Information-theoretic treatment of tripartite systems and quantum channels

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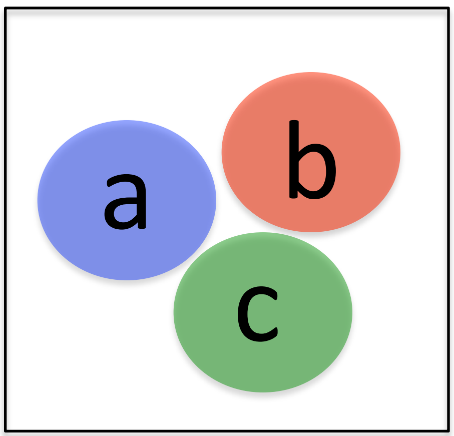

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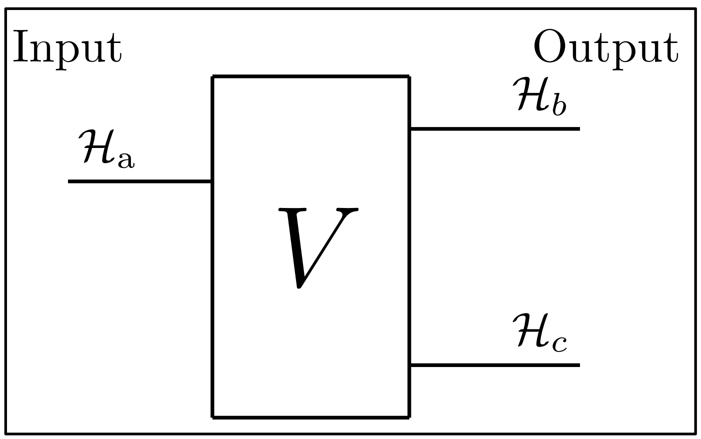
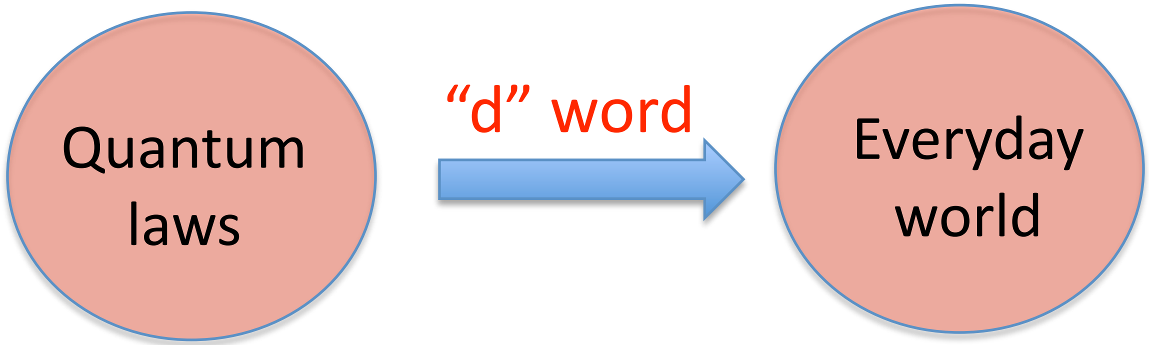


Research program



Approach

~~1.) Dynamics of prototypical systems~~
~~(Consistent histories)~~

2.) Information theory

How is quantum information unusual?

- *More general* than classical information

- Classical information: always possible to combine two logical statements to make a new logical statement

  "Spain won the Euro-Cup AND Spain won the World Cup."

- "The z-component of an electron spin is +1/2 AND its x-component is -1/2."
  NONSENSE

  x and z are incompatible types of information

- *Challenge: finding relations between incompatible types of information*

- Strong tradeoff in transmitting incompatible (i.e. quantum) information
  - through a channel and its complementary channel
  - to distinct parties of a multipartite state

Types and location of information

<u>Type of information</u>

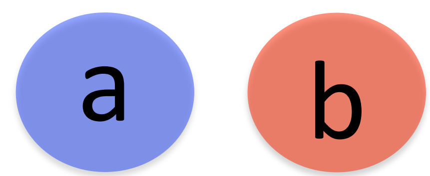
Technically: Decomposition of the identity

$$I_a = \sum_j P_{aj}$$

where $\{P_{aj}\}$ is set of orthogonal projectors

we also consider more general decompositions (POVMS)

Information of some type about *a* can be located inside *b* (in that some property of *b* is correlated to this property of *a*)

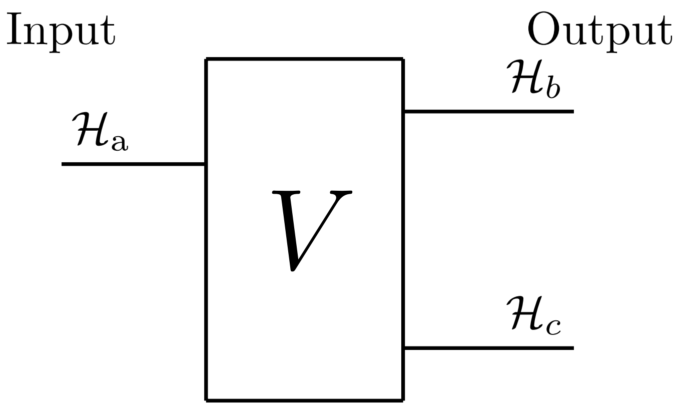


Are the conditional density operators on b, associated with the P_a information, distinguishable?

$$p_j \rho_{bj} = \mathrm{Tr}_a(P_{aj}\rho_{ab}) \qquad p_j := \Pr(P_{aj})$$

.... we aim to quantify this

Quantum Channel Problem

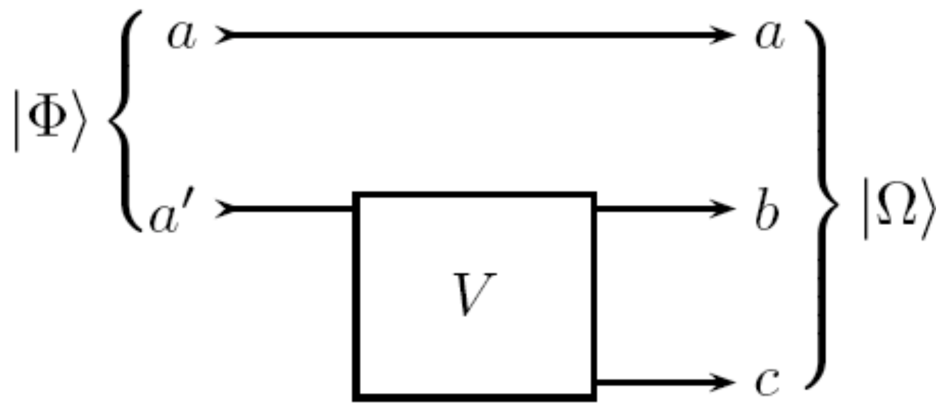


$$V^\dagger V = I_a$$

complementary channels

- D word
- Cryptography

Introduce reference system



$|\Phi\rangle$ is maximally entangled

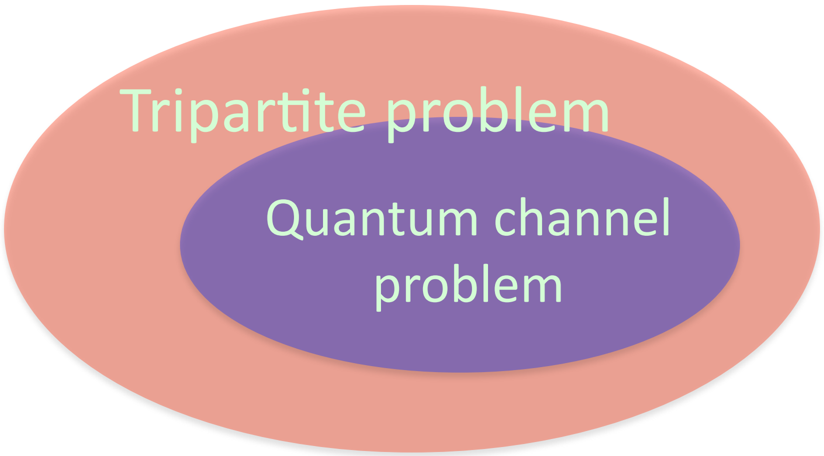
$$|\Omega\rangle = (I_a \otimes V)|\Phi\rangle$$

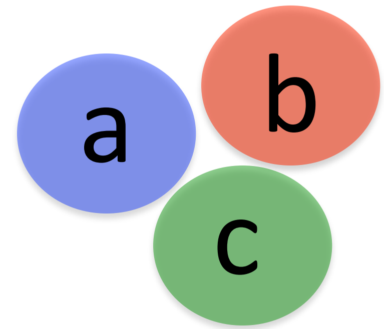
$$\rho_a = \mathrm{Tr}_{bc}(|\Omega\rangle\langle\Omega|) = I_a/d_a$$

Or start from the tripartite pure state,
use map-state duality to construct isometry

$$\sqrt{d_a}|\Omega\rangle = \sum_j |a_j\rangle \otimes |s_j\rangle \qquad V = \sum_j |s_j\rangle\langle a_j|$$

Three-party problem





$$\mathcal{H}_{abc} = \mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{H}_c$$

$$d_a,\ d_b,\ d_c$$

$$I_a = \sum_j P_{aj}, \quad I_b = \sum_k Q_{bk}, \quad I_c = \sum_l R_{cl}$$

What can we say about the probability distribution?

$$\Pr(P_{aj}, Q_{bk}, R_{cl}) = \mathrm{Tr}(P_{aj} Q_{bk} R_{cl} \rho_{abc})$$

<u>All-or-nothing theorems</u>

e.g. all information about *a* in *b*, then none in *c*

Goal: generalize all-or-nothing results to case of partial information

Information measures

General form

$$\chi_K(\{p_j, \rho_j\}) = S_K(\sum_j p_j \rho_j) - \sum_j p_j S_K(\rho_j)$$

$$\chi_K(P_a, b) := S_K(\rho_b) - \sum_j p_j S_K(\rho_{bj})$$

Particular entropy functions

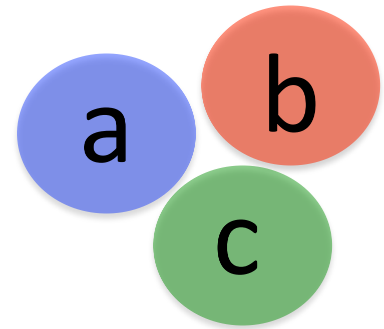
$$S_V(\rho) = -\mathrm{Tr}(\rho \log \rho),$$

$$S_R(\rho) = \frac{1}{1-q} \log \mathrm{Tr}(\rho^q), \quad 0 < q \leqslant 1,$$

$$S_T(\rho) = \frac{1}{1-q} [\mathrm{Tr}(\rho^q) - 1], \quad 0 < q \leqslant \infty$$

$$S_Q(\rho) = 1 - \mathrm{Tr}(\rho^2).$$

Basis invariance of information difference



Definitions

Entropy bias $\Delta S_K^{bc} := S_K(\rho_b) - S_K(\rho_c)$

Information bias $\Delta_K^{bc}(P_a) := \chi_K(P_a, b) - \chi_K(P_a, c)$

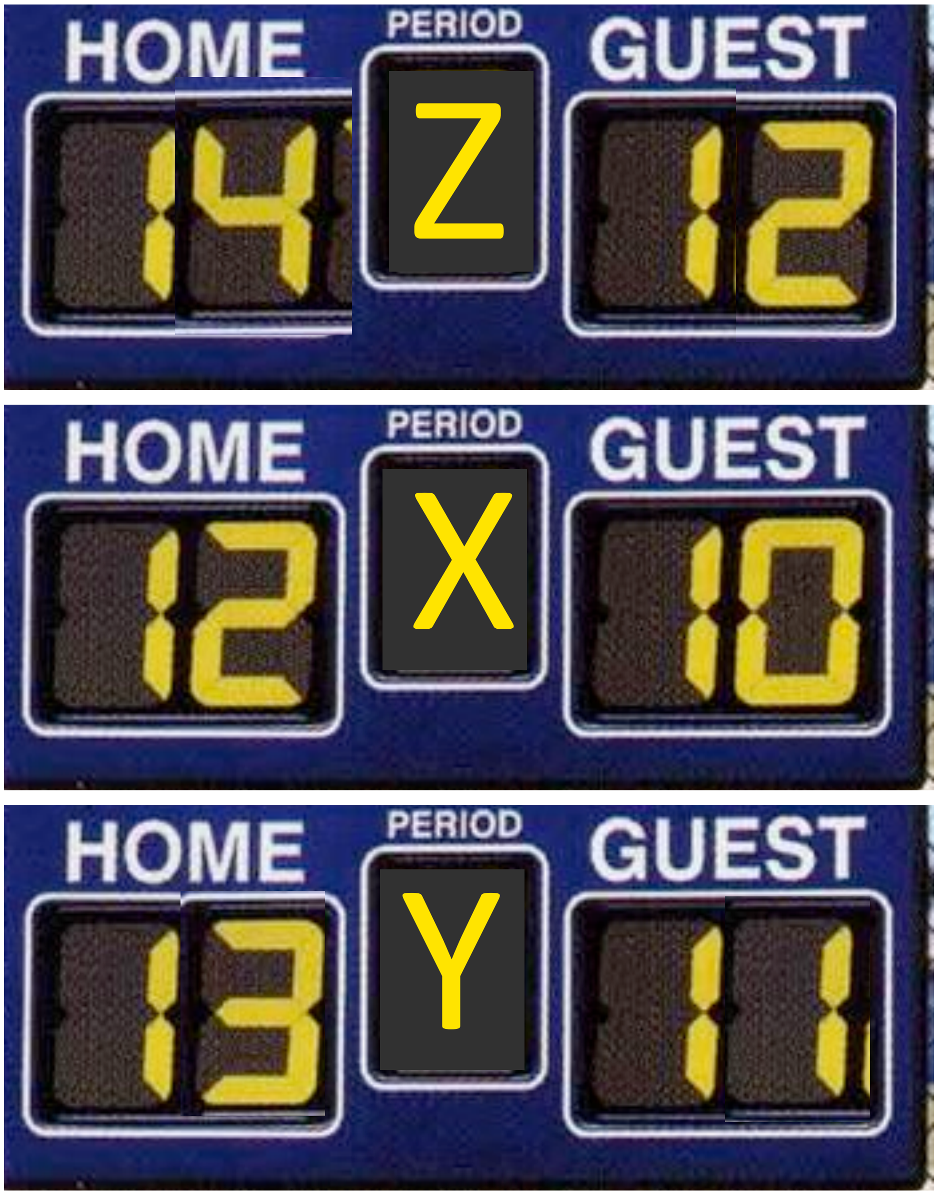
Theorem

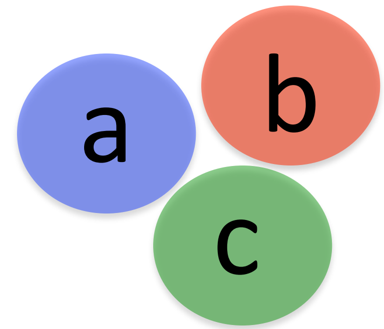
Consider orthonormal bases *u* and *w* for system *a*

$\rho_{abc} = |\Omega\rangle\langle\Omega|$ (pure state)

$$\begin{aligned}\Delta_K^{bc}(w) &= \chi_K(w, b) - \chi_K(w, c) \\ &= S_K(\rho_b) - S_K(\rho_c) = \Delta S_K^{bc}\end{aligned}$$

Basis invariance of information difference



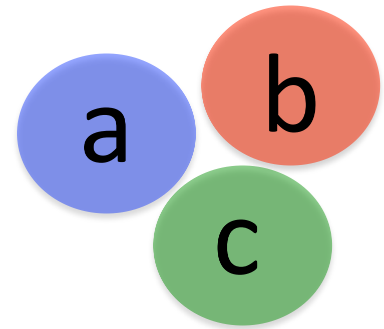


$$\rho_{abc} = |\Omega\rangle\langle\Omega|$$

Difference between Bob's and Charlie's scores is the same every game!

Example

$$\chi_V(z,b) - \chi_V(z,c) = \chi_V(x,b) - \chi_V(x,c)$$
$$\chi_V(z,b) - \chi_V(x,b) = \chi_V(z,c) - \chi_V(x,c)$$



$$\rho_{abc} = |\Omega\rangle\langle\Omega|$$

Suppose Bob has perfect classical information about Alice

$$\chi_V(z,b) = \log d_a$$
$$\chi_V(x,b) = 0$$

Then it follows that:

$$\chi_V(z,c) = \log d_a$$
$$\chi_V(x,c) = 0$$

So classical information always gets copied

Shannon and von Neumann measures

Classical entropy: $$H(P) = H(\{p_j\}) = -\sum_j p_j \log p_j$$

Classical mutual information: $$H(P:Q) = H(P) + H(Q) - H(P,Q)$$

A relation between classical and quantum entropies:

$$\chi_V(\{p_j, \rho_j\}) = S_V(\sum_j p_j \rho_j) - \sum_j p_j S_V(\rho_j) \leqslant H(\{p_j\})$$

$$\chi_V(\{p_j, \rho_j\}) = H(\{p_j\})$$ iff all ρ_j are orthogonal

$$\theta(P_a, b) := H(P_a) - \chi_V(P_a, b)$$ *is a positive quantity* *"missing information"*

Uncertainty Principle

Robertson. Phys. Rev. (1929)

$$\Delta A \, \Delta B \geqslant \frac{1}{2} |\langle \psi | [A, B] | \psi \rangle|$$

$$\Delta X = \sqrt{\langle \psi | X^2 | \psi \rangle - \langle \psi | X | \psi \rangle^2}$$

Right-hand-side depends on the state.

Can be zero,
e.g. A=Z, B=X,
$|\psi\rangle$ = z-eigenstate.

Entropy: alternative measure of spread

Maassen, Uffink. PRL (1988)

$$H(u) + H(w) \geqslant -\log(r^2)$$

$$r = \max_{j,k} |\langle u_j | w_k \rangle|$$

Mutually unbiased bases (MUBs)

$$r = 1/\sqrt{d} \qquad H(x) + H(z) \geq \log d$$

The main result

ArXiv: 1006.4859

Theorem

POVMs on $\mathcal{H}_a$

$P_a = \{P_{aj}\}$

$\bar{P}_a = \{\bar{P}_{ak}\}$

$$\theta(P_a, b) := H(P_a) - \chi_V(P_a, b)$$

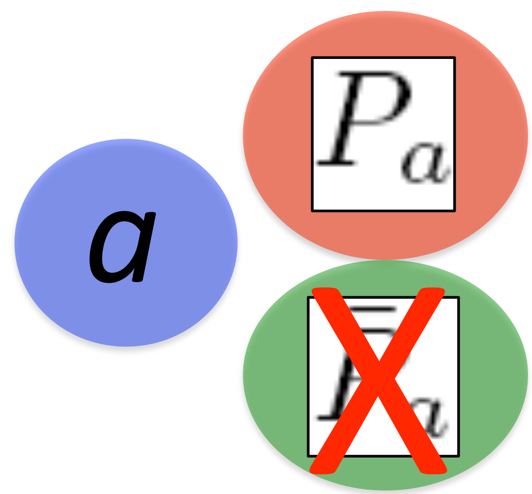
"missing information"

$$\boxed{\theta(P_a, b) + \theta(\bar{P}_a, c) \geqslant -\log \max_{j,k} \mathrm{Tr}[P_{aj}\bar{P}_{ak}]}$$

Very general, Very strong uncertainty relation

Presence of P_a *information in b*

EXCLUDES $\bar{P}_a$ *information from c*



Appreciating this result

Orthonormal bases

$$u = \{|u_j\rangle\langle u_j|\}$$
$$w = \{|w_k\rangle\langle w_k|\}$$

$$\boxed{\begin{gathered}\theta(u,b) + \theta(w,c) \geqslant -\log r^2 \\ r = \max_{j,k} |\langle u_j | w_k\rangle|\end{gathered}}$$

Mutually unbiased bases (MUBs)

$$\theta(u,b) + \theta(w,c) \geqslant \log d_a$$

Both an entropic uncertainty relation AND information exclusion relation

$$H(u) + H(w) \geqslant \chi_V(u,b) + \chi_V(w,c) + \log d_a$$

Suppose $\theta(u,b)=0$…. then $H(w)=\log d_a$ AND $\chi_V(w,c)=0$

$$\theta(u_a, b) = S_V(u_a|b)$$
$$S_V(u_a|b) := S_V[\mathcal{U}_a(\rho_{ab})] - S_V(\rho_b)$$
$$\mathcal{U}_a(\rho_{ab}) = \sum_j P_{aj}\rho_{ab}P_{aj}$$

Equivalent to "strong complementary information tradeoff" conjectured by Renes, Boileau (PRL 2009), proven by Berta et al. (Nature Physics 2010)

Corollaries

Strengthened uncertainty relations for mixed states

Maassen, Uffink (1988) $$H(u) + H(w) \geqslant -\log(r^2)$$

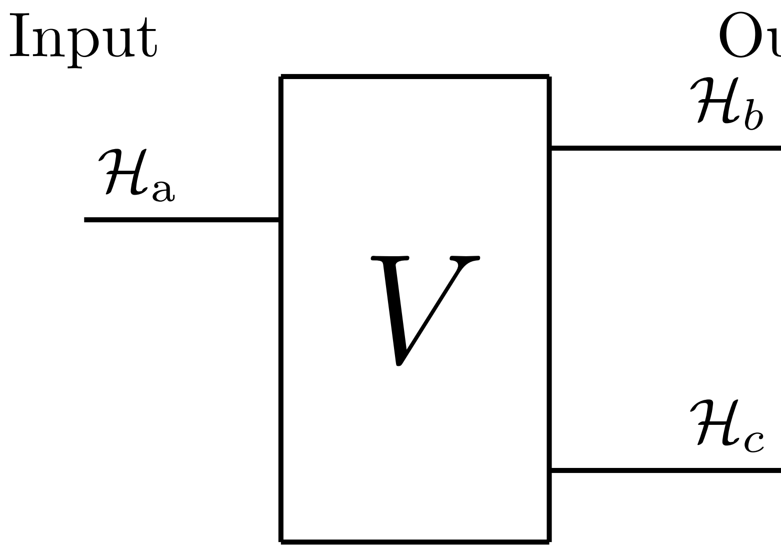
Corollary of our result $$H(u) + H(w) \geqslant -\log r^2 + S_V(\rho_a)$$

$\boxed{d_a = 2}$ For qubits, *x, y,* and *z* form a complete set of MUBs

Sanchez-Ruiz (1995) $$H(x) + H(y) + H(z) \geqslant 2\log 2$$

Corollary of our result $$H(x) + H(y) + H(z) \geqslant 2\log 2 + S_V(\rho_a)$$

The *dynamic* uncertainty principle



$\mathcal{E}$: channel to b

$\mathcal{F}$: channel to c

- Feed in *w* basis states
- Input probabilities $\{p_j\}$

<u>Output density operators</u>

$$\begin{aligned}\rho_{bj} &= \mathcal{E}(|w_j\rangle\langle w_j|) \\ &= \mathrm{Tr}_c(V|w_j\rangle\langle w_j|V^\dagger)\end{aligned}$$

<u>Quantify distinguishability at the output</u>

$$\chi_K(\{p_j\}, w, \mathcal{E}) = S_K\left(\sum p_j \rho_{bj}\right) - \sum p_j S_K(\rho_{bj})$$

The *dynamic* uncertainty principle

Quantify distinguishability at the output

$$\chi_K(\{p_j\}, w, \mathcal{E}) = S_K\left(\sum p_j \rho_{bj}\right) - \sum p_j S_K(\rho_{bj})$$

Corollary of our result

Arbitrary bases *u* and *w*, $r = \max_{j,k} |\langle u_j | w_k \rangle|$

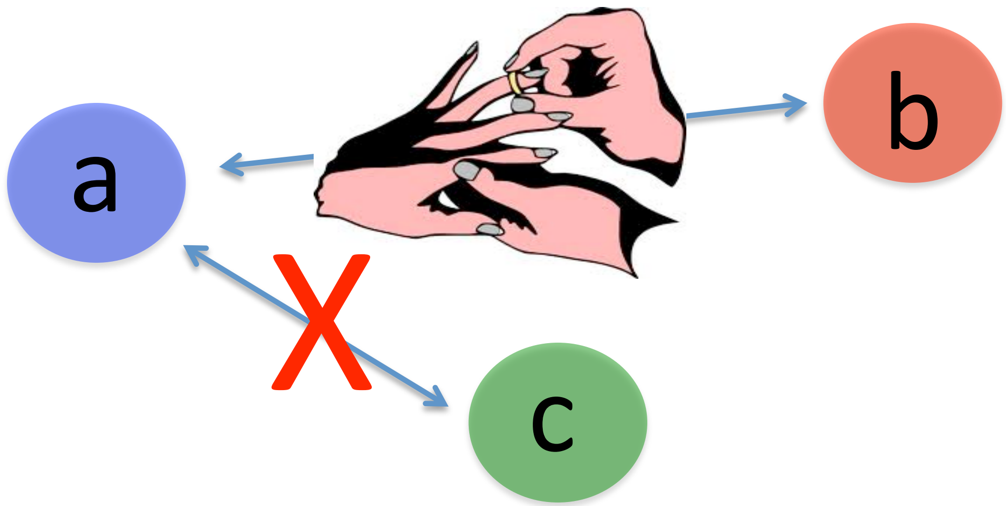
complementary quantum channels $\mathcal{E}$ and $\mathcal{F}$.

$$\chi_V(\{1/d_a\}, u, \mathcal{E}) + \chi_V(\{1/d_a\}, w, \mathcal{F}) \leqslant 2\log(d_a r)$$

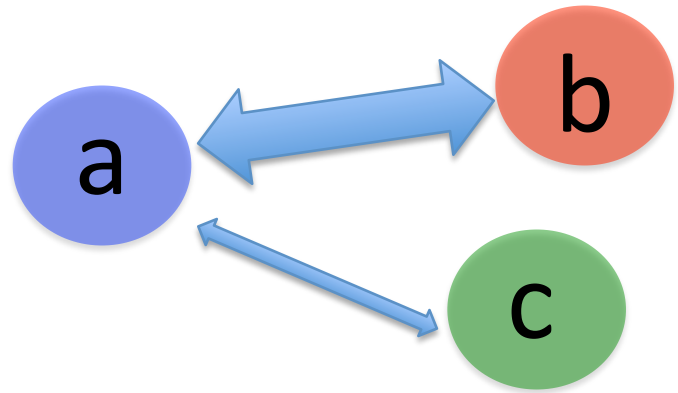
Can't build a machine that can send *z*-info to Bob and *x*-info to Charlie

No copying or No splitting or Monogamy

Monogamy of entanglement



Gradual approach to monogamy



If $\theta(w,b) \leqslant \alpha$ every orthonormal basis w of $\mathcal{H}_a$

then $\chi_V(w,c) \leqslant \alpha$ every orthonormal basis w of $\mathcal{H}_a$

Proof: Every basis has at least one MUB

But do we really have to know that Bob has every type *w* of information about Alice to ensure Charlie has none?

Two-type presence

Alice & Bob
Date #1

Alice & Bob
Date #2

(provided dates are sufficiently different)

Alice & Bob

Charlie (no Alice)

Quantum mutual information

$$I(a:b) = S_V(\rho_a) + S_V(\rho_b) - S_V(\rho_{ab})$$

$$I(a:b) \geq 2\log d_a - 2[\theta(x,b) + \theta(z,b)]$$

$$I(a:c) \leq \theta(x,b) + \theta(z,b)$$

More general form, for arbitrary bases, on ArXiv

One-type presence/absence

Alice & Bob
Date #1: Drinks

Alice & Charlie
Date #1: Drinks

Charlie (no Alice)

Charlie completely decoupled from Alice!

One-type presence/absence

Quantitative version

$$I(a:c) \leq \chi_V(z,c) + \theta(z,b)$$

Suppose the *z* type of information about *a* is perfectly present in *b*:

$$\theta(z,b) = 0$$

… and absent from *c*:

$$\chi_V(z,c) = 0$$

Then *a* and *c* are completely uncorrelated:

$$\rho_{ac} = \rho_a \otimes \rho_c.$$

All-or-nothing theorem not previously known?

Results for Tripartite states

- All-or-nothing theorems
- Theorems for MUBs

- Partial information theorems
- More general types of information

- Basis invariance of information bias $\Delta_K(w) = \Delta S_K$
- Uncertainty principle $\theta(x,b) + \theta(z,c) \geq \log d_a$
- Monogamy (No copying) $I(a:b) \geq 2\log d_a - 2[\theta(x,b) + \theta(z,b)]$
- Two type presence $I(a:c) \leq \theta(x,b) + \theta(z,b)$
- One type presence / absence $I(a:c) \leq \chi_V(z,c) + \theta(z,b)$

All of these results apply to complementary quantum channels!

One would have never stumbled upon our results using a global measure of entanglement, it is crucial to look at *individual types of information* to study these phenomena

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