Aspects of $\mathcal{W}$-Extended Logarithmic Minimal Models

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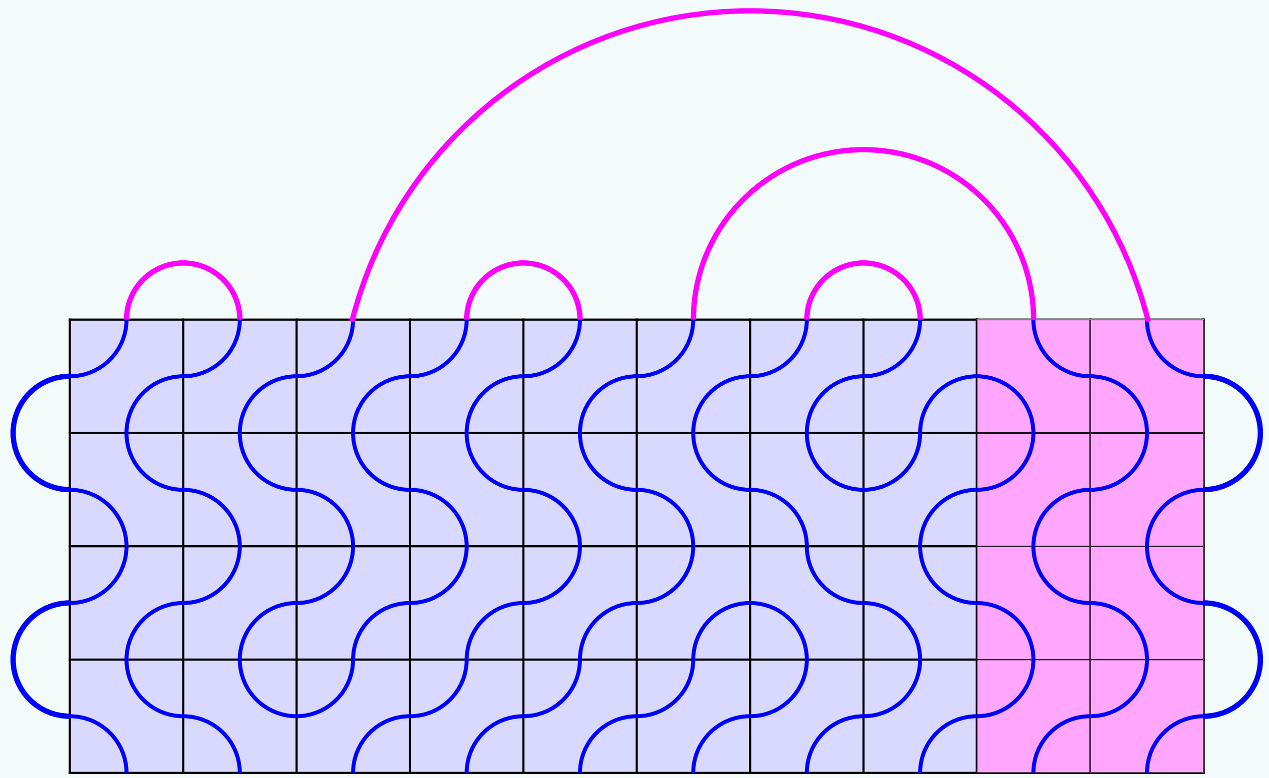


Collaborators: Paul A. Pearce, Philippe Ruelle, Yvan Saint-Aubin, Jean-Bernard Zuber

Some Background and Motivation

- A **Log CFT** may be characterized by the presence of a **non-diagonalizable** L_0

$$L_0 \begin{pmatrix} |\Psi\rangle \\ |\Phi\rangle \end{pmatrix} = \begin{pmatrix} \Delta & 1 \\ 0 & \Delta \end{pmatrix} \begin{pmatrix} |\Psi\rangle \\ |\Phi\rangle \end{pmatrix} = \begin{pmatrix} \Delta|\Psi\rangle + |\Phi\rangle \\ \Delta|\Phi\rangle \end{pmatrix}$$

(here a Jordan block of rank 2)

For so-called quasi-primary fields, the global conformal Ward identities then imply

$$\langle\Phi(z)\Phi(w)\rangle = 0, \qquad \langle\Phi(z)\Psi(w)\rangle = \frac{A}{(z-w)^{2\Delta}}, \qquad \langle\Psi(z)\Psi(w)\rangle = \frac{B - 2A\ln(z-w)}{(z-w)^{2\Delta}}$$

Conventional lattice approaches to CFT (Potts models, RSOS models, . . .)

local degrees of freedom $\Rightarrow$ symmetric transfer matrices $\Rightarrow$ diagonalizable transfer matrices $\Rightarrow$ no rank ≥ 2 Virasoro representations $\Rightarrow$ non-logarithmic CFT

- Statistical systems with local "point" degrees of freedom yield rational CFTs.

Paradigm shift

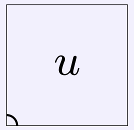
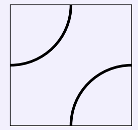
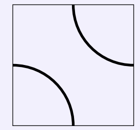
- Polymers and percolation have non-local stringy degrees of freedom (polymers, connectivities) and are associated with Logarithmic CFTs.

logarithmic theory $\Rightarrow$ non-local degrees of freedom

- Logarithmic CFTs have arisen, or found applications, in statistical systems such as polymers, percolation, symplectic fermions, the Abelian sandpile model, etc., as well as in string theory, AdS/CFT correspondence, chiral gravity, WZW models, quantum Hall effect, etc.

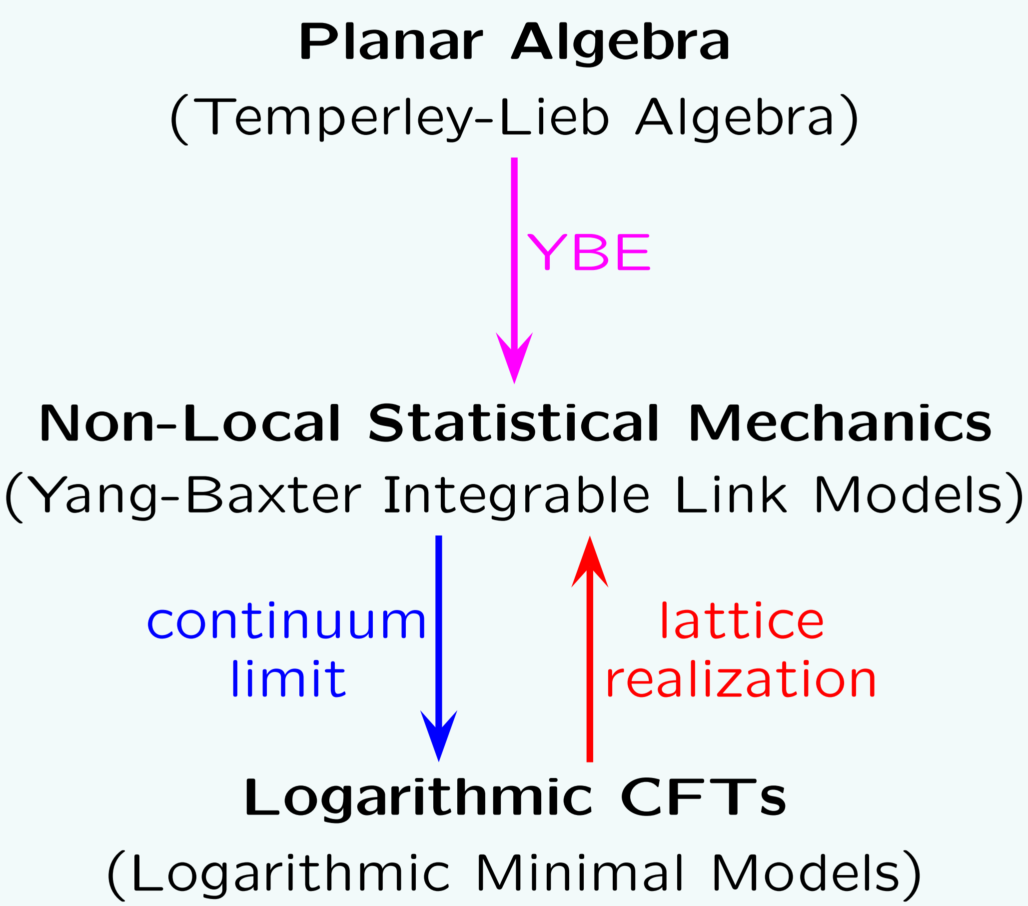
Logarithmic Minimal Models $\mathcal{LM}(p,p')$

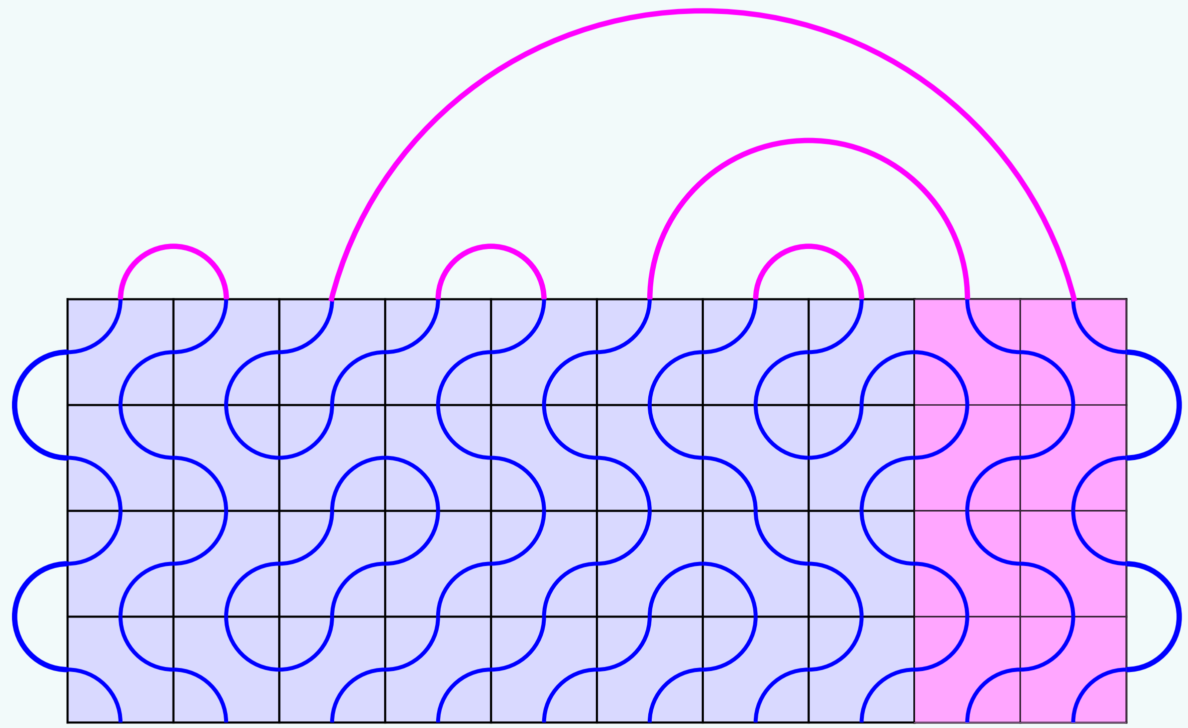
Face operators defined in planar Temperley-Lieb algebra (Jones 1999)

$X(u) =$  $= \dfrac{\sin(\lambda-u)}{\sin\lambda}$  $+ \dfrac{\sin u}{\sin\lambda}$  ; $X_j(u) = \dfrac{\sin(\lambda-u)}{\sin\lambda}\, I + \dfrac{\sin u}{\sin\lambda}\, e_j$

$1 \le p < p'$ coprime integers, $\lambda = \dfrac{(p'-p)\pi}{p'}$ = crossing parameter

u = spectral parameter, $\beta = 2\cos\lambda$ = fugacity of loops

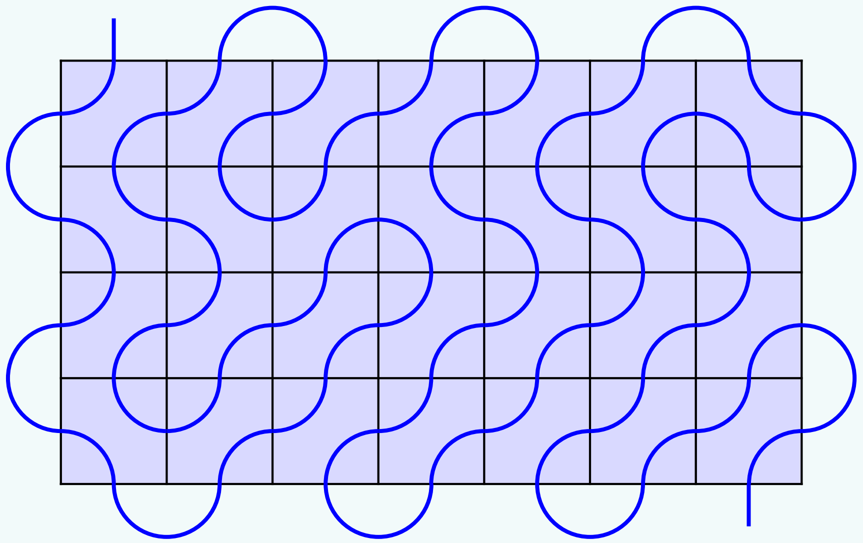




non-local degrees of freedom

Polymers and Percolation on the Lattice

- **Critical dense polymers** $(p,p')=(1,2), \qquad \lambda=\frac{\pi}{2}$

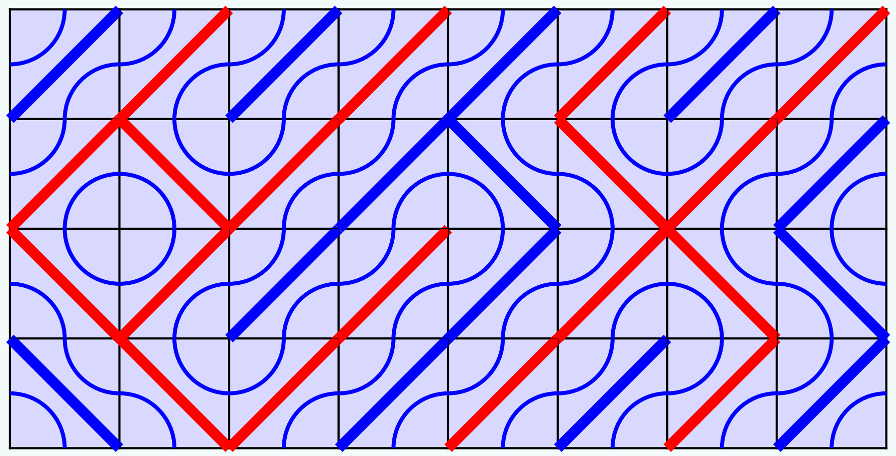


$$d_{path}^{SLE}=2-2\Delta_{p,p'-1}=2, \qquad \kappa=\frac{4p'}{p}=8$$

$\Delta_{1,1}=0$ lies outside rational $\mathcal{M}(1,2)$ Kac table

$\beta=0 \;\Rightarrow\;$ no loops $\;\Rightarrow\;$ space-filling dense polymer

- **Critical percolation** $(p,p')=(2,3), \qquad \lambda=\frac{\pi}{3}, \qquad u=\frac{\lambda}{2}=\frac{\pi}{6}$ (isotropic)



$$d_{path}^{SLE}=2-2\Delta_{p,p'-1}=\frac{7}{4}, \qquad \kappa=\frac{4p'}{p}=6$$

$\Delta_{2,2}=\frac{1}{8}$ lies outside rational $\mathcal{M}(2,3)$ Kac table

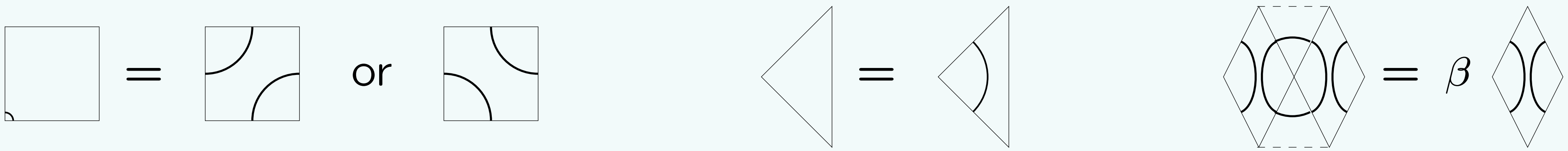
Bond percolation on the blue square lattice:

Critical probability $=p_c=\sin(\lambda-u)=\sin u=\frac{1}{2}$

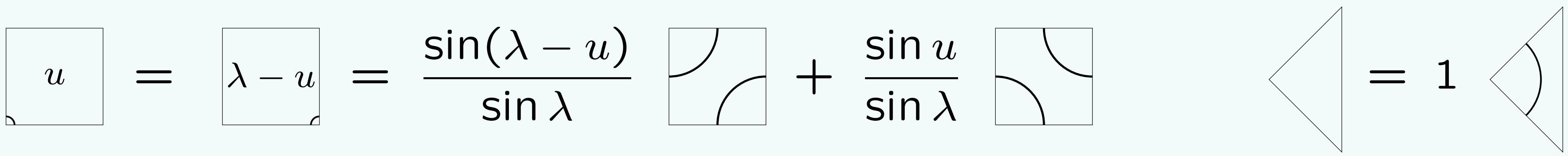
$\beta=1 \qquad\Rightarrow\qquad$ local stochastic process

Planar Temperley-Lieb Algebra

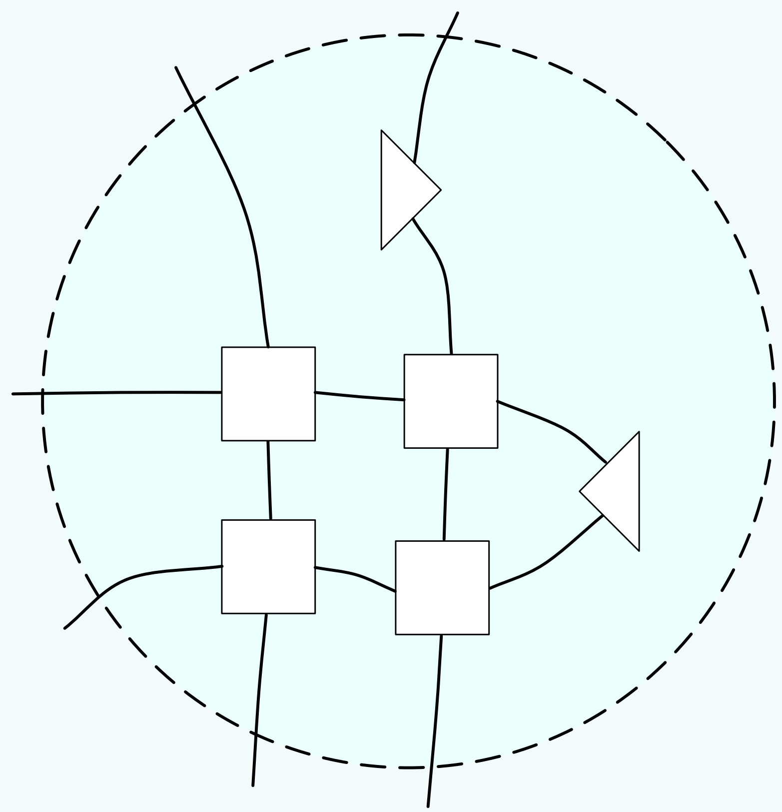
- The planar Temperley-Lieb algebra is a **diagrammatic algebra** generated by elementary 2-boxes (oriented monoids) and elementary 1-triangles



- The 2-boxes and 1-triangles occur with weights given by

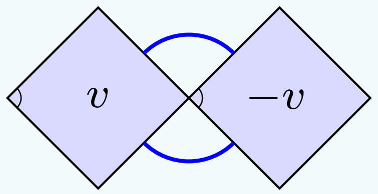


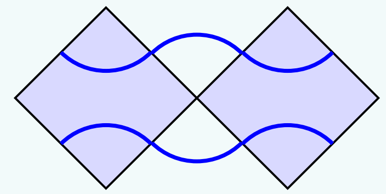
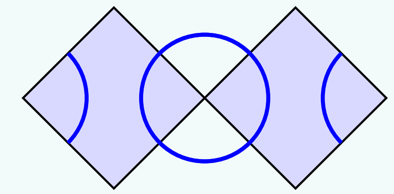
- Example 3-tangle: Any 3 consecutive strings can be taken as "in-states", the other 3 are then "out-states". As a planar operator, the 3-tangle can act in "6 different directions".

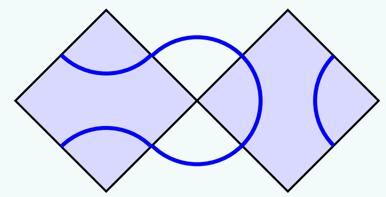
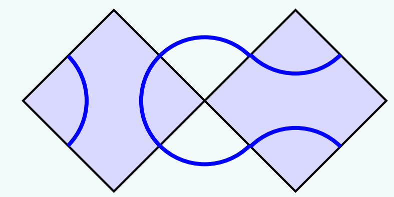


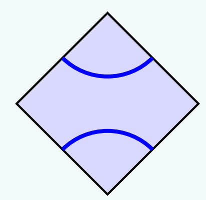
- Two N-tangles are equal if they have the same connectivities with the same weights.

Local Inversion Relation



$$= \frac{\sin(\lambda - v)\sin(\lambda + v)}{\sin^2\lambda}$$  $$+ \frac{\sin v \sin(-v)}{\sin^2\lambda}$$ 

$$+ \frac{\sin(\lambda - v)\sin(-v)}{\sin^2\lambda}$$  $$+ \frac{\sin v \sin(\lambda + v)}{\sin^2\lambda}$$ 

$$= \frac{\sin(\lambda - v)\sin(\lambda + v)}{\sin^2\lambda}$$ 

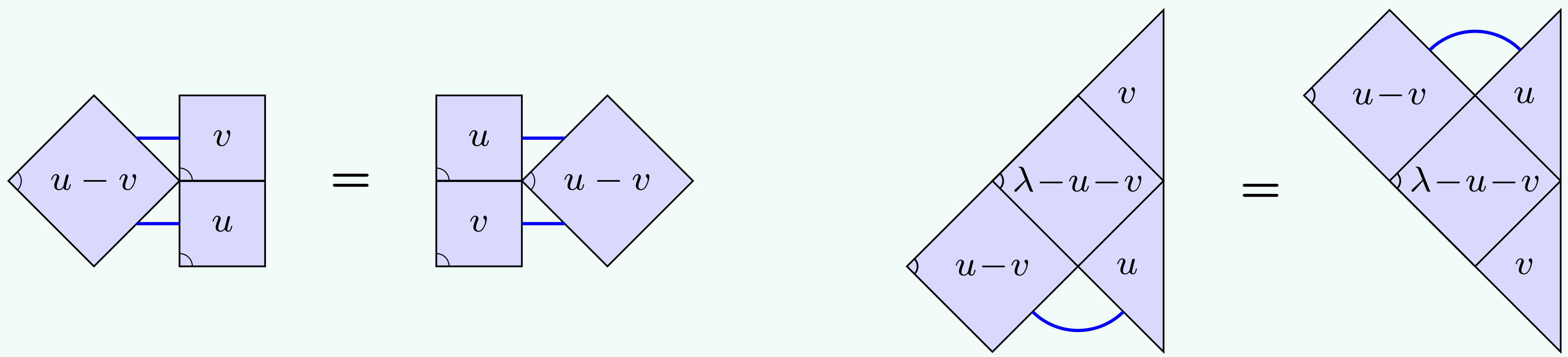
since

$$\beta \sin v \sin(-v) + \sin(\lambda - v)\sin(-v) + \sin v \sin(\lambda + v) \;=\; 0$$

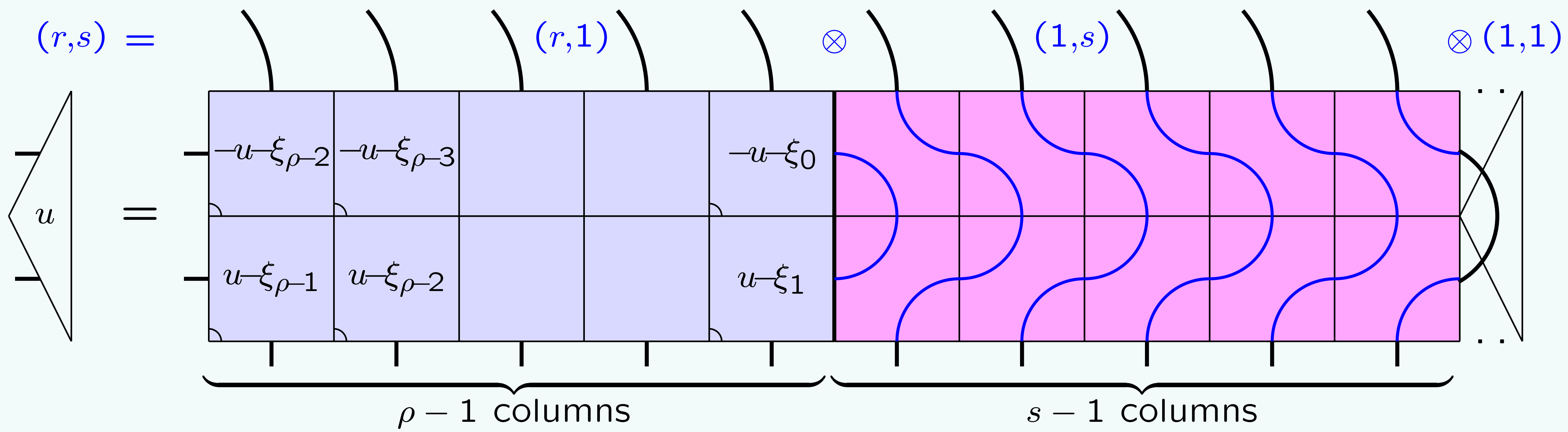
This equality is an equality of 2-tangles.

Yang-Baxter Equations and Boundary Conditions

Yang-Baxter equations



- Equality is the equality of N-tangles.
- (r, s) **solution** $(r, s \in \mathbb{N}$, ρ is related to r, and ξ_k is linear in $\lambda)$

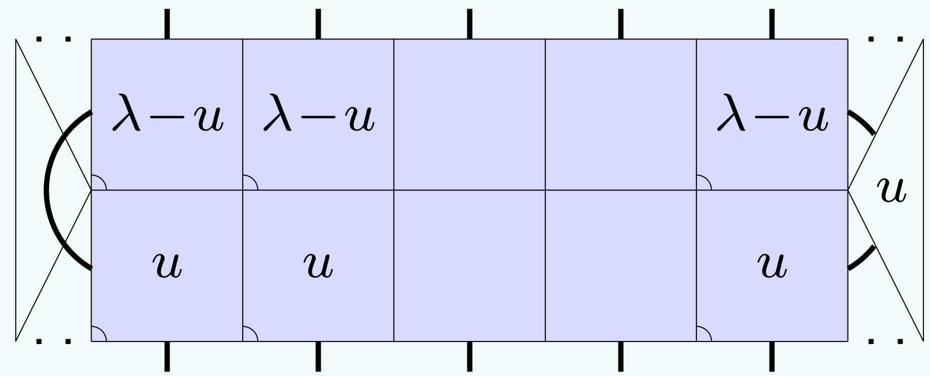


- Left boundary conditions are constructed similarly.

Double-Row Transfer Matrix

- For a strip with N columns, the double-row transfer "matrix" is the N-tangle

$\boldsymbol{D}(u) =$



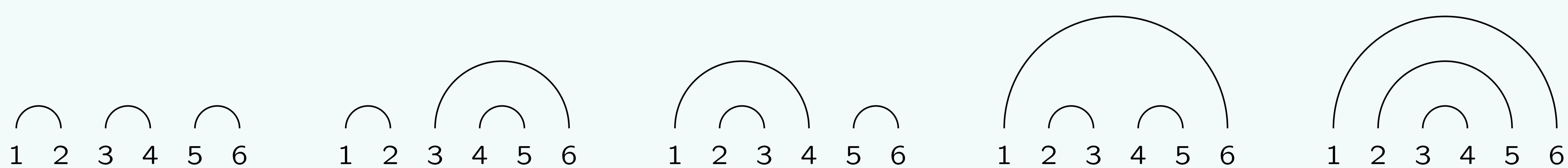
- Using the Yang-Baxter and Boundary Yang-Baxter Equations in the planar Temperley-Lieb algebra, it can be shown that, for any (r,s), the double-row transfer tangles **commute** and are **crossing symmetric**

$$\boldsymbol{D}(u)\boldsymbol{D}(v) = \boldsymbol{D}(v)\boldsymbol{D}(u), \qquad \boldsymbol{D}(u) = \boldsymbol{D}(\lambda - u)$$

- Multiplication is vertical concatenation of diagrams.
- **Matrix realizations** and their spectra are obtained by acting on **vector spaces**.

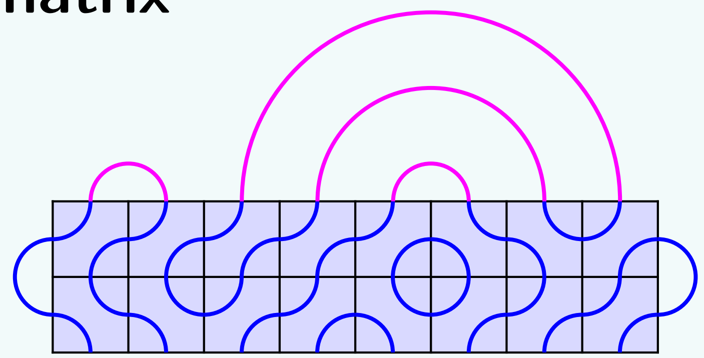
Planar link diagrams

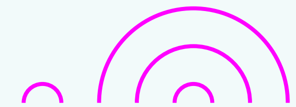
- The planar N-tangles act on a vector space $\mathcal{V}_N$ of **planar link diagrams**. The dimension of $\mathcal{V}_N$ is given by Catalan numbers. For $N = 6$, there is a basis of 5 link diagrams:



Link States and Defects

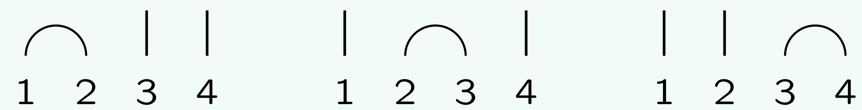
Transfer matrix



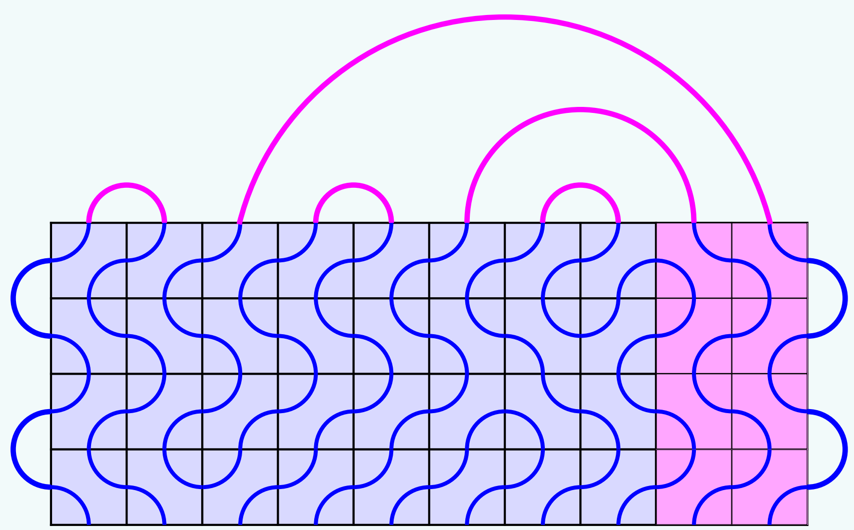
initial state: 

resulting state: β^2 

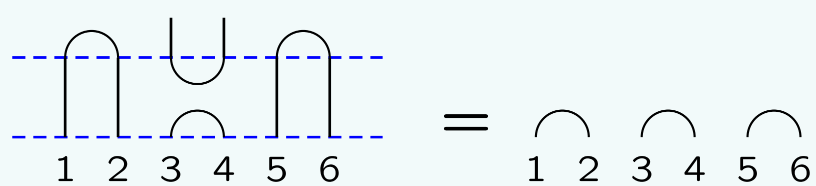
- Generally, the vector space of states $\mathcal{V}_N^{(\ell)}$ can contain ℓ **defects**

$N = 4, \; \ell = 2:$ 

- The ℓ defects can be closed on the right or the left. In this way, the number of defects propagating in the bulk is controlled by the boundary conditions. For a $(1, s)$ boundary condition, in particular, the $\ell = s - 1$ defects simply propagate along the boundary



- Defects in the bulk can be annihilated in pairs but not created under the action of TL



etc.

- The transfer matrices are thus **block-triangular** with respect to the number of defects.

Conformal Field Theory and Kac Representations

- With only one non-trivial (r,s)-type boundary condition, the double-row transfer matrix is found to be **diagonalizable**.
- In the continuum scaling limit, each logarithmic minimal model gives rise to a CFT

$$\boldsymbol{D}(u) \sim e^{-u\mathcal{H}}, \qquad -\mathcal{H} \to L_0 - \frac{c}{24}, \qquad Z_{r,s}(q) = \mathrm{Tr}\, \boldsymbol{D}(u)^P \to q^{-c/24}\, \mathrm{Tr}\, q^{L_0} = \chi_{r,s}(q)$$

  where q is the modular parameter.
- Associated to the boundary condition (r,s) is the so-called **Kac representation** (r,s).
- As representations of the Virasoro algebra, the Kac representations fall in three groups:
  (i) irreducible representations,
  (ii) reducible yet indecomposable representations,
  (iii) fully reducible representations.
- The **identity** representation is $(1,1)$. It is $\begin{cases} \text{irreducible,} & p = 1 \\ \text{reducible yet indecomposable,} & p \geq 2 \end{cases}$
- There are **infinitely** many distinct Kac representations.
- This infinite set of representations is associated to an infinitely extended Kac table.
- The Kac representations are the building blocks for fusion.

Critical Dense Polymer Kac Table

Central charge $(p, p') = (1, 2)$

$$c = 1 - \frac{6(p-p')^2}{pp'} = -2$$

Conformal weights

$$\Delta_{r,s} = \frac{(p'r - ps)^2 - (p-p')^2}{4pp'} = \frac{(2r-s)^2 - 1}{8}, \qquad r, s \in \mathbb{N}$$

Kac representation characters

$$\chi_{r,s}(q) = q^{-c/24} \frac{q^{\Delta_{r,s}}(1 - q^{rs})}{\prod_{n=1}^{\infty}(1-q^n)}$$

Irreducible representations

There is an irreducible representation for each distinct conformal weight. The Kac representations which happen to be irreducible are marked with a red quadrant.

s							
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
10	$\frac{63}{8}$	$\frac{35}{8}$	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\cdots$
9	6	3	1	0	0	1	$\cdots$
8	$\frac{35}{8}$	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\cdots$
7	3	1	0	0	1	3	$\cdots$
6	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\cdots$
5	1	0	0	1	3	6	$\cdots$
4	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\frac{63}{8}$	$\cdots$
3	0	0	1	3	6	10	$\cdots$
2	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\frac{63}{8}$	$\frac{99}{8}$	$\cdots$
1	0	1	3	6	10	15	$\cdots$
	1	2	3	4	5	6	r

Critical Percolation Kac Table

Central charge $(p,p')=(2,3)$

$$c \;=\; 1-\frac{6(p-p')^2}{pp'} \;=\; 0$$

Conformal weights

$$\Delta_{r,s} \;=\; \frac{(p'r-ps)^2-(p-p')^2}{4pp'} \;=\; \frac{(3r-2s)^2-1}{24}, \qquad r,s\in\mathbb{N}$$

Kac representation characters

$$\chi_{r,s}(q)=q^{-c/24}\,\frac{q^{\Delta_{r,s}}(1-q^{rs})}{\prod_{n=1}^{\infty}(1-q^n)}$$

Irreducible representations

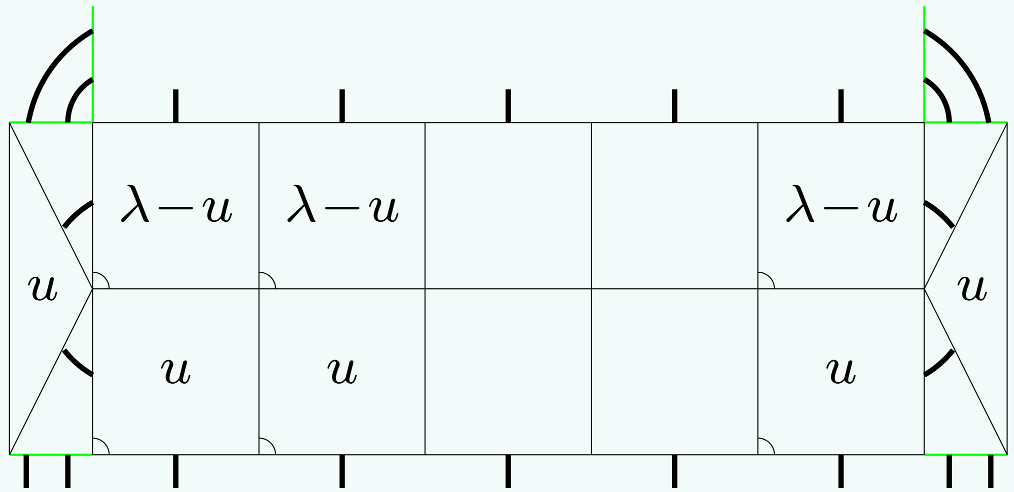
There is an irreducible representation for each distinct conformal weight. The Kac representations which happen to be irreducible are marked with a red quadrant.

s							
s	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
10	12	$\frac{65}{8}$	5	$\frac{21}{8}$	1	$\frac{1}{8}$	$\cdots$
9	$\frac{28}{3}$	$\frac{143}{24}$	$\frac{10}{3}$	$\frac{35}{24}$	$\frac{1}{3}$	$-\frac{1}{24}$	$\cdots$
8	7	$\frac{33}{8}$	2	$\frac{5}{8}$	0	$\frac{1}{8}$	$\cdots$
7	5	$\frac{21}{8}$	1	$\frac{1}{8}$	0	$\frac{5}{8}$	$\cdots$
6	$\frac{10}{3}$	$\frac{35}{24}$	$\frac{1}{3}$	$-\frac{1}{24}$	$\frac{1}{3}$	$\frac{35}{24}$	$\cdots$
5	2	$\frac{5}{8}$	0	$\frac{1}{8}$	1	$\frac{21}{8}$	$\cdots$
4	1	$\frac{1}{8}$	0	$\frac{5}{8}$	2	$\frac{33}{8}$	$\cdots$
3	$\frac{1}{3}$	$-\frac{1}{24}$	$\frac{1}{3}$	$\frac{35}{24}$	$\frac{10}{3}$	$\frac{143}{24}$	$\cdots$
2	0	$\frac{1}{8}$	1	$\frac{21}{8}$	5	$\frac{65}{8}$	$\cdots$
1	0	$\frac{5}{8}$	2	$\frac{33}{8}$	7	$\frac{85}{8}$	$\cdots$
	1	2	3	4	5	6	r

Lattice Implementation of Fusion

- **Fusion** is implemented on the lattice by taking non-trivial boundary conditions on the left and right $(r',s')\otimes(r,s)$

$\boldsymbol{D}(u) =$



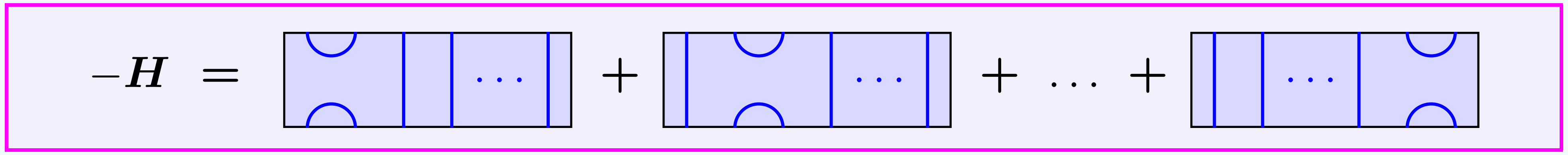
- In general, these fusion transfer matrices are **non-diagonalizable** as they can exhibit **non-trivial Jordan blocks**.
- In terms of representations, such examples correspond to reducible representations $\mathcal{R}$ of **rank greater than 1** $\Rightarrow$ **Logarithmic CFT**. There are infinitely many of these representations; all of rank 2 or 3 and all associated to the infinitely extended Kac table.

Hamiltonian Limit

Expansion of $D(u)$ for a $(1,s)$-type boundary condition

$$\boldsymbol{D}(u) = \boldsymbol{I} - 2u\boldsymbol{H} + \mathcal{O}(u^2)$$

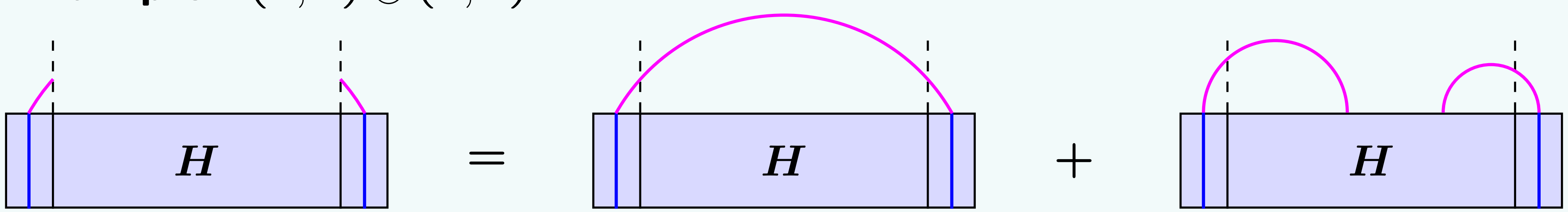
It follows that



- In terms of the generators of the **linear** TL algebra, this corresponds to

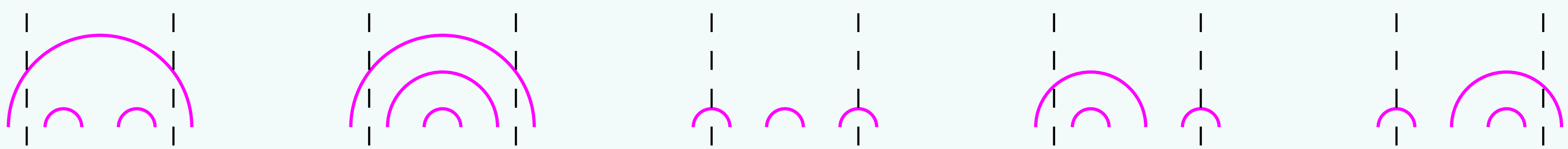
$$\boldsymbol{H} = -\sum_{j=1}^{N-1} e_j$$

Fusion Example: $(1,2)\otimes(1,2)$



$(1,2)\otimes(1,2)$ $(1,1)$ $(1,3)$

- For $N = 4$, there are 5 link states:



An Indecomposable Representation of Rank 2

- For $\mathcal{LM}(1,2)$, the fusion "$(1,2)\otimes(1,2)=(1,1)+(1,3)$" yields a reducible yet indecomposable representation of rank 2.
- For $N=4$, the Hamiltonian reads

$$\boldsymbol{D}(u)\sim e^{-u\mathcal{H}} \qquad -\mathcal{H} = \sum_j e_j \sim \left(\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) + \sqrt{2}\,I \qquad -\mathcal{H} \mapsto L_0 - \frac{c}{24}$$

- The Jordan canonical form of $\mathcal{H}$ has rank-2 Jordan blocks

$$-\mathcal{H} \sim \left(\begin{array}{cc|ccc} 0 & 0 & 1 & 0 & 0 \\ 0 & \sqrt{8} & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{8} \end{array}\right) \sim \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{8} & 1 \\ 0 & 0 & 0 & 0 & \sqrt{8} \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} = L_0^{(4)}$$

- As $N\to\infty$, the eigenvalues of $-\mathcal{H}$ approach the integer energies indicated in $L_0^{(4)}$.
- For $N=4$, the finitized partition function is (q is the modular parameter)

$$Z^{(N)}_{(1,2)|(1,2)}(q) = \underbrace{\chi^{(N)}_{(1,1)}(q)}_{0\text{ defects}} + \underbrace{\chi^{(N)}_{(1,3)}(q)}_{2\text{ defects}} = q^{-c/24}[(1+q^2)+(1+q+q^2)] = q^{-c/24}(2+q+2q^2)$$

- The resulting rank-2 representation may be viewed as an indecomposable sum

$$(1,2)\otimes(1,2) = \mathcal{R}_1 = (1,1)\oplus_i(1,3)$$

Dense Polymer Virasoro Fusion Algebra

- The fundamental Virasoro fusion algebra of critical dense polymers $\mathcal{LM}(1,2)$ is

$$\Big\langle (2,1),(1,2) \Big\rangle \;=\; \Big\langle (r,1),(1,2k),\mathcal{R}_k;\;\; r,k\in\mathbb{N} \Big\rangle$$

- With the identifications $(k,2k')\equiv(k',2k)$, the fusion rules obtained empirically from the lattice are **commutative**, **associative** and agree with Gaberdiel & Kausch (1996)

$$(r,1)\otimes(r',1) \;=\; \bigoplus_{j=|r-r'|+1,\ \text{by } 2}^{r+r'-1} (j,1)$$

$$(1,2k)\otimes(1,2k') \;=\; \bigoplus_{j=|k-k'|+1,\ \text{by } 2}^{k+k'-1} \mathcal{R}_j$$

$$(1,2k)\otimes\mathcal{R}_{k'} \;=\; \bigoplus_{j=|k-k'|}^{k+k'} \delta^{(2)}_{j,\{k,k'\}}(1,2j)$$

$$\mathcal{R}_k\otimes\mathcal{R}_{k'} \;=\; \bigoplus_{j=|k-k'|}^{k+k'} \delta^{(2)}_{j,\{k,k'\}}\,\mathcal{R}_j$$

$$(r,1)\otimes(1,2k) \;=\; \bigoplus_{j=|r-k|+1,\ \text{by } 2}^{r+k-1} (1,2j) = (r,2k)$$

$$(r,1)\otimes\mathcal{R}_k \;=\; \bigoplus_{j=|r-k|+1,\ \text{by } 2}^{r+k-1} \mathcal{R}_j$$

s							
	⋮	⋮	⋮	⋮	⋮	⋮	⋰
10	$\frac{63}{8}$	$\frac{35}{8}$	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	⋯
9	6	3	1	0	0	1	⋯
8	$\frac{35}{8}$	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	⋯
7	3	1	0	0	1	3	⋯
6	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	⋯
5	1	0	0	1	3	6	⋯
4	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\frac{63}{8}$	⋯
3	0	0	1	3	6	10	⋯
2	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\frac{63}{8}$	$\frac{99}{8}$	⋯
1	0	1	3	6	10	15	⋯
	1	2	3	4	5	6	r

$\mathcal{R}_k = (1,2k-1)\oplus_i(1,2k+1)$ (indecomposable rep of rank 2)

$$\delta^{(2)}_{j,\{k,k'\}} = 2-\delta_{j,|k-k'|}-\delta_{j,k+k'}$$

$\mathcal{W}$-Extended Vacuum of $\mathcal{WLM}(1,2)$

- Critical dense polymers $\mathcal{LM}(1,2)$ in the $\mathcal{W}$-extended picture is identified with the so-called **symplectic fermions** or **triplet model**.
- The $\mathcal{W}$-extended vacuum character of symplectic fermions is known to be

$$\hat{\chi}_{1,1}(q) = \sum_{n=1}^{\infty} (2n-1)\, \chi_{2n-1,1}(q)$$

- The BYBE is *not* linear and sums of solutions do *not* usually give new solutions. Rather, the **BYBE is closed under fusions**. We thus consider the triple fusion

$$(2n-1,1) \otimes (2n-1,1) \otimes (2n-1,1) \;=\; (1,1) \oplus 3(3,1) \oplus 5(5,1) \oplus \ldots \oplus (2n-1)(2n-1,1) \oplus \ldots$$

For large n, the coefficients stabilize and reproduce the extended vacuum character $\hat{\chi}_{1,1}(q)$.

- The **$\mathcal{W}$-Extended Vacuum** is thus defined by

$$(1,1)_{\mathcal{W}} := \lim_{n\to\infty} (2n-1,1) \otimes (2n-1,1) \otimes (2n-1,1) = \bigoplus_{n=1}^{\infty} (2n-1)\,(2n-1,1)$$

- In general, we denote by $\mathcal{WLM}(p,p')$ the logarithmic minimal model $\mathcal{LM}(p,p)$ viewed in the $\mathcal{W}$-extended picture.

$\mathcal{W}$-Extended Boundary Conditions and Fusion

- The $\mathcal{W}$-extended vacuum $(1,1)_\mathcal{W}$ of $\mathcal{WLM}(1,2)$ should act as the identity. In particular,

$$(1,1)_\mathcal{W}\,\hat{\otimes}\,(1,1)_\mathcal{W} \;=\; (1,1)_\mathcal{W}$$

where $\hat{\otimes}$ denotes the fusion multiplication in the extended picture.

- The $\mathcal{W}$-extended vacuum has the **stability property**

$$(2n-1,1)\otimes(1,1)_\mathcal{W} \;=\; (2n-1)\,(1,1)_\mathcal{W}$$

- The $\mathcal{W}$-extended fusion $\hat{\otimes}$ is therefore defined by

$$(1,1)_\mathcal{W}\,\hat{\otimes}\,(1,1)_\mathcal{W} := \lim_{n\to\infty}\Big(\frac{1}{(2n-1)^3}(2n-1,1)\otimes(2n-1,1)\otimes(2n-1,1)\otimes(1,1)_\mathcal{W}\Big) = (1,1)_\mathcal{W}$$

- Additional stability properties enable us to define

$$\begin{aligned}
(1,s)_\mathcal{W} &:= (1,s)\otimes(1,1)_\mathcal{W} = \bigoplus_{n=1}^{\infty}(2n-1)\,(2n-1,s), \qquad s=1,2\\
(2,s)_\mathcal{W} &:= \tfrac{1}{2}(2,s)\otimes(1,1)_\mathcal{W} = \bigoplus_{n=1}^{\infty}2n\,(2n,s), \qquad s=1,2\\
\hat{\mathcal{R}}_1\equiv(\mathcal{R}_1)_\mathcal{W} &:= \mathcal{R}_1\otimes(1,1)_\mathcal{W} = \bigoplus_{n=1}^{\infty}(2n-1)\,\mathcal{R}_{2n-1}\\
\hat{\mathcal{R}}_0\equiv(\mathcal{R}_2)_\mathcal{W} &:= \tfrac{1}{2}\mathcal{R}_2\otimes(1,1)_\mathcal{W} = \bigoplus_{n=1}^{\infty}2n\,\mathcal{R}_{2n}
\end{aligned}$$

- The ensuing representation content: 4 $\mathcal{W}$-irreducible representations and 2 $\mathcal{W}$-reducible yet $\mathcal{W}$-indecomposable representations of rank 2.

Fusion Rules for $\mathcal{WLM}(1,2)$

- The $\mathcal{W}$-extended fusion rules follow from the Virasoro fusion rules combined with stability

$\hat{\otimes}$	0	1	$-\frac{1}{8}$	$\frac{3}{8}$	$\hat{\mathcal{R}}_0$	$\hat{\mathcal{R}}_1$
0	0	1	$-\frac{1}{8}$	$\frac{3}{8}$	$\hat{\mathcal{R}}_0$	$\hat{\mathcal{R}}_1$
1	1	$\mathbf{0}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\hat{\mathcal{R}}_1$	$\hat{\mathcal{R}}_0$
$-\frac{1}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\hat{\mathcal{R}}_0$	$\hat{\mathcal{R}}_1$	$2(-\frac{1}{8})+2(\frac{3}{8})$	$2(-\frac{1}{8})+2(\frac{3}{8})$
$\frac{3}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\hat{\mathcal{R}}_1$	$\hat{\mathcal{R}}_0$	$2(-\frac{1}{8})+2(\frac{3}{8})$	$2(-\frac{1}{8})+2(\frac{3}{8})$
$\hat{\mathcal{R}}_0$	$\hat{\mathcal{R}}_0$	$\hat{\mathcal{R}}_1$	$2(-\frac{1}{8})+2(\frac{3}{8})$	$2(-\frac{1}{8})+2(\frac{3}{8})$	$2\hat{\mathcal{R}}_0+2\hat{\mathcal{R}}_1$	$2\hat{\mathcal{R}}_0+2\hat{\mathcal{R}}_1$
$\hat{\mathcal{R}}_1$	$\hat{\mathcal{R}}_1$	$\hat{\mathcal{R}}_0$	$2(-\frac{1}{8})+2(\frac{3}{8})$	$2(-\frac{1}{8})+2(\frac{3}{8})$	$2\hat{\mathcal{R}}_0+2\hat{\mathcal{R}}_1$	$2\hat{\mathcal{R}}_0+2\hat{\mathcal{R}}_1$

where the 4 $\mathcal{W}$-irreducible representations are represented by their conformal weights.

Example Consider the $\mathcal{W}$-extended fusion rule $1 \,\hat{\otimes}\, 1 = \mathbf{0}$:

$$\begin{aligned}
(2,1)_{\mathcal{W}} \,\hat{\otimes}\, (2,1)_{\mathcal{W}} &= \Big(\tfrac{1}{2}(2,1)\otimes(1,1)_{\mathcal{W}}\Big) \,\hat{\otimes}\, \Big(\tfrac{1}{2}(2,1)\otimes(1,1)_{\mathcal{W}}\Big) \\
&= \tfrac{1}{4}\Big((2,1)\otimes(2,1)\Big)\otimes\Big((1,1)_{\mathcal{W}} \,\hat{\otimes}\, (1,1)_{\mathcal{W}}\Big) \\
&= \tfrac{1}{4}\Big((1,1)\oplus(3,1)\Big)\otimes(1,1)_{\mathcal{W}} \\
&= \tfrac{1}{4}(1+3)(1,1)_{\mathcal{W}} \\
&= (1,1)_{\mathcal{W}}
\end{aligned}$$

- **For general $\mathcal{WLM}(1,p')$**, the $\mathcal{W}$-extended fusion rules and characters agree with Gaberdiel & Kausch (1996) and Gaberdiel & Runkel (2008).

Summary of $\mathcal{WLM}(p, p')$

	Number	Symplectic Fermions	Critical Percolation
$\mathcal{W}$-indec reps	$6pp' - 2p - 2p'$	6	26
Rank 1	$2p + 2p' - 2$	4	8
Rank 2	$4pp' - 2p - 2p'$	2	14
Rank 3	$2(p-1)(p'-1)$	0	4
$\mathcal{W}$-irred chars	$2pp' + \frac{1}{2}(p-1)(p'-1)$	4	13
$\mathcal{W}$-proj reps	$2pp'$	4	12
$\mathcal{W}$-proj chars	$\frac{1}{2}(p+1)(p'+1)$	3	6

- The **finitely** many $\mathcal{W}$-indecomposable reps close under fusion with respect to $\hat{\otimes}$.
- For $p \geq 2$, this fusion algebra has **no identity**. A canonical algebraic extension exists.
- A "disentangling procedure" is employed when identifying the various representations.
- The $\mathcal{W}$-*projective* representations form a fusion sub-algebra. Here a $\mathcal{W}$-projective representation is a "maximal" $\mathcal{W}$-indecomposable representation in the sense that it does not appear as a subfactor of any other $\mathcal{W}$-indecomposable representation.
- **For general** $\mathcal{WLM}(p, p')$, the $\mathcal{W}$-characters agree with Feigin, Gainutdinov, Semikhatov & Tipunin (2006). **For** $\mathcal{WLM}(2, 3)$, the fusion rules agree with Gaberdiel, Runkel & Wood (2009).

Polynomial Fusion Ring of $\mathcal{WLM}(p,p')$

- Notation: $\kappa\in\mathbb{Z}_{1,2}$, $a\in\mathbb{Z}_{1,p-1}$, $b\in\mathbb{Z}_{1,p'-1}$, $\alpha\in\mathbb{Z}_{0,p-1}$, $\beta\in\mathbb{Z}_{0,p'-1}$
- Rank-1 representations: $\{(\mathcal{R}_{a,\kappa p'}^{0,0})_{\mathcal{W}},(\mathcal{R}_{\kappa p,b}^{0,0})_{\mathcal{W}},(\mathcal{R}_{\kappa p,p'}^{0,0})_{\mathcal{W}}\}$ $\sharp=2p+2p'-2$

  Rank-2 representations: $\{(\mathcal{R}_{\kappa p,b}^{a,0})_{\mathcal{W}},(\mathcal{R}_{\kappa p,p'}^{a,0})_{\mathcal{W}},(\mathcal{R}_{a,\kappa p'}^{0,b})_{\mathcal{W}},(\mathcal{R}_{\kappa p,p'}^{0,b})_{\mathcal{W}}\}$ $\sharp=4pp'-2p-2p'$

  Rank-3 representations: $\{(\mathcal{R}_{\kappa p,p'}^{a,b})_{\mathcal{W}}\}$ $\sharp=2(p-1)(p'-1)$
- The polynomials (T_n and U_n are Chebyshev polynomials of the first and second kind)

$$\begin{aligned}
\mathrm{pol}_{(\mathcal{R}_{\kappa p,b}^{\alpha,0})_{\mathcal{W}}}(X,Y) &= \tfrac{2-\delta_{\alpha,0}}{\kappa}T_\alpha\big(\tfrac{X}{2}\big)U_{\kappa p-1}\big(\tfrac{X}{2}\big)U_{b-1}\big(\tfrac{Y}{2}\big)\\
\mathrm{pol}_{(\mathcal{R}_{a,\kappa p'}^{0,\beta})_{\mathcal{W}}}(X,Y) &= U_{a-1}\big(\tfrac{X}{2}\big)\tfrac{2-\delta_{\beta,0}}{\kappa}T_\beta\big(\tfrac{Y}{2}\big)U_{\kappa p'-1}\big(\tfrac{Y}{2}\big)\\
\mathrm{pol}_{(\mathcal{R}_{\kappa p,p'}^{\alpha,\beta})_{\mathcal{W}}}(X,Y) &= \tfrac{2-\delta_{\alpha,0}}{\kappa}T_\alpha\big(\tfrac{X}{2}\big)U_{\kappa p-1}\big(\tfrac{X}{2}\big)\big(2-\delta_{\beta,0}\big)T_\beta\big(\tfrac{Y}{2}\big)U_{p'-1}\big(\tfrac{Y}{2}\big)
\end{aligned}$$

generate an **ideal** of the quotient polynomial ring

$$\mathbb{C}[X,Y]\Big/\Big(P_p(X),P_{p'}(Y),P_{p,p'}(X,Y)\Big)$$

where

$$\begin{aligned}
P_n(x) &= U_{3n-1}\big(\tfrac{x}{2}\big)-3U_{n-1}\big(\tfrac{x}{2}\big) = 2\Big(T_{2n}\big(\tfrac{x}{2}\big)-1\Big)U_{n-1}\big(\tfrac{x}{2}\big) = (x^2-4)U_{n-1}^3\big(\tfrac{x}{2}\big)\\
P_{n,n'}(x,y) &= \Big(T_n\big(\tfrac{x}{2}\big)-T_{n'}\big(\tfrac{y}{2}\big)\Big)U_{n-1}\big(\tfrac{x}{2}\big)U_{n'-1}\big(\tfrac{y}{2}\big)
\end{aligned}$$

- The $\mathcal{W}$-extended fusion algebra of $\mathcal{WLM}(p,p')$ is **isomorphic** to this ideal.
- By algebraic completion, this offers a canonical introduction of the identity, subsequently corroborated by Gaberdiel, Runkel and Wood for $\mathcal{WLM}(2,3)$ (2009).

Graph Fusion Algebra

Fusion matrices (regular representation)

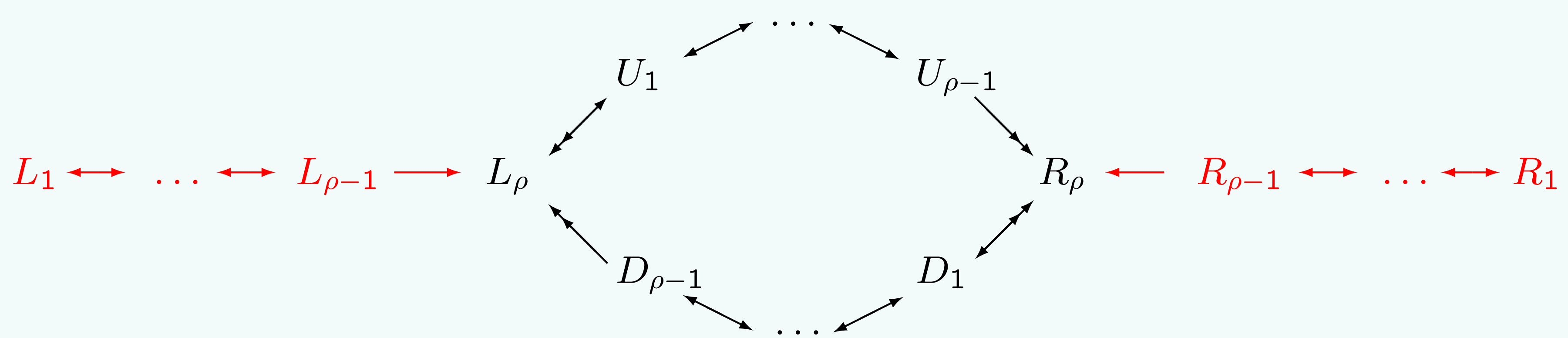
$$\mu\otimes\nu \;=\; \bigoplus N_{\mu,\nu}{}^{\lambda}\lambda, \qquad N_\mu N_\nu \;=\; \sum N_{\mu,\nu}{}^{\lambda}N_\lambda, \qquad [N_\mu]_\nu^\lambda \;=\; N_{\mu,\nu}{}^{\lambda}\in\mathbb{Z}_\geq$$

- Fusion alg isomorphic to **proper ideal** of pol-ring $\Rightarrow X,Y$ **not** images of fus-alg generators.
- For $p>2$, X and Y are thus **auxiliary** fusion matrices.
- Decomposing $X\text{pol}_\mu(X,Y)$ yields the entries in the μ'th row of the matrix realization of X.

Adjacency matrices of graphs

$$X \;=\; \text{diag}\Big(\underbrace{C_p,\ldots,C_p}_{p'-1},\underbrace{E_p,\ldots,E_p}_{p'}\Big), \qquad Y \;=\; \mathcal{P}^{-1}\text{diag}\Big(\underbrace{C_{p'},\ldots,C_{p'}}_{p-1},\underbrace{E_{p'},\ldots,E_{p'}}_{p}\Big)\mathcal{P}$$

with cycle C_ρ and eye-patch E_ρ graphs given by $(\rho=p,p')$



Spectral decompositions (→ Verlinde-like formulas)

$$Q^{-1}XQ \;=\; J_X, \qquad Q^{-1}YQ \;=\; P^{-1}J_YP$$

$$\begin{aligned} Q^{-1}N_\mu Q \;&=\; Q^{-1}\text{pol}_\mu(X,Y)Q \;=\; \text{pol}_\mu(Q^{-1}XQ,Q^{-1}YQ) \\ &=\; \text{pol}_\mu(J_X,P^{-1}J_YP) \;=\; \text{pol}_\mu^{(x)}(J_X)\,P^{-1}\text{pol}_\mu^{(y)}(J_Y)P \end{aligned}$$

Summary and Outlook

- Infinite series of Yang-Baxter integrable lattice models of non-local statistical mechanics
- Logarithmic CFT with infinitely many (higher-rank) indecomposable representations
- (Inversion identity and exact solution for critical dense polymers $\mathcal{LM}(1,2)$)
- Empirical Virasoro fusion rules for $\mathcal{LM}(p,p')$

Checks:
1. $\mathcal{LM}(p,p')$ fusion rules agree with partial results of Eberle & Flohr (2006)
2. Vertical fusion subalgebras agree with Read & Saleur (2007) and Mathieu & Ridout (2008)
3. Associativity

- $\mathcal{W}$-extended picture with finitely many (higher-rank) indecomposable representations
- Inferred $\mathcal{W}$-algebra fusion rules for $\mathcal{WLM}(p,p')$

Checks:
1. $\mathcal{WLM}(1,p')$ fusion rules agree with Gaberdiel & Kausch (1996) and Gaberdiel & Runkel (2008)
2. $\mathcal{WLM}(p,p')$ characters agree with Feigin et al (2006)
3. $\mathcal{WLM}(2,3)$ fusion rules agree with Gaberdiel, Runkel & Wood (2009)
4. Associativity

- Polynomial fusion rings; identity introduced by algebraic completion; graph fusion algebras
- Verlinde formulas from spectral decompositions: Grothendieck ring (with Pearce & Ruelle); Fusion algebra
- Critical dense polymers on a cylinder (with Pearce & Villani) – cf Paul's talk last week
- Torus partition functions (with Pearce) $\rightarrow$ ¿modular invariance?
- Exact solution of critical percolation (with Pearce)