent information.*
- *Definition of QD.*
- *State merging and its total entanglement consumption.*
- *Operational interpretation of QD.*
- *Asymmetry of QD.*
- *QD, state merging and the quantum advantage of dense coding.*
- *Asymptotic regularization and concluding remarks.*

Quantum conditional entropy and coherent information

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- *The classical (Shannon) entropy measures the (average) uncertainty in the value of a classical random variable a:*

$$H(a) \equiv H(\{p_i^a\}) := -\sum_i p_i^a \log_2 p_i^a$$

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• *The von Neumman entropy is the quantum counterpart:* $S(\rho) := -\mathrm{Tr}[\rho \log_2 \rho]$

• *Notation (for the reduced state of part X):* $S(X) = S(\varrho_X)$

• *The classical conditional entropy measures the uncertainty left - on average - for the value of a given that the value of b has been discovered:*

$$H(a|b) := H(a,b) - H(b)$$

• *Classical info theory:* $H(a|b)$ *is the average amount of (partial) classical information that A must give to B (who already knows the value of b) so that the latter gains full knowledge of (a,b)* [Slepian & Wolf (71)].

• *Given this interpretation,* $H(a|b)$ *is of course non-negative. And, in fact, it can also be expressed as*

$$H(a|b) = \sum_j p_j^b H(a|b=j),$$

where $H(a|b=j)$ *is the entropy of the conditional probability* $p^a_{i|b=j} := p^{ab}_{ij}/p^b_j$

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but, in contrast, it can take negative values!!!

• *The possible negativity was for a long time a hard obstacle to an operational interpretation for* $S(A|B)$

• *As a matter of fact, its opposite was even given a name of its own. The coherent information* $I(A\rangle B) := - S(A|B)$.

• *Originally introduced in quantum info as purely-quantum quantity to measure the amount of quantum info conveyable by a quantum channel* [*Schumacher & Nielsen* (*96*)].

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• *A "remedy" to negative quantum conditional entropy is* [*Henderson & Vedral* (*01*), *Ollivier & Zurek* (*01*)]:

$$S(A|B_c) := \min_{\{N_j\}} \sum_j p_j^B S(A|B=j),$$

over a positive rank-1 decomposition of the identity:

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- *The quantum discord of AB with measurements on B* [*Ollivier & Zurek* (*01*)]:

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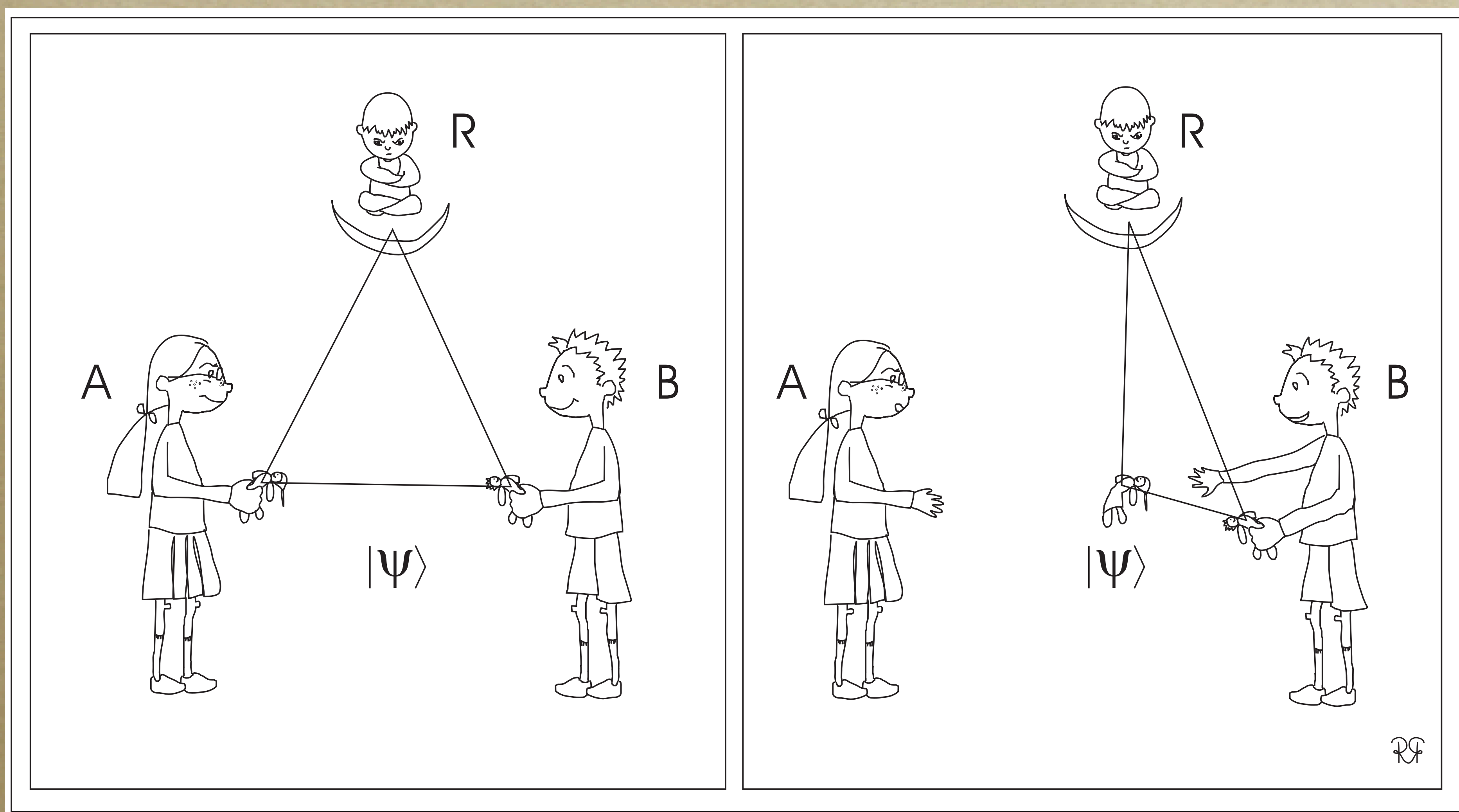
[A Ferraro, L. Aolita, D. Cavalcanti, F. M. Cucchietti, & A. Acín, PRA (2010)]

State merging and entanglement consumption

• *A satisfactory operational interpretation for* $S(A|B)$ *- and therefore also* $I(A\rangle B)$ *- was found* in the context of state merging:



[*Horodecki, Oppenheim & Winter, Nature* (*05*)]

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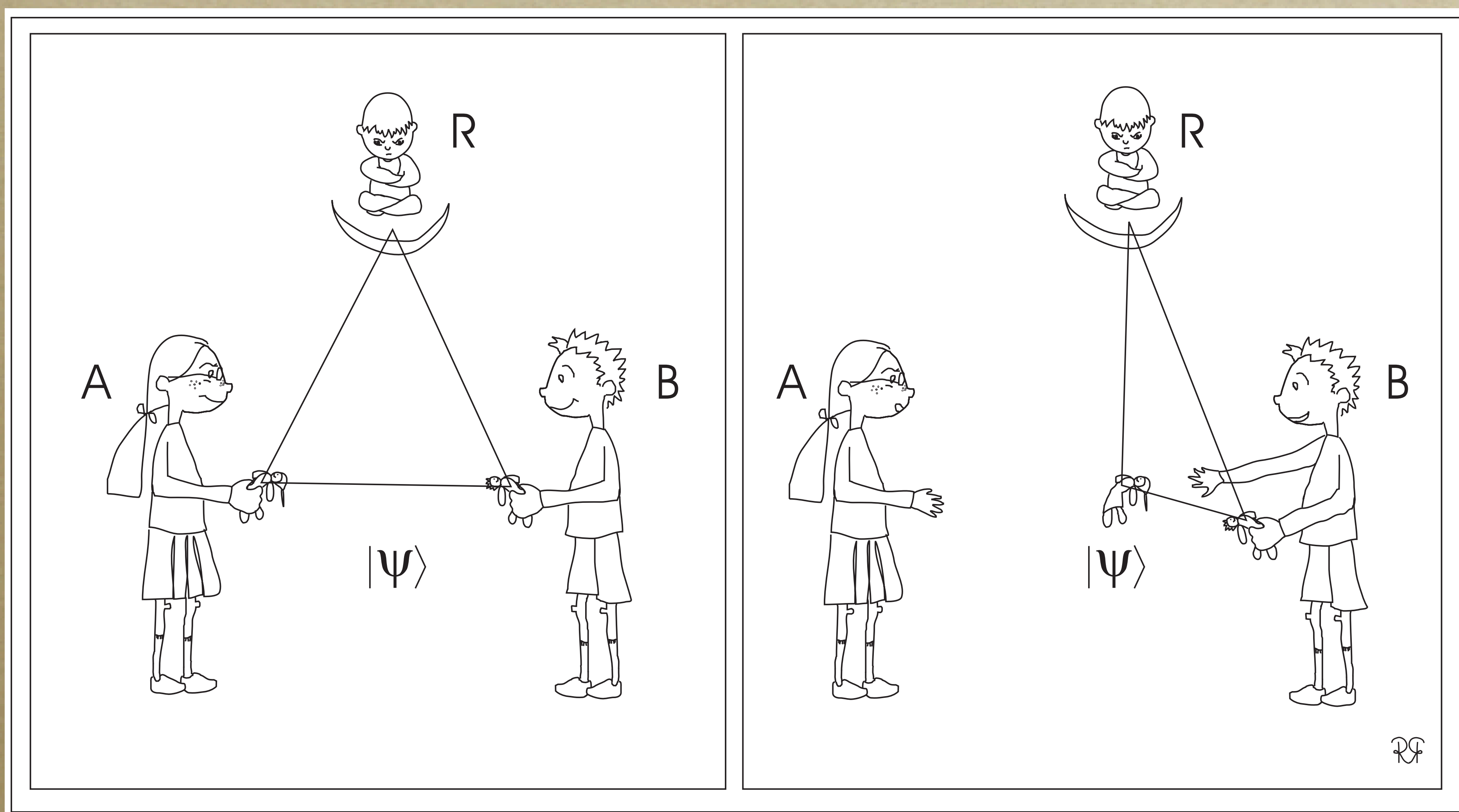
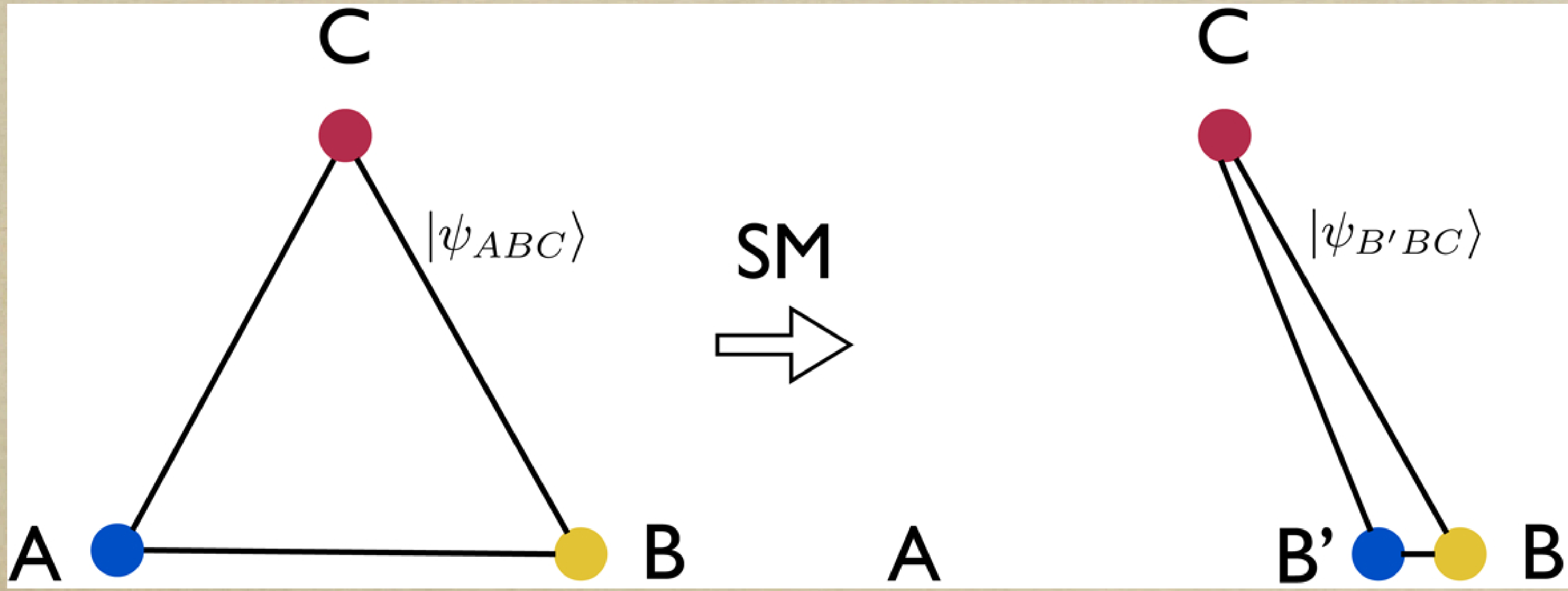


• *$S(A|B)$ quantifies exactly the optimal amount of uses of a perfect quantum channel!!!*

• *In a sense this is similar to what happened with $H(a|b)$...*

[*Horodecki, Oppenheim & Winter, Nature* (*05*)]

More technically,



- *A and B know the state they have,* ϱ_{AB}.
- *C is a neutral (inactive) reference system.*
- *LOCCs are for free, but quantum channels (singlets!) are expensive.*
- *Acting on N copies of the state, their goal is to end up with* $|\psi_{B'BC}\rangle^{\otimes N}$,

such that $|\psi_{B'BC}\rangle \rightarrow |\psi_{ABC}\rangle$, *for* $N \rightarrow \infty$.

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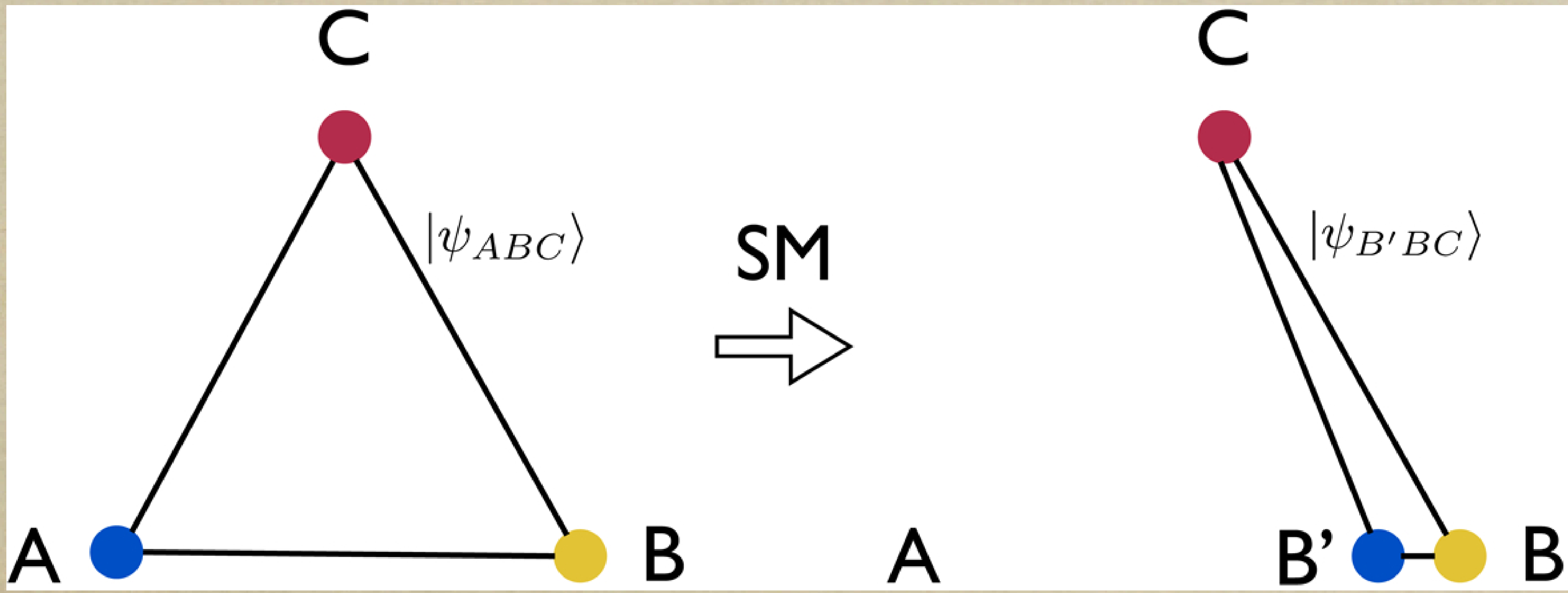

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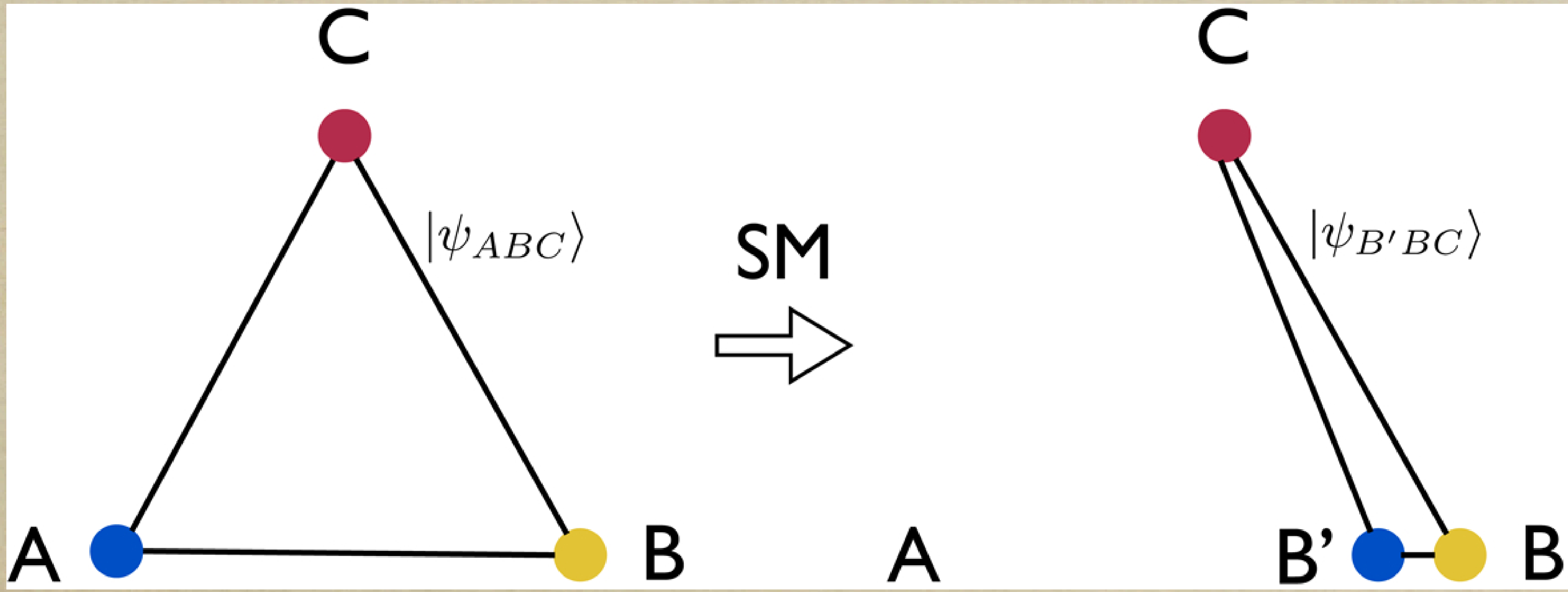

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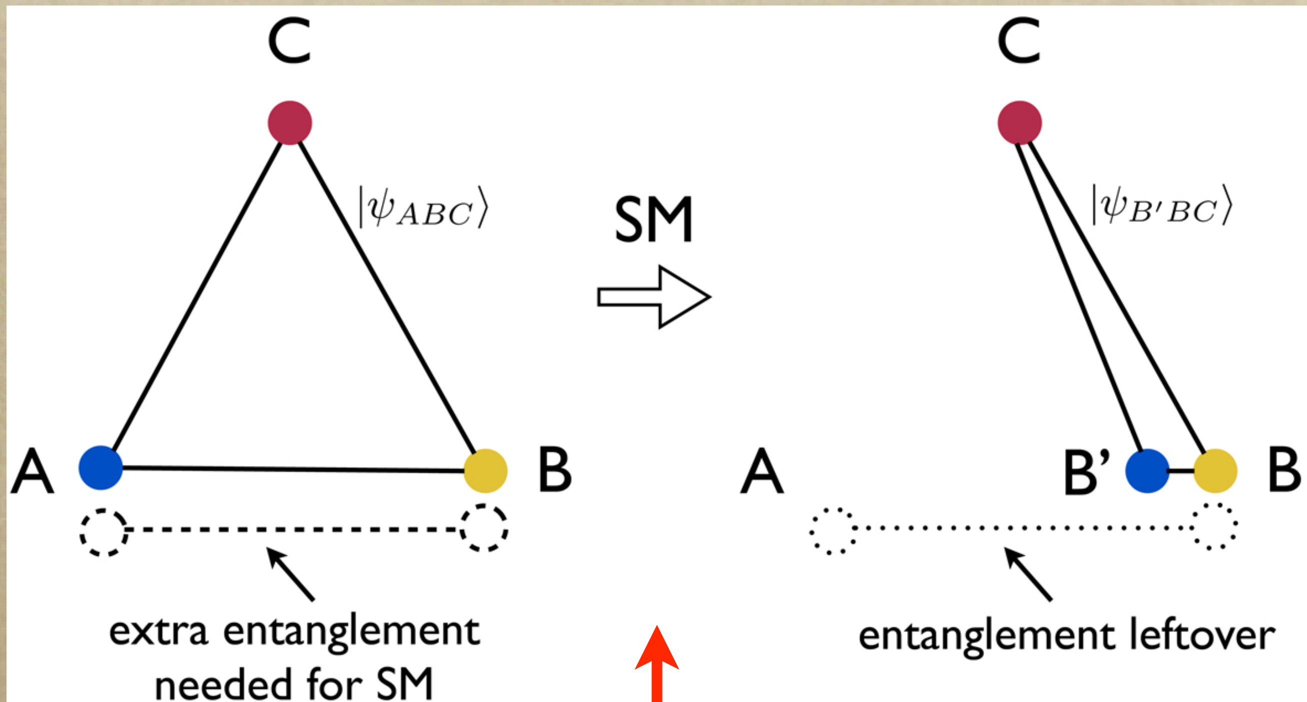
• *For S(A|B) ≥ 0, A and B consume S(A|B) extra singlets (per copy of the state) and end up fully uncorrelated.*

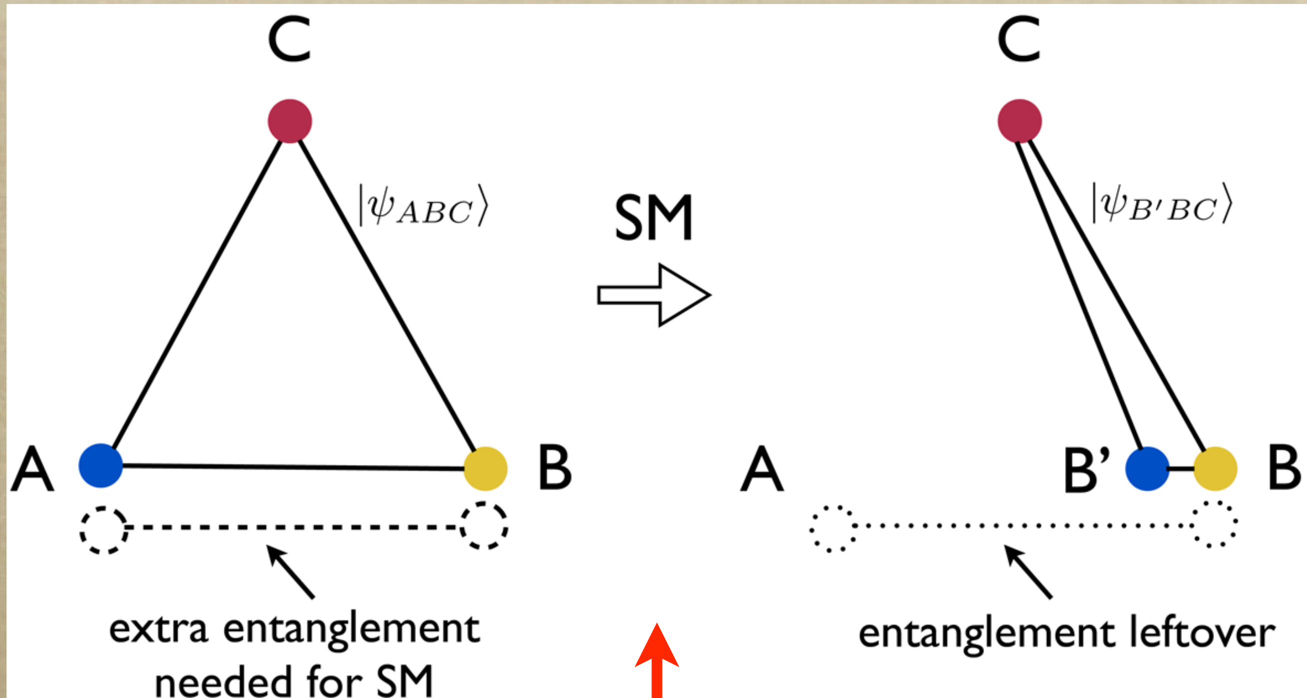
• *For S(A|B) < 0, not only do they perform the SM for free but they also retain -S(A|B)= I(A⟩B) singlets (per copy of the state), which they can use for future merings.*

Thursday, September 9, 2010


C
$|\psi_{ABC}\rangle$
SM
A
B
extra entanglement needed for SM
C
$|\psi_{B'BC}\rangle$
A
B'
B
entanglement leftover

INGLETS


Reserve Bank



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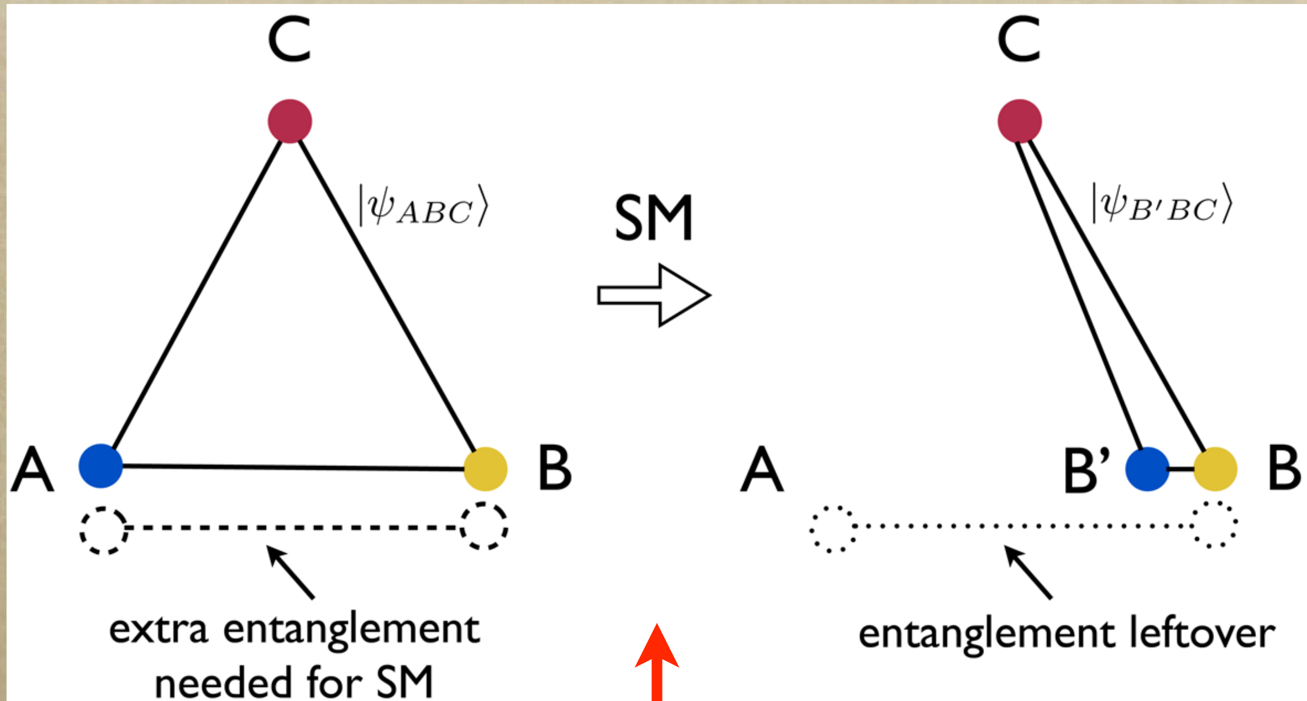

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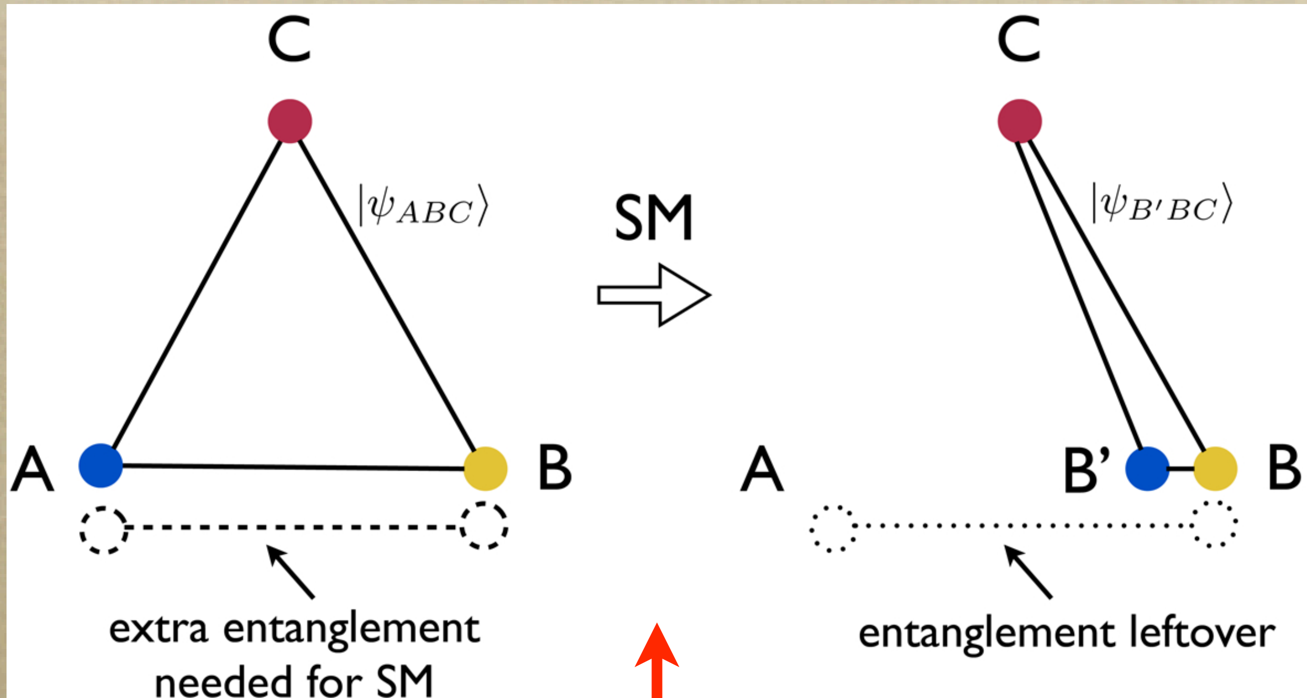

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- *The balance of the SM operation is given precisely by* $I(A\rangle B)$.
- *But what about the initial entanglement between A and B????*

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• *Therefore, the total entanglement consumption during the process of SM has to take this initial amount into account:*

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with $E_F(A:B) := \min_{\{p_i,\psi_i^{AB}\}} \sum_i p_i S\big(\mathrm{Tr}_A[\psi_i^{AB}]\big)$, *for* $\varrho_{AB} = \sum_i p_i |\psi_i^{AB}\rangle\langle\psi_i^{AB}|$.

[*D. Cavalcanti, L. Aolita, S. Boixo, K. Modi, M. Piani, & A. Winter, arXiv:*1008.3205]

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• *The entanglement of formation quantifies the minimum amount of pure-state entanglement that A and B consume to create (asymptotically many copies of)* ϱ_{AB} *by LOCC with strategies where each pure-state member of the ensemble is prepared independently.*

•*Thus,* $\Gamma(A\rangle B)$ *quantifies the total entanglement consumed, by taking into account the amount A and B would have needed to prepare* ϱ_{AB} *by LOCC - and ``lost'' during SM - plus the amount used by the process of SM itself.* >>>> *EXTENDED STATE MERGING.*

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Then $$D(A|C) = E_F(A:B) + S(A|B) := \Gamma(A\rangle B)!!!$$

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... for the first time a physical scenario where the values of QD provide concrete quantitative info about which of two possible strategies is most convenient!!!

QD, dense coding, and extended state merging

• *DC: by sending her part of* ϱ_{AB}, *A transmits classical info more efficiently than she could was the system classical* [Bennett & Wiesner (92)].

• *Conventional (pure-state) DC scenario: each letter in an alphabet is associated to a unitary rotation.*

• *Then the correction to the classical capacity* (*rate of information transmission per shared state used*) *is exactly the coherent information* $I(A\rangle B)$ [Horodecki et al. (01), Winter (02), Bruss et al. (04)].

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• *The most general (mixed-state) DC scenario: A's optimal encoding also consists of unitary rotations, but preceded by a pre-processing general quantum operation*

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Then the (single-shot) capacity is $$\chi(A\rangle B) := \log_2 d'_A + \max_{\Lambda_A} I(A'\rangle B)$$

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And the quantum advantage [Horodecki & Piani (07)]: $$\Delta_{DC}(A\rangle B)$$

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QD imbalance quantifies how much more efficient it is for C to do DC toward A as compared to toward B!!!!!

• *And in fact, if C sends always the same subsystem then it is:*

$$\Rightarrow D(A|C) - D(B|C) = \chi_{DC}(C\rangle A) - \chi_{DC}(C\rangle B)!!!$$

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- *All the relations we have found can be recast in their regularized version:*

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$$\Gamma^\infty(A\rangle B) := \lim_{n\to\infty} \Gamma(A\rangle B)_{\rho_{AB}^{\otimes n}}/n = E_C(A:B) + S(A|B)$$

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Conclusions

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- *We introduced extended state merging and its total entanglement consumption.*

- *QD quantifies the total singlet consumption in ESM.*

- *The intrinsic asymmetry in QD plays a natural role in this scenario: it tells us which of two possible strategies is cheapest.*

- *QD imbalance (with the measured system as the one in common) quantifies the difference in efficiency gain between DC toward two different receivers.*

- *These results define for the first time a physical scenario where the values of QD provide concrete quantitative info about the efficiency or cost involved in physical protocols.*

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THANKS!