The Relativistic Avatars of Giant Magnons

The symmetric space sine-Gordon theories

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Based on `arXiv:0808:3365, 0902.2405, 0905.2534, and 1006.3667`

The AdS/CFT correspondence is remarkable

- It is a working quantum theory of gravity which is completely well-defined
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It has a hidden integrability which emerges on the string world sheet!

- Existence of an infinite tower of hidden conserved charges on both sides of the correspondence.
- Implies exact spectrum of string states/anomalous dimensions.
- Enables the quantitative investigation of the conjectured duality.

Outline

The SSSG theories

Symmetric space sine-Gordon (SSSG) theories

- Two-dimensional Integrable relativistic theories.
- Obtained from sigma models via the Pohlmeyer reduction.
- Relevant for the investigation of the AdS/CFT correspondence.
- Admit soliton solutions $\longrightarrow$ Giant Magnons

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- Relevant for the investigation of the AdS/CFT correspondence.
- Admit soliton solutions $\longrightarrow$ Giant Magnons
- Quantum S-matrices were never found in the old days except for sine-Gordon and complex sine-Gordon cases!

Pohlmeyer reduction of symmetric space sigma models

1. Take a symmetric space F/G
   - Involution: $\sigma^2 = 1, \quad \sigma(G) = G$
   - Lie algebra decomposition: $\mathfrak{f} = \mathfrak{g} \oplus \mathfrak{p}, \quad [\mathfrak{g}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{g}$
2. Define a sigma model with target F/G:

$$\boxed{\mathscr{L} = \text{Tr}\left(\partial_\mu \mathcal{F} \partial^\mu \mathcal{F}^{-1}\right)}$$

   with $\mathcal{F} \in F$ and $\sigma(\mathcal{F}) = \mathcal{F}^{-1}$.
3. Impose the constraints (breaking conformal and relativistic invariance)

$$\boxed{T_{++} = T_{--} = \mu^2} \quad \Rightarrow \quad \boxed{\partial_\pm \mathcal{F} \mathcal{F}^{-1} = f_\pm \Lambda f_\pm^{-1}}$$

   where $\sigma(\Lambda) = -\Lambda \in \mathfrak{p}$, and $f_\pm \in F$.

The Reduced Model

- The constrained model can be re-formulated in terms of

$$\boxed{\gamma = f_-^{-1} f_+ \in G}$$

- There is a $H_L \times H_R$ gauge symmetry arising from $f_\pm \to f_\pm h_\pm$ giving

$$\gamma \to h_-^{-1} \gamma h_+, \quad \text{for} \quad h_\pm \in H \subset G \quad \text{such that} \quad h_\pm \Lambda h_\pm^{-1} = \Lambda$$

- The equations of the reduced model are zero-curvature conditions

$$\boxed{\left[\partial_+ + \gamma^{-1}\partial_+\gamma + \gamma^{-1} A_+^{(L)} \gamma - z\Lambda \,,\; \partial_- + A_-^{(R)} - z^{-1}\gamma^{-1}\Lambda\gamma\right] = 0}$$

($z =$spectral parameter) $\Rightarrow$ Classical integrability.

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(z =spectral parameter) $\Rightarrow$ Classical integrability.

★ These are relativistic equations!

The Reduced Model

- Fixing the gauge $A_{-}^{(R)} = A_{+}^{(L)} = 0$, the SSSG equations become the non-abelian affine Toda equations

Pohlmeyer, Eichenherr, Forger, D'Auria, Regge, ...'79-81

$$\boxed{\partial_{-}(\gamma^{-1}\partial_{+}\gamma) = [\Lambda, \gamma^{-1}\Lambda\gamma]}$$

Leznov-Saveliev'83
Ferreira-Miramontes-SanchezGuillen'97
Nirov-Razumov'07
....

associated to the affine Lie algebra

$$\hat{\mathfrak{f}} = \bigoplus_{n\in\mathbb{Z}} \left(z^{2n} \otimes \mathfrak{g} + z^{2n+1} \otimes \mathfrak{p}\right)$$

- $\gamma \in G$ and $\Lambda \in \mathfrak{p}$.

Lagrangian formalism

Bakas-Park-Shin'95
Grigoriev-Tseytlin'08
JLM'08

★ Choosing partial gauge fixing conditions

$$H_L \times H_R \;\longrightarrow\; H_{\text{vec}}$$

the SSSG equations can be derived from a relativistic Lagrangian

$$\boxed{\mathscr{L} = \mathscr{L}_{\text{gWZW}}(G/H) + \text{Tr}(\Lambda \gamma^{-1} \Lambda \gamma)}$$

with gauge group $\gamma \to h^{-1}\gamma h$, for $h \in H$.

Some features

- Degenerate vacuum $\gamma_0 \in$ Cartan Torus of H.
- Solitons carry a topological charge $\gamma(x = +\infty)\gamma^{-1}(x = -\infty)$ in the Cartan Torus of H.
- No conventional perturbative expansion around the vacuum.
- Coupling constant is the level of WZW.
- Natural interpretation as perturbed CFT.

Examples

Pohlmeyer'76

$F/G = SO(3)/SO(2) \simeq S^2, \ \ H = \emptyset$

$\longrightarrow$ sine-Gordon theory

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \cos\phi$$

$F/G = SO(4)/SO(3) \simeq S^3, \ \ H = SO(2)$

$\longrightarrow$ complex sine-Gordon theory

$$\mathscr{L} = \partial_\mu\phi\partial^\mu\phi + \cot^2\phi\,\partial_\mu\theta\partial^\mu\theta + \cos 2\phi$$

Zamolodchikov and Zamolodchikov'79
Dorey-TJH'95

- Both these theories are exactly solved in terms of solitons: Exact spectrum and S-matrix.

Strings on curved space-times are described by worldsheet sigma models

On the string world sheet gauge fixing leads naturally to the Pohlmeyer constraints

Tseytlin'03

$$\boxed{\text{Virasoro constraints on } \mathbb{R}_t \times \mathfrak{M}} \xrightarrow{X^0=\mu t} \boxed{T^{\mathfrak{M}}_{\pm\pm} = \mu^2}$$

Examples of compact symmetric spaces

$$S^n = SO(n+1)/SO(n) \longrightarrow \boxed{\mathbb{R}_t \times S^n \subset \text{AdS}_5 \times S^5}$$

$$\mathbb{C}P^n = SU(n+1)/U(n) \longrightarrow \boxed{\mathbb{R}_t \times \mathbb{C}P^n \subset \text{AdS}_4 \times \mathbb{C}P^3}$$

Examples of non-compact symmetric spaces $\longrightarrow$ different types of Pohlmeyer reductions

$$\mathrm{AdS}_n = SO(2,n-1)/SO(1,n-1)$$

(i) $\mu^2 > 0 \;\rightarrow\; \mathrm{AdS}_n \times \mathbb{R}_t$

(ii) $\mu^2 < 0 \;\rightarrow\; \boxed{\mathrm{AdS}_n \times S^1 \subset \mathrm{AdS}_n \times S^5 \quad \text{or} \quad \mathrm{AdS}_n \times \mathbb{C}P^3}$

Alday-Maldacena'09

(iii) $\mu^2 = 0 \;\rightarrow\; \boxed{\mathrm{AdS}_n}$ $\longrightarrow$ gluon scattering amplitudes

(ii) and (iii) relevant for the AdS/CFT correspondence!

Giant magnons

Minahan-Zarembo'04

- On the CFT side, integrability is manifested by the appearance of an integrable spin chain whose Hamiltonian provide the spectrum of exact scaling/conformal dimensions Δ.

Hofman-Maldacena'06

- In the limit where Δ and a conserved charge J become infinite, with the difference $\Delta - J$ and the 't Hooft coupling held fixed, the string dual of the fundamental magnon excitations are lump-like solutions known as Giant magnons, which propagate in an infinite long string.
- Giant magnons describe the classical motion of (bosonic) strings on curved space-times of the form $R_t \times \mathfrak{M}$, with $\mathfrak{M} = F/G$ a symmetric space $\longrightarrow$ SSSG theories

Staudacher'04
Beisert'05
Arutyunov-Frolov-Zamaklar'06
Ahn-Nepomechie'08
....

- For $\text{AdS}_5 \times S^5$ and $\text{AdS}_4 \times \mathbb{C}P^3$, the spectrum and S-matrix of giant magnons is already known.
- The S-matrix is complicated by the fact that the worldsheet theory is non-relativistic.
- The non-relativistic giant magnons map to a relativistic soliton "avatar" in the SSSG theory via the (complicated) Pohlmeyer map.

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★ The equivalence between the gauged fixed worldsheet theory and the SSSG theory is at the classical level but they have different symplectic structures.

★ Quantum equivalence may hold in the full (conformal invariant) theory with all the fermions included!

Generalized Pohlmeyer reduction for $AdS_5 \times S^5$

Grigoriev-Tseytlin'08
Mikhailov-SchaferNakemi'08

Virasoro constraints

$$T_{\pm\pm} = T_{\pm\pm}^{AdS_5} + T_{\pm\pm}^{S^5} = 0 \rightsquigarrow \boxed{\begin{array}{l} T_{\pm\pm}^{S^5} = +\mu^2 \\ T_{\pm\pm}^{AdS_5} = -\mu^2 \;\leftarrow \end{array}}$$

$\longrightarrow$ Lorentz invariant Lagrangian action for $AdS_5 \times S^5$ superstring theory

$$\boxed{\mathscr{L} = \mathscr{L}_{gWZW}\left[\frac{Sp(2,2)}{SU(2)\times SU(2)} \times \frac{Sp(4)}{SU(2)\times SU(2)}\right] + \text{potential} + \textbf{fermions}}$$

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Outstanding problem:

Find the exact relativistic S-matrix of the SSSG theories

An approach to quantization of generic SSSG theories

- Focus not so much on the Lagrangian and perturbation theory but rather on the solitons themselves: perturbative fields re-appear.
- The route to the spectrum and S-matrix is to use semi-classical methods (novelty: solitons carry non-abelian internal d-o-f):

1. Quantize the moduli space dynamics of the solitons yielding the semi-classical spectrum.
2. Conjecture S-matrix by imposing all the axioms of S-matrix theory and solve the bootstrap (account for all the bound state poles).
3. Check using semi-classical limit

$$\boxed{\lim_{k\to\infty} S(E) \sim \exp\left[i\int^{E} dE'\,\Delta t(E')\right]}$$

Giant magnons and solitons

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Reformulate equations of sigma model as auxiliary linear problem

$$\left(\partial_\pm - \frac{\partial_\pm \mathcal{F}\mathcal{F}^{-1}}{1\pm\lambda}\right)\Psi(\lambda) = 0$$

$\lambda =$ spectral parameter, equations-of-motion

$$\left[\partial_+ - \frac{\partial_+ \mathcal{F}\mathcal{F}^{-1}}{1+\lambda}, \partial_- - \frac{\partial_- \mathcal{F}\mathcal{F}^{-1}}{1-\lambda}\right] = 0$$

with $\mathcal{F} = \Psi(0)$.

Dressing transformation

Zakharov-Mikhailov'78
Harnad-SaintAubin-Shnider'84

$$\boxed{\Psi(\lambda) = \chi(\lambda)\Psi_0(\lambda)}$$

$$\chi(\lambda) = 1 + \frac{\vec{F}_k \Gamma^{-1}_{kj} \vec{F}^\dagger_j}{\lambda - \xi_j}$$

$$\vec{F}_j = \Psi_0(\xi^*_j)\vec{\varpi}_j, \qquad \Gamma_{jk} = \vec{F}^\dagger_j \cdot \vec{F}_k / (\xi_j - \xi^*_k)$$

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Dressing data

- ξ_j determine rapidity and the mass.
- The vectors $\vec{\varpi}_j$ are collective coordinates (position plus internal d-o-f).

The key fact

TJH-Miramontes'09

If the "vacuum" satisfies the Pohlmeyer constraints

$$\Psi_0(\lambda) = \exp\left(\frac{x_+}{1+\lambda} + \frac{x_-}{1-\lambda}\right)\Lambda$$

then the dressed solution also satisfies Pohlmeyer constraints

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The dressing determines both the giant magnon and the soliton

$$\mathcal{F}_{\text{magnon}} = \chi(0)e^{2t\Lambda}$$

$$\gamma_{\text{soliton}} = e^{-t\Lambda}\chi(1)^{-1}\chi(-1)e^{t\Lambda}$$

Hence the magnon and soliton are 2 views of the same underlying object.

$\mathbb{C}P^{n+1}$ giant magnons and their solitonic avatars

The $\mathbb{C}P^{n+1}$ symmetric space

$$\boxed{\mathbb{C}P^{n+1} = F/G = SU(n+2)/U(n+1)} \qquad \boxed{H = U(n)}$$

$$\Lambda = \left(\begin{array}{cc|c} 0 & -1 & \mathbf{0} \\ 1 & 0 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & 0 \end{array}\right) \quad G = \left(\begin{array}{c|cc} e^{i\alpha} & 0 & \mathbf{0} \\ \hline 0 & * & * \\ \mathbf{0} & * & * \end{array}\right) \quad H = \left(\begin{array}{cc|c} e^{i\alpha} & 0 & \mathbf{0} \\ 0 & e^{i\alpha} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & * \end{array}\right)$$

Involution $\sigma(M) = \theta M \theta$ with $\theta = \mathrm{diag}(-1, 1, \ldots, 1)$.

Simplest case: $\mathbb{C}P^2$ SSSG

- $\gamma \in U(2)$ and we fix the $H = U(1)$ gauge by taking the slice

$$\gamma = \begin{pmatrix} e^{i\psi/2} & 0 & 0 \\ 0 & \cos\theta e^{i\varphi + i\psi/2} & e^{-i\psi/2}\sin\theta \\ 0 & -e^{i\psi/2}\sin\theta & \cos\theta e^{-i\varphi - i\psi} \end{pmatrix}$$

- The Lagrangian is

Eichenherr-Honerkamp'81

$$\mathscr{L} = \partial_\mu\theta\partial^\mu\theta + \tfrac{1}{4}\partial_\mu\psi\partial^\mu\psi + \cot^2\theta\partial_\mu(\psi + \varphi)\partial^\mu(\psi + \varphi) + 2\mu^2\cos\theta\cos\varphi$$

with moduli space of vacua $\theta = \varphi = 0$, and $0 \leq \psi < 4\pi$.

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with moduli space of vacua $\theta = \varphi = 0$, and $0 \leq \psi < 4\pi$.

General $\mathbb{C}P^{n+1}$ case

Integrable perturbation of the $U(n+1)_k/U(n)_k$ gauged WZW model.

The elementary (non relativistic) $\mathbb{C}P^{n+1}$ giant magnons

TJH-Miramontes'09
Abbott-Aniceto-Sax'09

- Constructed using the dressing method with two poles:

$$\xi_1 = r\, e^{ip/2}, \qquad \xi_2 = 1/\xi_1, \qquad 0 \leq p \leq 2\pi, \qquad r > 0$$

- $\vec{\varpi}_2 = \theta \vec{\varpi}_1$ and use shifts in x and t to fix

$$\vec{\varpi}_1 = (1, i, \boldsymbol{\Omega}) \qquad |\boldsymbol{\Omega}| = 1$$

  where the complex n vector $\boldsymbol{\Omega} \sim e^{i\alpha}\boldsymbol{\Omega}$, so internal moduli space is $\boldsymbol{\Omega} \in \mathbb{C}P^{n-1}$.

- Magnon has $SU(n+2)$ Noether charges:

$$\boxed{\Delta \mathcal{Q} = \int dt\, \partial_0 \mathcal{F}\mathcal{F}^{-1} - \text{vac} = J_\Lambda \Lambda + J_H h_{\boldsymbol{\Omega}}}$$

$$J_\Lambda = -\frac{1+r^2}{r}\left|\sin\frac{p}{2}\right|, \quad J_H = -\frac{1-r^2}{r}\left|\sin\frac{p}{2}\right|, \quad h_{\boldsymbol{\Omega}} = i\left(\begin{array}{c|c} \mathbf{1} & \mathbf{0} \\ \hline \mathbf{0} & -2\boldsymbol{\Omega}\boldsymbol{\Omega}^\dagger \end{array}\right)$$

Non-relativistic dispersion relation

$$\Delta - \frac{1}{2}J = -\sqrt{\frac{\lambda}{2}}\,J_{\Lambda}, \qquad \frac{1}{2}Q = \sqrt{\frac{\lambda}{2}}J_H, \qquad \lambda = \text{'t Hooft coupling}$$

$$\Rightarrow \boxed{\Delta - \frac{1}{2}J = \sqrt{\frac{1}{4}Q^2 + 2\lambda\sin^2\frac{p}{2}}}$$

Beisert'05
Chen-Dorey-Okamura'06

- Consequence of centrally extended $SU(2|2)$ symmetry.
- Bound state of Q elementary giant magnons of charge $Q = 1$.

The $\mathbb{C}P^{n+1}$ (relativistic) soliton avatars

TJH-Miramontes'09

- $\boxed{(\xi = re^{ip/2}, \boldsymbol{\Omega})} \longrightarrow \tan q = \frac{2r}{1-r^2} \sin \frac{p}{2}$

$$\rightarrow \begin{cases} \text{Mass:} & m = \frac{4k}{\pi} \big| \sin q \big| \\ \text{Topological charge:} & \gamma^{-1}(-\infty)\gamma(+\infty) = \exp\left(-2qh_{\boldsymbol{\Omega}}\right) \\ \text{Rapidity:} & \tanh \vartheta = \frac{2r}{1+r^2} \cos \frac{p}{2} \end{cases}$$

- Solitons carry an internal collective coordinate $\boldsymbol{\Omega} \in \mathbb{C}P^{n-1}$ hidden in the algebra element $h_{\boldsymbol{\Omega}}$.

How do we deal with $\mathbf{\Omega}$?

- Under H

$$\boxed{\mathbf{\Omega}\mathbf{\Omega}^\dagger \to U\mathbf{\Omega}\mathbf{\Omega}^\dagger U^{-1}}$$

so moduli space is a (co-)adjoint orbit of an element of the Lie algebra $\mathfrak{h}$

$$\boxed{\mathbf{\Omega}\mathbf{\Omega}^\dagger = \mathrm{diag}(1, 0, \ldots, 0) \sim \vec{\omega}_1 \cdot \vec{H}}$$

- The orbit is $SU(n)/U(n-1) = \mathbb{C}P^{n-1}$.
- But $H = U(n)$ is gauged: is the orbit physical?
- Moduli space dynamics: allow $U \to U(t)$ and substitute into action to get effective quantum mechanics on the orbit.

Because soliton is a kink there is a boundary term $\int dt\, L$ coming from the WZ term

$$\boxed{L = \frac{2iqk}{\pi}\,\mathrm{Tr}\left(U^{-1}\dot{U}\,h_{\Omega}\right)}$$

Balachandran et al'01

- Leads to quantization of the co-adjoint orbit (fuzzy geometry).
- Coordinates $U = e^{i\lambda_i\theta^i}$ and conjugate momenta give constraints

$$\pi_i = \frac{\partial L}{\partial\dot{\theta}^i} \approx \frac{2iqk}{\pi}\,\mathrm{Tr}\left(U^{-1}\frac{\partial U}{\partial\theta^i}h_{\Omega}\right)$$

- Using $U^{-1}dU = -i\lambda_i E_{ij}d\theta^j$, define for $a \in \mathfrak{h}$

$$\Lambda_a = -\pi_j(E^{-1})_{ji}\,\mathrm{Tr}(a\lambda_i)$$

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- Poisson brackets

$$\boxed{\{\Lambda_a, U\} = -iUa\ , \qquad \{\Lambda_a, \Lambda_b\} = \Lambda_{[a,b]}}$$

- $H_0 = U(n-1)$ stability group of h_{Ω}, so $\mathfrak{h}_0 = \{a\}$ such that $[h_{\Omega}, a] = 0$ then

$$\boxed{\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{r} \qquad \mathfrak{h}_0 = \mathfrak{h}_{\Omega} \oplus \tilde{\mathfrak{h}}_0}$$

and the Lie algebra has the structure

$$[\mathfrak{h}_0, \mathfrak{h}_0] = \tilde{\mathfrak{h}}_0 \ , \qquad [\mathfrak{h}_0, \mathfrak{r}] = \mathfrak{r} \ , \qquad [\mathfrak{r}, \mathfrak{r}] = \mathfrak{h}_{\Omega} \oplus \tilde{\mathfrak{h}}_0 \ ,$$

- Constraints

$$\boxed{\Lambda_a \approx 0 \quad \text{except} \quad \Lambda_{h_{\Omega}} \approx \frac{2kq}{\pi}}$$

- So constraints for Λ_a $a \in \mathfrak{h}_0$ are first class and for $a \in \mathfrak{r}$ are second class.

- Quantize: set of $\mathcal{L}^2$ functions on H are (Peter-Weyl Theorem)

$$\psi(U) = \langle \rho_l | U | \rho_r \rangle \ ,$$

$$\boxed{\text{constraints: } \hat{\Lambda}_a \psi(U) = \langle \rho_l | U a | \rho_r \rangle}$$

★ For first class $a \in \mathfrak{h}_0 = \{\vec{H}, E_{\vec{\alpha}}\}$, with $\vec{\alpha} \cdot \vec{\omega}_1 = 0$,

$$E_{\vec{\alpha}} | \rho_r \rangle = 0 \ , \qquad \vec{H} | \rho_2 \rangle = \frac{2kq}{\pi} \vec{\omega}_1 | \rho_r \rangle$$

★ Second class $a \in \mathfrak{r} = \{E_{\vec{\alpha}}\}$, with $\vec{\alpha} \cdot \vec{\omega}_1 \neq 0$: split $a = E_{\pm\vec{\alpha}}$ then impose

$$E_{\text{sign}(q)\vec{\alpha}} | \rho_r \rangle = 0$$

So $|\rho_r\rangle$ is the highest (lowest) weight state with weight $a\vec{\omega}_1$ for $a \in \mathbb{Z}$

There is a quantization

$$\frac{2kq}{\pi} = a \in \mathbb{Z} \ .$$

The Hilbert space consists of modules with highest (lowest) weight $\pm a\vec{\omega}_1$ for $a = 1, 2, \ldots$ and

$$\boxed{q = \frac{\pi a}{N} \qquad m_a = \frac{4k}{\pi} \sin \frac{\pi |a|}{N}}$$

$N = 2k$ ($N = n + 2k$ exactly).

- So solitons have kink charges which are weights of the symmetric representations of $H = U(n-1)$ (this is not a Noether symmetry)!

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> Note: a must be fixed as $k \to \infty$. But what happens beyond the semi-classical limit; how does the tower of states truncate?

Back in the 20th century...

Ahn-Bernard-LeClair'90
TJH'90
deVega-Fateev'91
. . . .

- S-matrices associated to trigonometric solution to Yang-Baxter equations, involving the affine quantum group $U_q(SU(n)^{(1)})$
- R-matrix for vector-vector scattering $q = -e^{i\omega}$

$$\boxed{R(\vartheta) \sim \sin(\omega + i\lambda\vartheta)\mathbb{P}_{\text{symm}} + \sin(\omega - i\lambda\vartheta)\mathbb{P}_{\text{anti-symm}}}$$

old choice $\omega/\lambda > 0$ and bound-state pole at $\vartheta = i\omega/\lambda$ corresponds to anti-symmetric rep

In the present context, take instead $\omega/\lambda < 0$ and have bound-state pole at $\vartheta = -i\omega/\lambda$ corresponds to symmetric rep

Spectrum naturally truncates if q =root of unity

- Kinks $a = 1, \ldots, k$ in completely symmetric rank-a representations of $SU(n)$ and their conjugates, S-matrix has $U_q(SU(n)^{(1)})$ symmetry with

$$q = -\exp\left(\frac{i\pi}{n+k}\right)$$

and

$$m_a = M\sin\left(\frac{\pi a}{n+2k}\right)$$

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and

$$m_a = M \sin\left(\frac{\pi a}{n+2k}\right)$$

★ Quantum group at $q^{2(n+k)} = 1$ ensures that tower of states truncates

For TBAers: kinks associated to blobs on A_{n+2k-1} Dynkin diagram except for a gap between $k+1, \dots, n+k$: each blob is a symm-a rep (or conjugate) of $SU(n)$

The S-matrix

- Involves the trigonometric solution of the Yang-Baxter equations associated to $U_q\big(SU(n)^{(1)}\big)$ with $q = -e^{i\omega}$, $\omega = \frac{\pi}{k+n}$

$$S_{11}(\vartheta) = Y_{11}(\vartheta)\left(\mathbb{P}_{\text{symm}} + \frac{\sin(\omega + i\lambda\vartheta)}{\sin(\omega - i\lambda\vartheta)}\mathbb{P}_{\text{anti-symm}}\right)$$

  $Y_{11}(\vartheta)$ infinite product of gamma functions.

- Crossing: $S_{ab}(i\pi - \vartheta) = S_{\bar{b}a}(\vartheta) \quad \Rightarrow \quad \lambda = \frac{2k+n}{2k+2n}$
- Fusing rules:

$$[a]\circ[b] = \begin{cases}[a+b] & a+b \leq k \\ 0 & a+b > k\end{cases}, \qquad [a]\circ[b^{n-1}] = \begin{cases}[a-b] & a > b \\ [(b-a)^{n-1}] & a < b.\end{cases}.$$

  subset of A_{n+2k-1} fusing rules.

Semiclassical limit

- The scattering amplitudes for the special (coherent) states $||\boldsymbol{\Omega}, a\rangle\rangle = \left(\Omega_i|\mathbf{e}_i\rangle\right)^{\otimes a}$ matches the classical time-delays

$$\boxed{\lim_{k\to\infty} S(E) \sim \exp\left[i\int^E dE'\,\Delta t(E')\right]}$$

In the semi-classical limit $k \to \infty$ solitons with a/k fixed.

★ The symmetric representations of $SU(n)$ can be thought of as fuzzy $\mathbb{C}P^{n-1}$s. In the semi-classical limit, the fuzzy $\mathbb{C}P^{n-1}$ becomes a closer approximation of $\mathbb{C}P^{n-1}$ itself, which matches the fact that classical solitons exhibit a $\mathbb{C}P^{n-1}$ moduli space of solutions.

Generalizations to other symmetric spaces

- $S^{n+1} = SO(n+2)/SO(n+1)$ with $H = SO(n)$. Co-adjoint orbit in this case is a real Grassmannian $SO(n)/SO(2) \times SO(n-2)$ and states transform in the symmetric rank-a representations of $SO(n)$, $a = 1, 2, \ldots, k$.

$$m_a = M \sin\left(\frac{\pi a}{n-2+2k}\right)$$

- $SU(2m+n)/S(U(m) \times U(m+n))$ in which case $H = U(n)$.

$$\Lambda = \left(\begin{array}{c|c|c} 0 & A & 0 \\ \hline -A & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \quad G = \left(\begin{array}{c|c|c} * & 0 & 0 \\ \hline 0 & * & * \\ \hline 0 & * & * \end{array}\right) \quad H = \left(\begin{array}{c|c|c} e^{i\alpha} & 0 & 0 \\ \hline 0 & e^{i\alpha} & 0 \\ \hline 0 & 0 & * \end{array}\right)$$

in $(m+m+n)^2$ block-form where $A = \mathrm{diag}(a_1, \ldots, a_m)$.

★ This case is like the homogeneous SG theories with masses that "float". TBA system should involve 3 algebras: A_{n-1}, A_{m-1} and A_{n+2k-1}.

$AdS_5 \times S^5$

Grigoriev-Tseytlin'08

- In this case $F = PSU(2,2|4)$ and the involution σ is replaced by a $\mathbb{Z}_4$ automorphism with $G = Sp(2,2) \times Sp(4)$.
- However the SSSG can be formulated in terms of a Lax pair in the graded affine algebra

$$\hat{\mathfrak{f}} = \bigoplus_{n\in\mathbb{Z}} \bigoplus_{j=0}^{3} z^{4n+j}\mathfrak{f}_j \ , \qquad \mathfrak{f} = \bigoplus_{j=0}^{3} \mathfrak{f}_j$$

  where $\mathfrak{f}_{0,2}$ are bosonic and $\mathfrak{f}_{1,3}$ are fermionic, $\mathfrak{f}_0 = sp(2,2) \oplus sp(4)$.
- Fields $\gamma \in G$ and fermions with gauge group $H = SU(2)^4$.
- It seems that the dressing transformation extends in a nice way.
- The solitons now have bosonic and Grassmann collective coordinates.

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Quantization?

Are magnons and solitons quantum equivalent?

- It is not obvious: magnons come in an infinite tower of symmetric representations whereas the soliton tower is truncated.
- Even if the quantum magnons and solitons come in the same representations how can a relativistic S-matrix be equivalent to a non-relativistic one?

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Other open problems

- Solve all the other classes of symmetric space sine-Gordon theories.
- Further checks of the conjectured S-matrix: TBA, etc.
- . . . .

Muchas Gracias