# On some exactly or efficiently solvable open quantum many-body systems far from equilibrium

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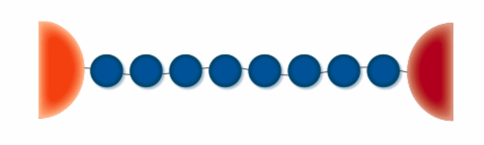

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  - XY spin 1/2 chain - *fermionic case*: transition to *long range order* due to local *boundary* opening (TP, NJP 2008, TP and I. Pižorn PRL 2008)
  - *Translationally invariant* fermionic/bosonic chains /w *bulk* noise/opening (/w J. Eisert, preprint)
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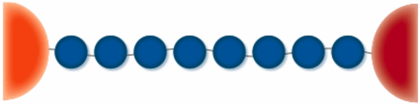

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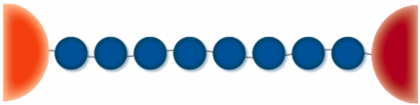

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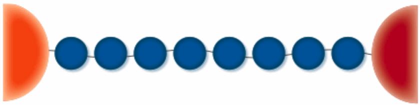

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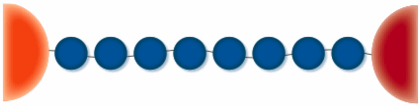

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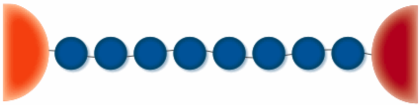

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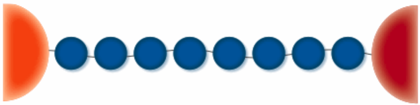

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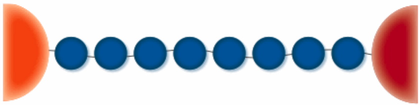

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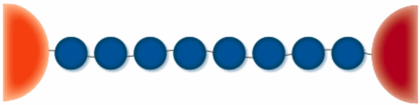

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$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \hat{\mathcal{L}}\rho := -\mathrm{i}[H,\rho] + \sum_{\mu} \left( 2L_\mu \rho L_\mu^\dagger - \{L_\mu^\dagger L_\mu, \rho\} \right)$$

where $H$ is a many-body (*Hamiltonian*) with $k-$**local couplings**,

$$H = \sum_{j=1}^{n-k+1} h_{[j,j+k-1]}$$

and $L_\mu$ are *Lindblad operators* which act **locally** (i.e. within some $[j, j+k-1]$), either near the **ends** of the chain (e.g. representing the baths), or in the **bulk** (e.g. representing dephasing noise).

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In the context of 1D quantum transport, the Lindblad model has been carefully derived and discussed in: Wichterich, Herich, Breuer and Gemmer, PRE 2007.

TP, New J. Phys. **10**, 043026 (2008)

Consider a general solution of the Lindblad equation:

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for a general *quadratic system* of $n$ fermions, or $n$ qubits (spins 1/2)

$$H = \sum_{j,k=1}^{2n} w_j H_{jk} w_k = \underline{w}\cdot\mathbf{H}\,\underline{w} \qquad L_{\mu} = \sum_{j=1}^{2n} l_{\mu,j} w_j = \underline{l}_{\mu}\cdot\underline{w}$$

where $w_j$, $j = 1,2,\ldots,2n$, are abstract *Hermitian* Majorana operators

$$\{w_j, w_k\} = 2\delta_{j,k} \qquad j,k = 1,2,\ldots,2n$$

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# Fock space of operators

Let us associate a Hilbert space structure $x \longrightarrow |x\rangle$ to a linear $2^{2n} = 4^n$ dimensional space $\mathcal{K}$ of operators, with basis

$$P_{\alpha_1,\alpha_2,\ldots,\alpha_{2n}} := w_1^{\alpha_1} w_2^{\alpha_2} \cdots w_{2n}^{\alpha_{2n}} \qquad \alpha_j \in \{0,1\}$$

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Define a set of $2n$ adjoint *annihilation linear maps* $\hat{c}_j$ over $\mathcal{K}$

$$\hat{c}_j |P_{\underline{\alpha}}\rangle = \delta_{\alpha_j,1} |w_j P_{\underline{\alpha}}\rangle$$

and derive the actions of their Hermitian adjoints - the *creation linear maps* $\hat{c}^\dagger$,
$\langle P_{\underline{\alpha}'}|\hat{c}_j^\dagger|P_{\underline{\alpha}}\rangle = \langle P_{\underline{\alpha}}|\hat{c}_j|P_{\underline{\alpha}'}\rangle^* = \delta_{\alpha'_j,1}\langle P_{\underline{\alpha}}|w_j P_{\underline{\alpha}'}\rangle^* = \delta_{\alpha_j,0}\langle P_{\underline{\alpha}'}|w_j P_{\underline{\alpha}}\rangle$:

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$$\hat{c}_j^\dagger |P_{\underline{\alpha}}\rangle = \delta_{\alpha_j,0} |w_j P_{\underline{\alpha}}\rangle$$

Clearly, $\hat{c}_j, \hat{c}_j^\dagger$ satisfy *canonical anti-commutation relations*

$$\{\hat{c}_j, \hat{c}_k\} = 0 \qquad \{\hat{c}_j, \hat{c}_k^\dagger\} = \delta_{j,k} \qquad j,k = 1,2,\ldots,2n$$

The generator of quantum Liouville equation can be expressed as:

$$\hat{\mathcal{L}} = \underline{\hat{a}} \cdot \mathbf{A} \underline{\hat{a}} - A_0 \hat{\mathbb{1}}$$

in terms of $4n$ Hermitian Majorana fermionic maps $\hat{a}_{1,j} := (\hat{c}_j + \hat{c}_j^\dagger)/\sqrt{2}$, $\hat{a}_{2,j} := \mathrm{i}(\hat{c}_j - \hat{c}_j^\dagger)/\sqrt{2}$ satisfying CAR

$$\{\hat{a}_{\nu,j}, \hat{a}_{\mu,k}\} = \delta_{\nu,\mu}\delta_{j,k}, \quad \nu,\mu = 1,2, \;\; j,k = 1,\ldots,2n.$$

where $\mathbf{A}$ is a $4n \times 4n$ complex *structure matrix*

$$\begin{aligned} \mathbf{A} &= -2\mathrm{i}\mathbb{1}_2 \otimes \mathbf{H} - 2\sigma^{\mathrm{y}} \otimes \mathbf{M}_{\mathrm{r}} - 2(\sigma^{\mathrm{x}} - \mathrm{i}\sigma^{\mathrm{z}}) \otimes \mathbf{M}_{\mathrm{i}} \\ \mathbf{M}_{\mathrm{r}} &:= \frac{1}{2}(\mathbf{M} + \bar{\mathbf{M}}) = \mathbf{M}_{\mathrm{r}}^T, \\ \mathbf{M}_{\mathrm{i}} &:= \frac{1}{2\mathrm{i}}(\mathbf{M} - \bar{\mathbf{M}}) = -\mathbf{M}_{\mathrm{i}}^T \end{aligned}$$

where $\mathbf{M} := \sum_\mu \underline{l}_\mu \otimes \underline{\bar{l}}_\mu$ is a positive semidefinite $\mathbf{M} \geq 0$ bath matrix, and $A_0 = 2\,\mathrm{tr}\,\mathbf{M}$.

The key element is a $2n \times 2n$ *real matrix* $\mathbf{X} := -2\mathrm{i}\mathbf{H} + \mathbf{M}_{\mathrm{r}}$ with Jordan canonical form

$$\mathbf{X} = \mathbf{P}\Delta\mathbf{P}^{-1} \tag{1}$$

where $\mathbf{P}$ is a non-singular matrix, and $\Delta = \bigoplus_{j,k} \Delta_{\ell_{j,k}}(\beta_j)$ is a direct sum of

$$\Delta_\ell(\beta) := \begin{pmatrix} \beta & 1 & & \\ & \beta & \ddots & \\ & & \ddots & 1 \\ & & & \beta \end{pmatrix}. \tag{2}$$

# Normal form: decomposition

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Then, the Liouvillean structure matrix $\mathbf{A}$ allows the decomposition:

$$\mathbf{A} = \mathbf{V}^T \begin{pmatrix} \mathbf{0} & \Delta \\ -\Delta^T & \mathbf{0} \end{pmatrix} \mathbf{V}$$

where the *eigenvector matrix*

$$\mathbf{V} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{P}^T(\mathbb{1}_{2n} - 4\mathrm{i}\mathbf{Z}) & -\mathrm{i}\mathbf{P}^T(\mathbb{1}_{2n} - 4\mathrm{i}\mathbf{Z}) \\ \mathbf{P}^{-1} & \mathrm{i}\mathbf{P}^{-1} \end{pmatrix}$$

satisfies the canonical normalization $\mathbf{V}\mathbf{V}^T = \sigma^{\mathrm{x}} \otimes \mathbb{1}_{2n}$, and the $2n \times 2n$ antisymmetric matrix $\mathbf{Z}$ is a solution to the *Lyapunov equation*

$$\mathbf{X}^T\mathbf{Z} + \mathbf{Z}\mathbf{X} = \mathbf{M}_{\mathrm{i}}.$$

Let us name the first $2n$ rows of $\mathbf{V}$ as $\underline{v}_{j,k,l}$, and the last $2n$ rows as $\underline{v}'_{j,k,l}$, which are exactly the generalized eigenvectors pertaining to $k$-th Jordan block of the eigenvalue $\beta_j$, and $-\beta_j$ respectively, and $l = 1, \ldots, \ell_{j,k}$ ($l = 1$ designates the *proper* eigenvector) where $\ell_{j,k}$ is the size of the Jordan block $(j, k)$.

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Then we introduce the *normal master mode* (NMM) maps as

$$\hat{b}_{j,k,l} := \underline{v}_{j,k,l} \cdot \underline{\hat{a}}, \qquad \hat{b}'_{j,k,l} := \underline{v}'_{j,k,l} \cdot \underline{\hat{a}},$$

satisfying the almost-CAR

$$\{\hat{b}_{j,k,l}, \hat{b}_{j',k',l'}\} = 0, \quad \{\hat{b}_{j,k,l}, \hat{b}'_{j',k',l'}\} = \delta_{j,j'}\delta_{k,k'}\delta_{l,l'}, \quad \{\hat{b}'_{j,k,l}, \hat{b}'_{j',k',l'}\} = 0.$$

so the Liouvillean acquires almost-diagonal normal form

$$\hat{\mathcal{L}} = -2\sum_{j,k}\left\{\beta_j \sum_{l=1}^{\ell_{j,k}} \hat{b}'_{j,k,l}\hat{b}_{j,k,l} + \sum_{l=1}^{\ell_{j,k}-1} \hat{b}'_{j,k,l+1}\hat{b}_{j,k,l}\right\}.$$

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There exist two vacua of such a Liouvillean:

- The trivial left vacuum (identity operator), $\langle 1|\hat{b}'_{j,k,l} = 0$,
- And the non-trivial right-vacuum (NESS), $\hat{b}_{j,k,l}|\mathrm{NESS}\rangle = 0$.

# Spectral theorem (J. Stat. Mech. (2010) P07020)

1. The complete spectrum of Liouvillean $\hat{\mathcal{L}}$ is given by the following integer linear combinations

$$\lambda_{\underline{m}} = -2\sum_{j,k} m_{j,k}\beta_j\,, \quad m_{j,k} \in \{0,1,\ldots,\ell_{j,k}\}.$$

2. The $4^n$ dimensional operator space, and its dual (the bra-space), admit the following decomposition $\mathcal{K} = \bigoplus_{\underline{m}} \mathcal{K}_{\underline{m}}, \mathcal{K}' = \bigoplus_{\underline{m}} \mathcal{K}'_{\underline{m}}$ in terms of $\dim \mathcal{K}_{\underline{m}} = \prod_{j,k} \binom{\ell_{j,k}}{m_{j,k}}$ dimensional invariant subspaces $\hat{\mathcal{L}}\mathcal{K}_{\underline{m}} \subseteq \mathcal{K}_{\underline{m}}, \mathcal{K}'_{\underline{m}}\hat{\mathcal{L}} \subseteq \mathcal{K}'_{\underline{m}}$ spanned by

$$\mathcal{K}_{\underline{m}} = L\left\{\prod_{j,k}\prod_{\eta=1}^{m_{j,k}} \hat{b}'_{j,k,l_\eta}|\mathrm{NESS}\rangle;\ 1 \le l_1 < \ldots < l_{m_{j,k}} \le \ell_{j,k}\right\},$$

$$\mathcal{K}'_{\underline{m}} = L\left\{\langle 1|\prod_{j,k}\prod_{\eta=1}^{m_{j,k}} \hat{b}_{j,k,l_\eta};\ 1 \le l_1 < \ldots < l_{m_{j,k}} \le \ell_{j,k}\right\}.$$

3. However, the dimension of the eigenspace (the number of proper eigenvectors corresponding to $\lambda_{\underline{m}}$) is smaller than $\dim \mathcal{K}_{\underline{m}}$ in the nontrivial case when at least one $\ell_{j,k} > 1$. The size of the largest Jordan block corresponding to $\lambda_{\underline{m}}$ is $1 + \sum_{j,k}(\ell_{j,k} - m_{j,k})m_{j,k}$.

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$|\mathrm{NESS}\rangle$ is a unique stationary state of open quantum dynamics if and only if all eigenvalues $\beta_j$ of **X** lie away from the imaginary line $\mathrm{Re}\,\beta_j > 0$.

If this is not the case, then:

1. For each zero rapidity $\beta_j = 0$,

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Then the expectation value of *any quadratic observable* $w_j w_k$ in a (unique) NESS can be explicitly computed as

$$\langle w_j w_k \rangle_{\mathrm{NESS}} = \delta_{j,k} + \langle 1 | \hat{c}_j \hat{c}_k | \mathrm{NESS} \rangle = \delta_{j,k} + 4\mathrm{i} Z_{j,k}$$

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Note an alternative representation of the observables

$$\langle w_j w_k \rangle_{\mathrm{NESS}} = \delta_{j,k} - \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega \, \mathbf{G}_{2j-1,2k-1}(\omega)$$

in terms of the resolvent ("non-equilibrium Green's function")

$$\mathbf{G}(\omega) = (\mathbf{A} - \mathrm{i}\omega \mathbb{1})^{-1}.$$

Consider magnetic and heat transport of a Heisenberg XY spin 1/2 chain, with arbitrary – either homogeneous or positionally dependent (e.g. disordered) – nearest neighbour interaction

$$H = \sum_{m=1}^{n-1} \left( J^{\mathrm{x}}_m \sigma^{\mathrm{x}}_m \sigma^{\mathrm{x}}_{m+1} + J^{\mathrm{y}}_m \sigma^{\mathrm{y}}_m \sigma^{\mathrm{y}}_{m+1} \right) + \sum_{m=1}^{n} h_m \sigma^{\mathrm{z}}_m \tag{3}$$

which is coupled to *two* thermal/magnetic baths *at the ends* of the chain, generated by two pairs of canonical Lindblad operators

$$\begin{aligned} L_1 &= \frac{1}{2}\sqrt{\Gamma^{\mathrm{L}}_1}\sigma^-_1 \qquad L_3 = \frac{1}{2}\sqrt{\Gamma^{\mathrm{R}}_1}\sigma^-_n \\ L_2 &= \frac{1}{2}\sqrt{\Gamma^{\mathrm{L}}_2}\sigma^+_1 \qquad L_4 = \frac{1}{2}\sqrt{\Gamma^{\mathrm{R}}_2}\sigma^+_n \end{aligned} \tag{4}$$

where $\sigma^{\pm}_m = \sigma^{\mathrm{x}}_m \pm \mathrm{i}\sigma^{\mathrm{y}}_m$ and $\Gamma^{\mathrm{L,R}}_{1,2}$ are positive coupling constants related to bath temperatures/magnetizations. e.g. if spins were non-interacting the bath temperatures $T_{\mathrm{L,R}}$ would be given with $\Gamma^{\mathrm{L,R}}_2/\Gamma^{\mathrm{L,R}}_1 = \exp(-2h_{1,\mathrm{n}}/T_{\mathrm{L,R}})$.

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Similar models were recently considered e.g. in Karevski and Platini PRL 2009, and Clark, Prior, Hartmann, Jaksch and Plenio, PRL2009 & 0907.5582

$$\mathbf{A} = \begin{pmatrix} \mathbf{B}_{\rm L} - h_1\mathbf{R} & \mathbf{R}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ -\mathbf{R}_1^T & -h_2\mathbf{R} & \mathbf{R}_2 & \ddots & \mathbf{0} \\ \mathbf{0} & -\mathbf{R}_2^T & -h_3\mathbf{R} & & \vdots \\ \vdots & \ddots & & \ddots & \mathbf{R}_{n-1} \\ \mathbf{0} & \mathbf{0} & \cdots & -\mathbf{R}_{n-1}^T & \mathbf{B}_{\rm R} - h_n\mathbf{R} \end{pmatrix}, \qquad A_0 = \Gamma_+^{\rm L} + \Gamma_+^{\rm R},$$

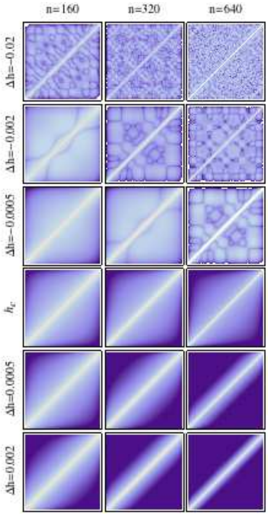
where $\mathbf{B}_{\rm L} := \mathbf{B}_{\Gamma_+^{\rm L},\Gamma_-^{\rm L}}$, $\mathbf{B}_{\rm R} := \mathbf{B}_{\Gamma_+^{\rm R},\Gamma_-^{\rm R}}$, $\Gamma_\pm^{\rm L,R} := \Gamma_2^{\rm L,R} \pm \Gamma_1^{\rm L,R}$, and

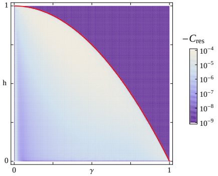
$$\mathbf{R}_m := \begin{pmatrix} 0 & 0 & J_m^{\rm y} & 0 \\ 0 & 0 & 0 & J_m^{\rm y} \\ -J_m^{\rm x} & 0 & 0 & 0 \\ 0 & -J_m^{\rm x} & 0 & 0 \end{pmatrix}, \qquad \mathbf{R} := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

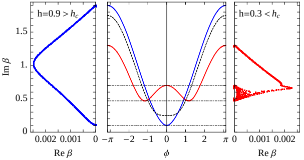
$$\mathbf{B}_{\Gamma_+,\Gamma_-} := \begin{pmatrix} 0 & \frac{\rm i}{2}\Gamma_+ & -\frac{\rm i}{2}\Gamma_- & \frac{1}{2}\Gamma_- \\ -\frac{\rm i}{2}\Gamma_+ & 0 & \frac{1}{2}\Gamma_- & \frac{\rm i}{2}\Gamma_- \\ \frac{\rm i}{2}\Gamma_- & -\frac{1}{2}\Gamma_- & 0 & \frac{\rm i}{2}\Gamma_+ \\ -\frac{1}{2}\Gamma_- & -\frac{\rm i}{2}\Gamma_- & -\frac{\rm i}{2}\Gamma_+ & 0 \end{pmatrix}$$

TP & I. Pižorn, PRL **101**, 105701 (2008)

$$
\begin{aligned}
J^x_m &= (1+\gamma)/2 \\
J^y_m &= (1-\gamma)/2, \\
h_m &= h \\
C(j,k) &= \langle \sigma^z_j \sigma^z_k \rangle - \langle \sigma^z_j \rangle \langle \sigma^z_k \rangle
\end{aligned}
$$




$$h_c = 1 - \gamma^2$$

# Spectral gap of Liouvillean

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We find explicit analytical result:

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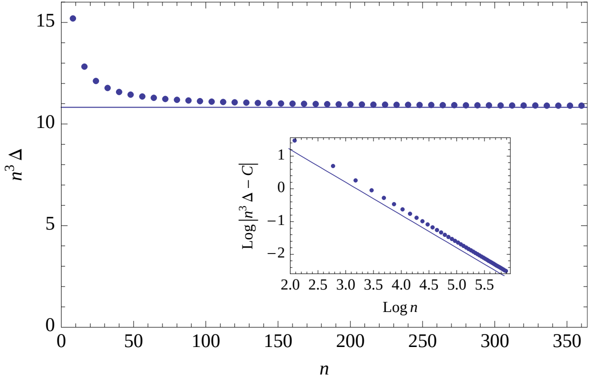
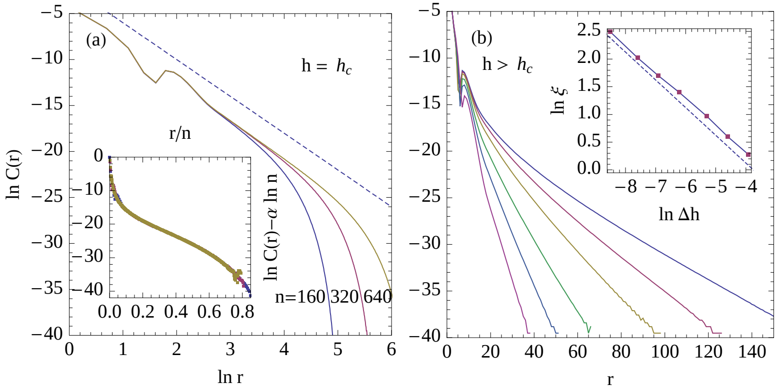
Comparing to numerics:



Saturation vs. exponential decay & power law critical scaling at the critical point.



Near QPT: Scaling variable $z = (h_c - h)n^2$

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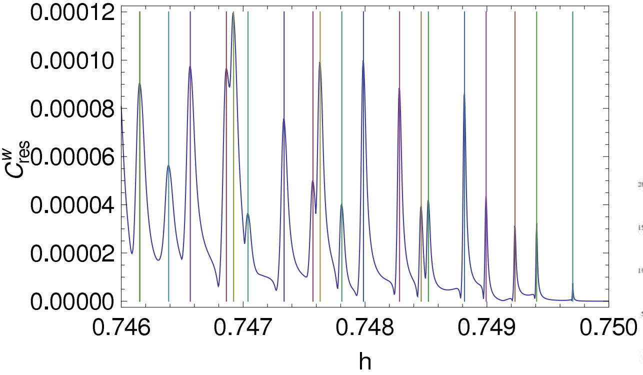
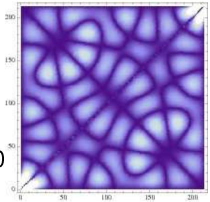
Scaling ansatz: $C_{2j+\alpha,2k+\beta} = \Psi^{\alpha,\beta}(x = j/n, y = k/n, z)$

# Fluctuation of spin-spin correlation in NESS and "wave resonators"

Near QPT: Scaling variable $z = (h_c - h)n^2$
Scaling ansatz: $C_{2j+\alpha,2k+\beta} = \Psi^{\alpha,\beta}(x = j/n, y = k/n, z)$
Certain combination $\Psi(x,y) = (\partial/\partial_x + \partial/\partial_y)(\Psi^{0,0}(x,y) + \Psi^{1,1}(x,y))$ obeys Helmoltz equation!!!

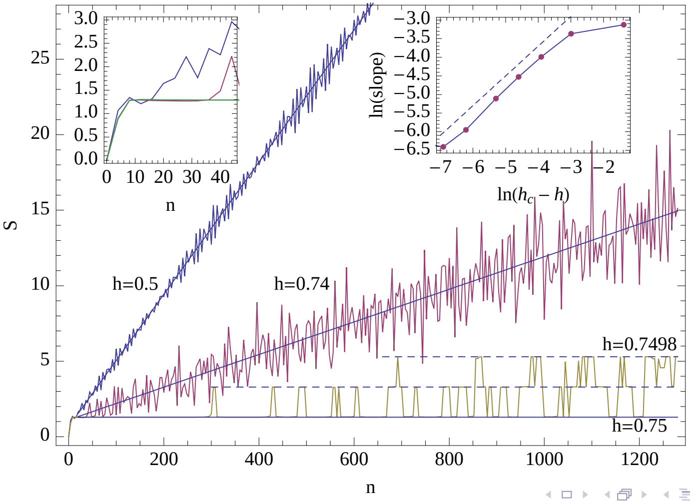
$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 4z\right)\Psi = \text{"octopole antenna sources"}$$



Von Neumann entropy of a bipartition of NESS as an element of a Fock space

Drastically different behaviour than for entanglement entropy of ground states of 1D critical/non-critical models!



Now we discuss the situation where the Hamiltonian and the set of Lindblad operators are *translationally invariant* (periodic), i.e.

$$
\begin{aligned}
H_{2j-1+\nu,2j'-1+\nu'} &=: h_{\nu,\nu'}(j-j'), \quad j,j'=1,\ldots,n, \quad \nu,\nu'=0,1 \\
l_{(\lambda,k),2j-1+\nu} &=: \omega_{(\lambda,\nu),j-k}, \qquad k=1,\ldots,n.
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The Hermitian *bath matrix* $\mathbf{M}=\sum_\mu \underline{l}_\mu \otimes \overline{\underline{l}_\mu}$ is, similarly to Hamiltonian, block $(2\times 2)$ circulant, $M_{2j-1+\nu,2j'-1+\nu'}=m_{\nu,\nu'}(j-j')$. Denoting 2-vectors $\underline{\omega}_{\lambda,k}=\begin{pmatrix}\omega_{(\lambda,0),k}\\ \omega_{(\lambda,1),k}\end{pmatrix}$ we write its $2\times 2$ blocks compactly in terms of a convolution

$$\mathbf{m}(j)=\sum_\lambda\sum_k \underline{\omega}_{\lambda,j+k}\otimes\overline{\underline{\omega}_{\lambda,k}}.$$

# Translationally invariant quasi-free fermionic semi-groups

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$$\begin{aligned} H_{2j-1+\nu,2j'-1+\nu'} &=: h_{\nu,\nu'}(j-j'), \quad j,j'=1,\ldots,n, \quad \nu,\nu'=0,1 \\ l_{(\lambda,k),2j-1+\nu} &=: \omega_{(\lambda,\nu),j-k}, \qquad k=1,\ldots,n. \end{aligned}$$

The Hermitian *bath matrix* $\mathbf{M}=\sum_\mu \underline{l}_\mu \otimes \overline{\underline{l}_\mu}$ is, similarly to Hamiltonian, block $(2\times 2)$ circulant, $M_{2j-1+\nu,2j'-1+\nu'}=m_{\nu,\nu'}(j-j')$. Denoting 2-vectors $\underline{\omega}_{\lambda,k}=\begin{pmatrix}\omega_{(\lambda,0),k}\\ \omega_{(\lambda,1),k}\end{pmatrix}$ we write its $2\times 2$ blocks compactly in terms of a convolution

$$\mathbf{m}(j)=\sum_\lambda\sum_k \underline{\omega}_{\lambda,j+k}\otimes\overline{\underline{\omega}_{\lambda,k}}.$$

Let us now define the *symbols*, the Fourier transformations of $2\times 2$ blocks and a $2-$vector

$$\begin{aligned} \tilde{\mathbf{h}}(\varphi) &:= \sum_{j\in\mathbb{Z}} \mathbf{h}(j)\exp(-\mathrm{i}\varphi j), \qquad \varphi\in[-\pi,\pi) \\ \underline{\tilde{\omega}}_\lambda(\varphi) &:= \sum_{j\in\mathbb{Z}} \underline{\omega}_{\lambda,j}\exp(-\mathrm{i}\varphi j), \\ \tilde{\mathbf{m}}(\varphi) &:= \sum_{j\in\mathbb{Z}} \mathbf{m}(j)\exp(-\mathrm{i}\varphi j)=\sum_\lambda \underline{\tilde{\omega}}_\lambda(\varphi)\otimes\overline{\underline{\tilde{\omega}}_\lambda(\varphi)}. \end{aligned}$$

Now, for a translationally invariant system, the spectrum of $\mathbf{X}$ is given by the two *Bloch bands* $\beta_\tau(\varphi)$, determined by the two eigenvalues of the $2 \times 2$ matrix valued symbol of $\mathbf{X}$, $\tilde{\mathbf{x}}(\varphi) = -2\mathrm{i}\tilde{\mathbf{h}}(\varphi) + 2\tilde{\mathbf{m}}_{\mathrm{r}}(\varphi)$. Since the correlation matrix is circulant as well

$$\mathrm{tr}\,\rho_{\mathrm{NESS}} w_{2j-1+\nu} w_{j'-1+\nu'} = \delta_{j,j'}\delta_{\nu,\nu'} + 4\mathrm{i} z_{\nu,\nu'}(j-j'),$$

the solution can be encoded in the symbol of the correlator

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The later satisfies a $2 \times 2$ matrix equation, obtained by block Fourier transforming the Lyapunov equation

$$\tilde{\mathbf{x}}^T(-\varphi)\tilde{\mathbf{z}}(\varphi) + \tilde{\mathbf{z}}(\varphi)\tilde{\mathbf{x}}(\varphi) = \tilde{\mathbf{m}}_{\mathrm{i}}(\varphi)$$

which is in fact a $4 \times 4$ linear system for elements of $\tilde{\mathbf{z}}(\varphi)$ (at fixed $\varphi$) which is solved explicitly.

Correlations decay *exponentially* $\mathbf{z}(j) = \mathcal{O}(\exp(-|j|/\xi))$ if $\tilde{\mathbf{z}}(\varphi)$ is analytic around the strip $|\mathrm{Im}\varphi| < \xi$. Note that that $\xi$ is always finite, but may not be bounded!

Let $H$ be fermionized XY spin chain with anysotropy $\gamma$ and magnetic field, and take the most general translationally invariant local noise with one Lindblad operator per site,

$$L_j = \epsilon_1(c_j + c_j^\dagger) + \epsilon_2 e^{\mathrm{i}\theta}\mathrm{i}(c_j - c_j^\dagger) = \epsilon_1 w_{2j-1} + \epsilon_2 e^{\mathrm{i}\theta} w_{2j}$$

parametrized with a triple of real parameters $\epsilon_1 > 0, \epsilon_2 > 0, \theta \in [0, \pi]$.

# Example 1: Spatially incoherent noise

Let $H$ be fermionized XY spin chain with anysotropy $\gamma$ and magnetic field, and take the most general translationally invariant local noise with one Lindblad operator per site,

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parametrized with a triple of real parameters $\epsilon_1 > 0, \epsilon_2 > 0, \theta \in [0,\pi]$. Then, the procedure above results in

$$\tilde{\mathbf{z}}(\varphi) = \frac{1}{d}\begin{pmatrix} a & b \\ -\bar{b} & c \end{pmatrix}$$

where $a, b, c, d$ are some trigonometric polynomials of $\varphi$, and in particular

$$d^* = \min_\varphi d(\varphi) = 2(\epsilon_1^2 + \epsilon_2^2)^2((|h| - 1)^2 + \epsilon_1^2\epsilon_2^2 \sin^2\theta)$$

meaning that the correlation length $\xi$ can diverge, only if the non-noisy model is critical $|h| = 1$ *and* if the noise satisfies the condition $\epsilon_1 = \epsilon_2, \theta \in \{0, \pi\}$.

As a second example we consider a special case of "coherent noise", namely Lindblad operators which couple two neighboring sites. Again, we take XY hamiltonian **H** and a single Linbdlad operator per site of the form

$$L_j = \epsilon_1 (c_j + c_j^\dagger) + \epsilon_2 e^{\mathrm{i}\theta} (c_{j+1} + c_{j+1}^\dagger) = \epsilon_1 w_{2j-1} + \epsilon_2 e^{\mathrm{i}\theta} w_{2j+1}$$

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The symbol of the correlator has now a simple general form

$$\tilde{\mathbf{z}}(\varphi) = \frac{\mathrm{i}\epsilon_1\epsilon_2 \sin\theta \sin\varphi}{\epsilon_1^2 + \epsilon_2^2 + 2\epsilon_1\epsilon_2 \cos\theta \cos\varphi} \mathbb{1}_2.$$

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The correlation exponent $\xi$ can be estimated from the location of the singularity as

$$\xi = \mathrm{Im}\,\operatorname{arccos} \frac{\epsilon_1^2 + \epsilon_2^2}{2\epsilon_1\epsilon_2 \cos\theta} = \mathrm{arcosh} \frac{\epsilon_1^2 + \epsilon_2^2}{2\epsilon_1\epsilon_2 \cos\theta}.$$

Correlation length **diverges** $\xi \to \infty$ when $\epsilon_1 = \epsilon_2$ and $\theta \to 0, \pi$, and **does not** depend on hamiltonian parameters at all!

Very similar development can be done for quasi-free bosonic case...

**First, some numerics to get the flavor of what is going on:**
tDMRG simulations of NESS for locally interacting boundary driven spin chains (method as described in TP & M. Žnidarič, JSTAT 2009).

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Example, toy model: Locally boundary driven XXZ spin 1/2 chain:

$$H = \sum_{j=1}^{n-1} h_{[j,j+1]}, \quad h_{[j,j+1]} = (\sigma_j^{\mathrm{x}}\sigma_{j+1}^{\mathrm{x}} + \sigma_j^{\mathrm{y}}\sigma_{j+1}^{\mathrm{y}} + \Delta\sigma_j^{\mathrm{z}}\sigma_{j+1}^{\mathrm{z}}) + \frac{1}{2}B(-1)^j(\sigma_j^{\mathrm{z}} + \sigma_{j+1}^{\mathrm{z}})$$

and symmetric magnetic-Lindblad boundary driving:

$$\begin{aligned} L_1^{\mathrm{L}} &= \sqrt{\frac{1}{2}(1-\mu)\Gamma}\sigma_1^+, \quad L_1^{\mathrm{R}} = \sqrt{\frac{1}{2}(1+\mu)\Gamma}\sigma_n^+, \\ L_2^{\mathrm{L}} &= \sqrt{\frac{1}{2}(1+\mu)\Gamma}\sigma_1^-, \quad L_2^{\mathrm{R}} = \sqrt{\frac{1}{2}(1-\mu)\Gamma}\sigma_n^-. \end{aligned}$$

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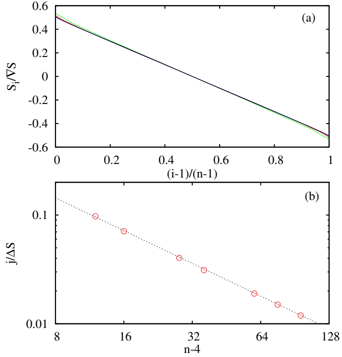
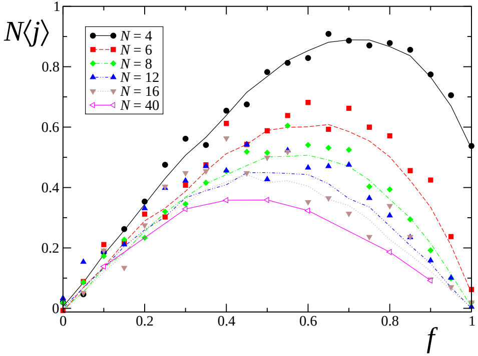
$$L_2^{\mathrm{L}} = \sqrt{\frac{1}{2}(1+\mu)\Gamma}\sigma_1^-, \quad L_2^{\mathrm{R}} = \sqrt{\frac{1}{2}(1-\mu)\Gamma}\sigma_n^-.$$

$H$ **integrable** if $B = 0$ and **non-integrable** if $B \neq 0$.

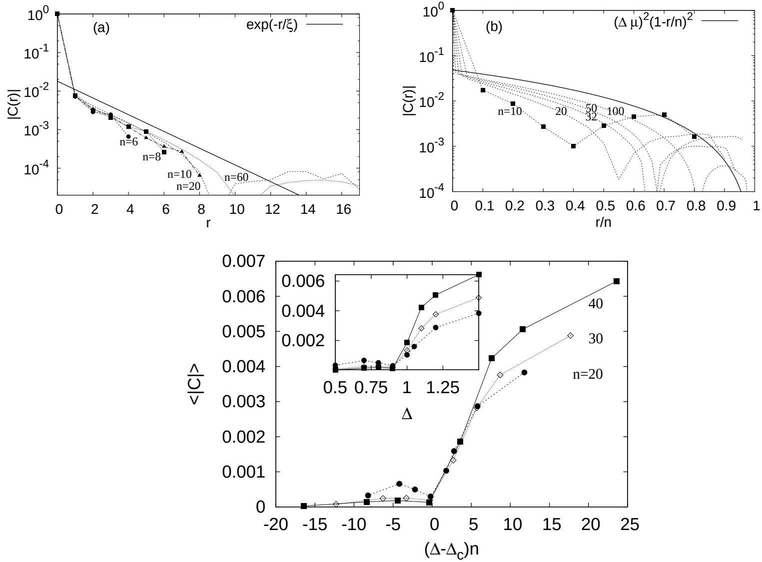
If $\Delta > 1$ (arbitrary $B$) the model exhibits diffusive transport for **small driving**, and negative differential conductance for **large driving** $\mu$.



# Transition to long-range order in NESS (PRL **105**, 060603 (2010))

$$C(r) = \langle \sigma^{\rm z}_{(n+r)/2} \sigma^{\rm z}_{(n-r)/2} \rangle - \langle \sigma^{\rm z}_{(n+r)/2} \rangle \langle \sigma^{\rm z}_{(n-r)/2} \rangle$$



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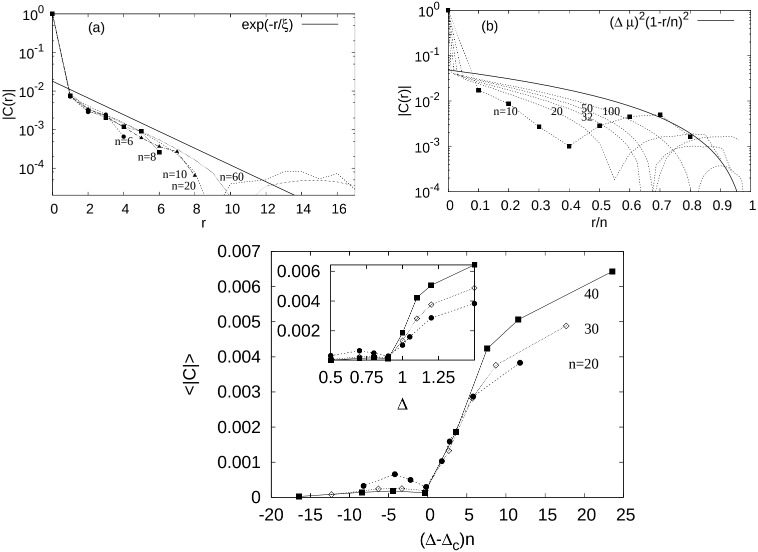


Critical anisotropy appears to be $\Delta_c \approx 0.91$ (!?)

We conclude by giving some exact results on NESS, as results of 'wild' guessing...

# XXZ spin 1/2 chain for weak coupling (small $\Gamma$) and strong driving $\mu = 1$

$$\rho_{\rm NESS} = \mathbb{1} + \Gamma(Z' - Z'') + \frac{1}{2}\Gamma^2(Z'^2 - 2Z'Z'' + Z''^2) + \mathcal{O}(\Gamma^3)$$

$$Z' = \sum_{s_1,s_2,\ldots,s_n\in\{-,0,+\}} (\mathbf{e}_{\rm L}\cdot \mathbf{A}_{s_1}\mathbf{A}_{s_2}\cdots\mathbf{A}_{s_n}\mathbf{e}_{\rm R})\prod_{j=1}^{n}\sigma_j^{s_j}$$

$$Z'' = \sum_{s_1,s_2,\ldots,s_n\in\{-,0,+\}} (\mathbf{e}_{\rm L}\cdot \mathbf{A}_{-s_1}\mathbf{A}_{-s_2}\cdots\mathbf{A}_{-s_n}\mathbf{e}_{\rm R})\prod_{j=1}^{n}\sigma_j^{s_j}$$

are Matrix Product Operators, w.r.t. auxiliary space basis $\{\mathbf{e}_{\rm L},\mathbf{e}_{\rm R},\mathbf{e}_1,\mathbf{e}_2,\ldots\}$

$$\mathbf{A}_0 = \mathbf{e}_{\rm L}\otimes\mathbf{e}_{\rm L} + \mathbf{e}_{\rm R}\otimes\mathbf{e}_{\rm R} + \sum_{k=1}^{\infty} T_k(\Delta)\mathbf{e}_k\otimes\mathbf{e}_k,$$

$$\mathbf{A}_+ = \mathbf{e}_{\rm L}\otimes\mathbf{e}_1 + \sqrt{2\Delta(\Delta^2-1)}\sum_{k=1}^{\infty} U^{(0)}_{\lfloor (k-1)/2\rfloor}(2\Delta^2-1)\,\mathbf{e}_k\otimes\mathbf{e}_{k+1},$$

$$\mathbf{A}_- = \mathbf{e}_1\otimes\mathbf{e}_{\rm R} + \sqrt{2\Delta(\Delta^2-1)}\sum_{k=1}^{\infty} U^{(1)}_{\lfloor k/2\rfloor}(2\Delta^2-1)\,\mathbf{e}_{k+1}\otimes\mathbf{e}_k,$$

$$T_0(x)=1,\ T_1(x)=x,\quad T_j(x) = 2xT_{j-1}(x) - T_{j-2}(x),$$

$$U^{(m)}_0(x)=1,\ U^{(m)}_1(x)=2x+m,\quad U^{(m)}_j(x) = 2xU^{(m)}_{j-1}(x) - U^{(m)}_{j-2}(x).$$

Take *boundary driven* XX spin chain ($\Delta = 0$) and in addition put local bulk dephasing with Lindblads $L_j = \gamma \sigma^{\mathrm{z}}_j$. [M. Žnidarič, JSTAT, L05002 (2010)]

$$\rho_{\mathrm{NESS}} = \mathbb{1} + \sum_{j=1}^{n} a_j \sigma^{\mathrm{z}} + b \sum_{j=1}^{n-1} J_j + \mathcal{O}(\mu^2)$$

where $J_j = \sigma^{\mathrm{x}}_j \sigma^{\mathrm{y}}_{j+1} - \sigma^{\mathrm{y}}_j \sigma^{\mathrm{x}}_{j+1}$ is the spin current and

$$a_1 = -b/\Gamma - \mu,\ a_j = -b(1/\Gamma + \Gamma + 2\gamma(j-1)) - \mu,\ a_n = -b(1/\Gamma + 2\Gamma + 2(n-1)\gamma) - \mu,$$

$$b = -\frac{\mu}{\Gamma + 1/\Gamma + (n-1)\gamma}.$$

# XX spin 1/2 chain with bulk dephasing: exact diffusive NESS

Take *boundary driven* XX spin chain ($\Delta = 0$) and in addition put local bulk dephasing with Lindblads $L_j = \gamma \sigma_j^{\mathrm{z}}$. [M. Žnidarič, JSTAT, L05002 (2010)]

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$$b = -\frac{\mu}{\Gamma + 1/\Gamma + (n-1)\gamma}.$$

The solution yields the spin Fick's law (spin diffusion),
$\langle(\sigma_j^{\mathrm{z}} - \sigma_k^{\mathrm{z}})\rangle \propto \frac{\mu(j-k)}{n}$, $\langle J_j \rangle \propto \frac{\mu}{n}$.

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The solution yields the spin Fick's law (spin diffusion),
$\langle(\sigma^z_j - \sigma^z_k)\rangle \propto \frac{\mu(j-k)}{n}$, $\langle J_j\rangle \propto \frac{\mu}{n}$.
The higher orders, say $\mathcal{O}(\mu^2)$ have also been calculated analytically and predict *'hydrodynamic long range order'* [observed in nonequilibrium classical exclusion processes (see e.g. Derrida JSTAT 2007)]

$$C_{j=xn,k=yn} = \frac{(2\mu)^2}{n} x(1-y)$$

# Conclusion, perspective

- *Long range order* seems to be abundant in quasi-free and interacting one dimensional quantum systems far from equilibrium
- Perhaps a systematic theory of *integrable* (interacting) many-body dynamical semigroups can be developed