

Four-loop Konishi for β -deformed SYM

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Review on AdS/CFT integrability

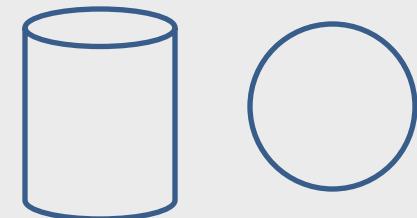
AdS/CFT duality

- [Maldacena, Gubser-Klebanov-Polyakov, Witten (1997)]
- Type IIB superstrings on $\text{AdS}_5 \times S^5$ are dual to 4d $N=4$ SU(N) SYM
- Energies of string states = Conformal dimensions of SYM operators
- Parameter relations:

$$g_s = \frac{4\pi\lambda}{N} \quad \text{and} \quad \frac{R^2}{\alpha'} = \sqrt{\lambda}$$

with 't Hooft coupling

$$\lambda = N g_{\text{YM}}^2$$



Planar Limit of SYM

- Free superstring theory corresponds to a planar limit of SYM

$$g_s \rightarrow 0 \quad \equiv \quad N \rightarrow \infty$$

- Energy of string = Conformal dimension Δ

Strong-weak coupling duality

- Quantitative check is difficult since it is a strong-weak duality
 - Classical string for $\alpha' \ll 1 \rightarrow \lambda \gg 1$
 - Perturbative SYM for $\lambda \ll 1 \rightarrow \alpha' \gg 1$
- Still very difficult to solve quantitatively for a general value of λ
- Need Nonperturbative method
- Breakthrough : Integrability has been discovered

Full sector Conjecture

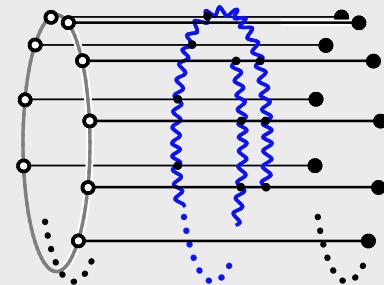
Beisert–Staudacher

$$O(x) = \text{Tr} \left[\dots Z \mathbf{X} \dots Z \mathbf{Y} \dots Z F_{\mu\nu} \dots Z \chi^\alpha \dots Z D_\mu Y \dots \right]$$

$$\begin{aligned}
 1 &= \prod_{k=1}^{M_2} \frac{u_{1j} - u_{2k} + \frac{i}{2}}{u_{1j} - u_{2k} - \frac{i}{2}} \prod_{k=1}^{M_4} \frac{1 - g^2/2x_{1j}x_{4k}^+}{1 - g^2/2x_{1j}x_{4k}^-} \\
 1 &= \prod_{k=1}^{M_2} \frac{u_{2j} - u_{2k} - i}{u_{2j} - u_{2k} + i} \prod_{k=1}^{M_3} \frac{u_{2j} - u_{3k} + \frac{i}{2}}{u_{2j} - u_{3k} - \frac{i}{2}} \prod_{k=1}^{M_1} \frac{u_{2j} - u_{1k} + \frac{i}{2}}{u_{2j} - u_{1k} - \frac{i}{2}} \\
 1 &= \prod_{k=1}^{M_2} \frac{u_{3j} - u_{2k} + \frac{i}{2}}{u_{3j} - u_{2k} - \frac{i}{2}} \prod_{k=1}^{M_4} \frac{x_{3j} - x_{4k}^+}{x_{3j} - x_{4k}^-} \\
 \left(\frac{x_{4j}^+}{x_{4j}^-} \right)^L &= \prod_{k=1}^{M_4} \sigma^2(x_{4j}, x_{4k}) \frac{x_{4j}^+ - x_{4k}^-}{x_{4j}^- - x_{4k}^+} \frac{1 - g^2/2x_{4j}^+x_{4k}^-}{1 - g^2/2x_{4j}^-x_{4k}^+} \\
 &\quad \times \prod_{k=1}^{M_1} \frac{1 - g^2/2x_{4j}^-x_{1k}}{1 - g^2/2x_{4j}^+x_{1k}} \prod_{k=1}^{M_3} \frac{x_{4j}^- - x_{3k}}{x_{4j}^+ - x_{3k}} \prod_{k=1}^{M_5} \frac{x_{4j}^- - x_{5k}}{x_{4j}^+ - x_{5k}} \prod_{k=1}^{M_7} \frac{1 - g^2/2x_{4j}^-x_{7k}}{1 - g^2/2x_{4j}^+x_{7k}} \\
 1 &= \prod_{k=1}^{M_6} \frac{u_{5j} - u_{6k} + \frac{i}{2}}{u_{5j} - u_{6k} - \frac{i}{2}} \prod_{k=1}^{M_4} \frac{x_{5j} - x_{4k}^+}{x_{5j} - x_{4k}^-} \\
 1 &= \prod_{k=1}^{M_6} \frac{u_{6j} - u_{6k} - i}{u_{6j} - u_{6k} + i} \prod_{k=1}^{M_5} \frac{u_{6j} - u_{5k} + \frac{i}{2}}{u_{6j} - u_{5k} - \frac{i}{2}} \prod_{k=1}^{M_7} \frac{u_{6j} - u_{7k} + \frac{i}{2}}{u_{6j} - u_{7k} - \frac{i}{2}} \\
 1 &= \prod_{k=1}^{M_6} \frac{u_{7j} - u_{6k} + \frac{i}{2}}{u_{7j} - u_{6k} - \frac{i}{2}} \prod_{k=1}^{M_4} \frac{1 - g^2/2x_{7j}x_{4k}^+}{1 - g^2/2x_{7j}x_{4k}^-}
 \end{aligned}$$

Wrapping problem

- High-order Feynman diagrams connect operators farther away
- When the length of a composite operator is shorter than the order of the perturbative expansion:
unphysical(“wrapping”) interactions appear
→ nonperturbative results valid only when the length is infinite

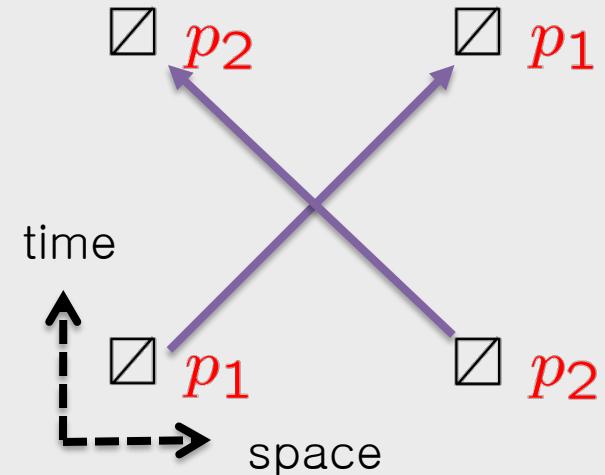


Symmetries and Spectrum

- Symmetries: $SU(2|2) \times SU(2|2)$ [Beisert]
- Each “ $SU(2|2)$ ”
$$\left(\begin{array}{c|c} \mathfrak{R}^a{}_b & \mathfrak{Q}^\alpha{}_b \\ \hline \mathfrak{S}^a{}_\beta & \mathfrak{L}^\alpha{}_\beta \end{array} \right)$$
- Spectrum: a tensor product of two Fundamental Rep.
 $\boxtimes = (\square, \mathbf{1}) \oplus (\mathbf{1}, \square) = (\phi^1, \phi^2 | \psi^1, \psi^2)$
 $(\boxtimes; \boxtimes) = \Phi_{a\dot{a}} \oplus \chi_\alpha^a \oplus \chi_\alpha^{\dot{a}} \oplus D_{\alpha\dot{\alpha}}$
- Focus on one $SU(2|2)$

S-matrix from Symmetries

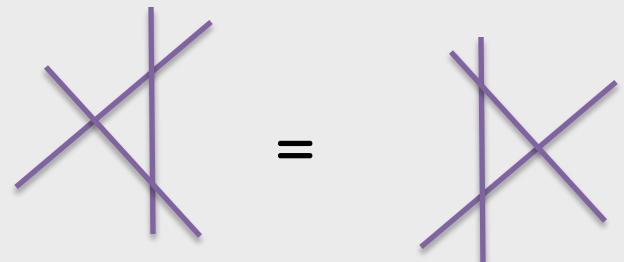
- Two particle S-matrix



- Commutativity with $SU(2|2)$ [Beisert]

$$\left[S(p_1, p_2), \left(\frac{\mathfrak{R}^a{}_b}{\mathfrak{S}^a{}_\beta} \Big| \frac{\mathfrak{Q}^\alpha{}_b}{\mathfrak{L}^\alpha{}_\beta} \right) \right] = 0$$

- Yang-Baxter equation [Arutyunov-Frolov-Zamaklar]

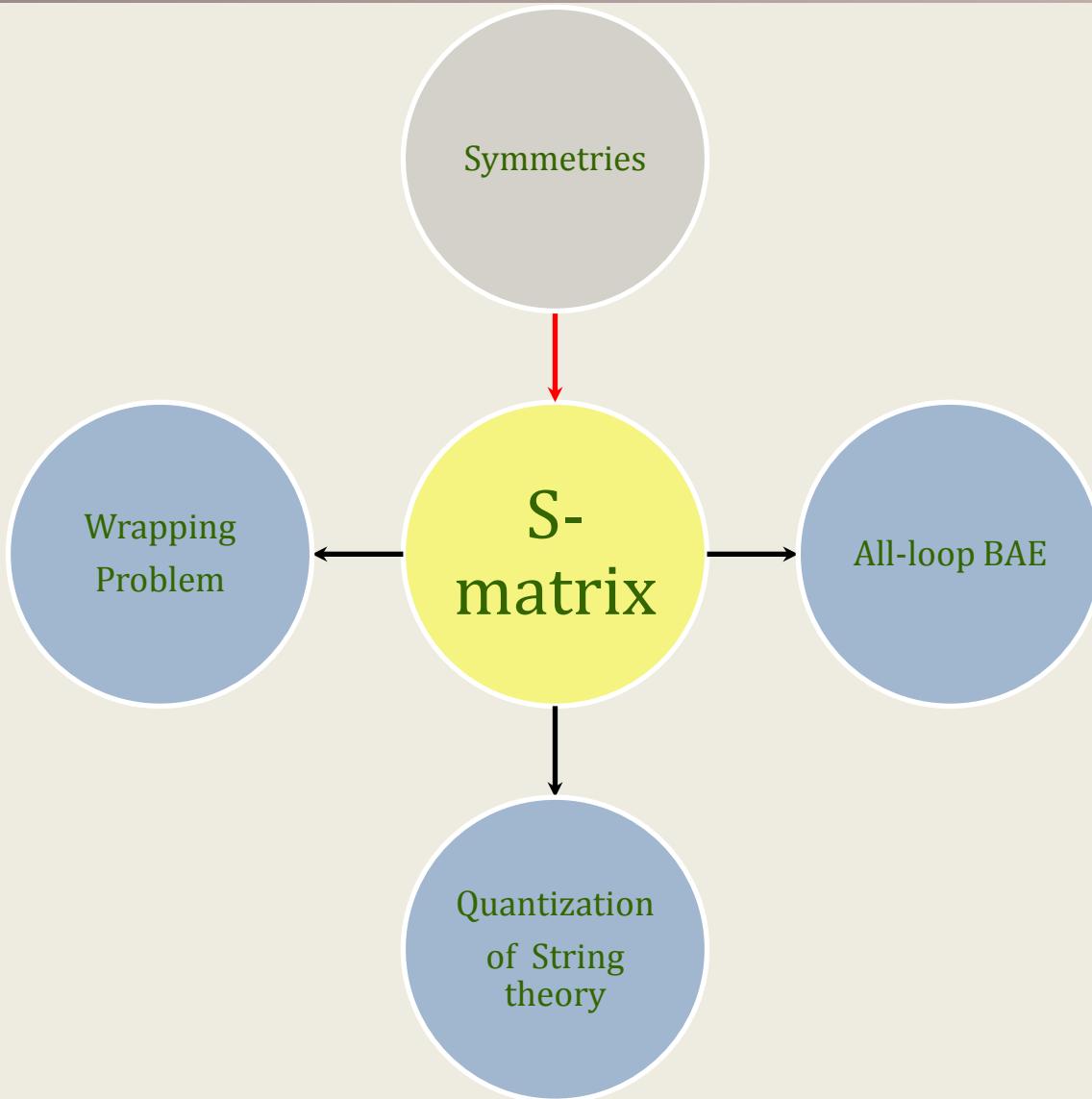


SU(2|2) S-matrix

$$\left(\begin{array}{cccc|cccc|cccc|cccc} a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{10} & 0 & 0 & a_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{10} & 0 & 0 & 0 & 0 & 0 & a_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_2 & 0 & 0 & -a_7 & 0 & 0 & a_7 & 0 & 0 & a_1 + a_2 & 0 & 0 & 0 & 0 \\ \hline 0 & a_5 & 0 & 0 & a_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_8 & 0 & 0 & -a_4 & 0 & 0 & a_3 + a_4 & 0 & 0 & a_8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_9 & 0 & 0 & 0 & 0 & 0 & a_5 & 0 & 0 & 0 \\ \hline 0 & 0 & a_5 & 0 & 0 & 0 & 0 & a_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_8 & 0 & 0 & a_3 + a_4 & 0 & 0 & -a_4 & 0 & 0 & -a_8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_9 & 0 & 0 & a_5 & 0 & 0 \\ \hline 0 & 0 & 0 & a_1 + a_2 & 0 & 0 & a_7 & 0 & 0 & -a_7 & 0 & 0 & -a_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_6 & 0 & 0 & 0 & 0 & 0 & a_{10} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_6 & 0 & 0 & a_{10} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 & 0 \end{array} \right)$$

$$a_1 = \frac{x_2^- - x_1^+}{x_2^+ - x_1^-}, \quad a_2 = \frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_2^- + x_1^+)}{(x_1^- - x_2^+)(x_1^- x_2^- - x_1^+ x_2^+)}, \dots$$

S-matrix program



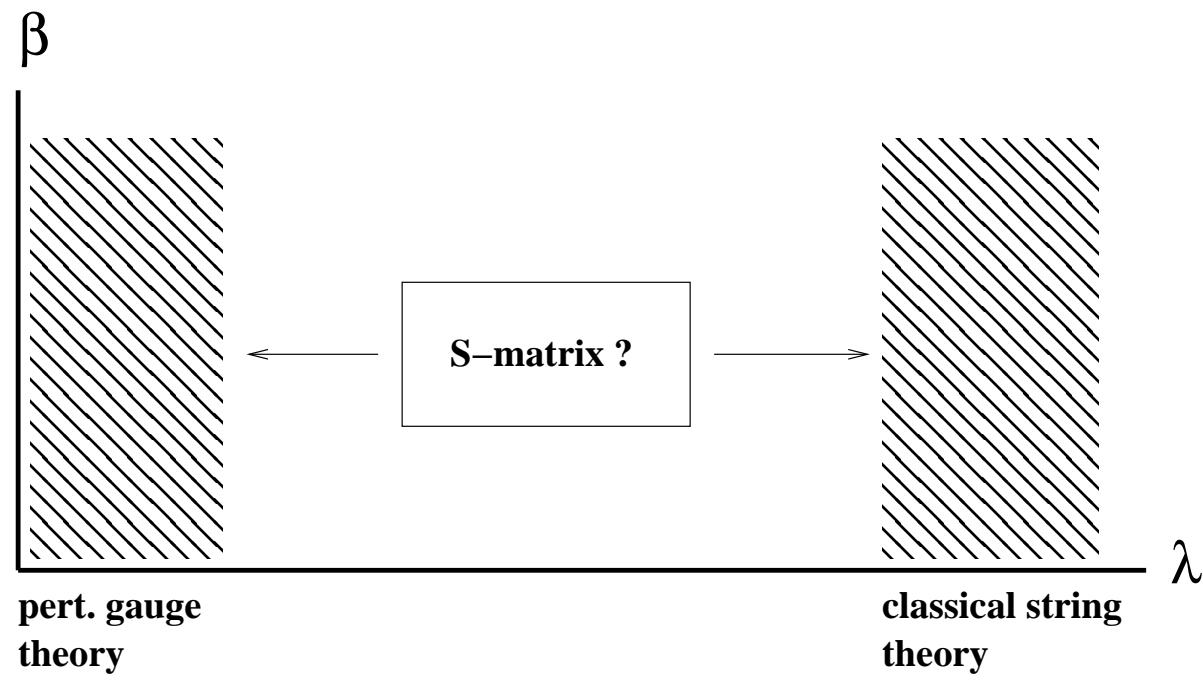
β -deformed SYM

- Superpotential:

$$W = \text{tr}(e^{i\pi\beta} \phi \psi Z - e^{-i\pi\beta} \phi Z \psi)$$

New AdS/CFT correspondence

- (Maybe) dual to type IIB string theory on TsT transformed $AdS_5 \times S^5$ (Lunin-Maldacena)
- Classical string analysis show that it still has giant magnon solution [Bykov-Frolov] and dyonic magnon solution [Bozhilov-CA]
- (Maybe) all-loop integrable: Bethe-ansatz was conjectured [Beisert-Roiban]
- What is the S -matrix, then?



Weak coupling of the β -deformed SYM

- Konishi operator

$$\text{Tr}(\phi Z \phi Z), \quad \text{Tr}(\phi \phi Z Z)$$

Fiamberti-Santambrogio-Sieg-Zanon perturbative results

$$\gamma^{(+)} = 4 + g^2 \gamma_1 + g^4 \gamma_2 + g^6 \gamma_3 + g^8 \gamma_4 + \dots$$

$$\gamma_1 = 6(1 + \Delta)$$

$$\gamma_2 = -\frac{3}{\Delta} - 15 - 21\Delta - 9\Delta^2$$

$$\gamma_3 = -\frac{3}{4\Delta^3} + \frac{153}{4\Delta} + 114 + \frac{495}{4}\Delta + 54\Delta^2 + \frac{27}{4}\Delta^3$$

$$\gamma_4 = -\frac{3}{8\Delta^5} + \frac{33}{2\Delta^3} - \frac{1701}{4\Delta} - 1230$$

$$- \frac{2427}{2}\Delta - 180\Delta^2 + 162\Delta^4 + \frac{2997}{8}\Delta^3$$

$$+ \left(-\frac{9}{\Delta} + 297 + 702\Delta + 234\Delta^2 - 405\Delta^3 - 243\Delta^4 \right) \zeta(3)$$

$$- 360(1 + \Delta)^2 \zeta(5)$$

$$\gamma^{(-)} = \gamma^{(+)}(\Delta \rightarrow -\Delta),$$

$$\Delta \equiv \frac{\sqrt{5 + 4 \cos(4\pi\beta)}}{3}$$

Beisert-Roiban Bethe ansatz equation

$$e^{ip_1 L} = e^{2\pi i \beta L} \cdot e^{2i\theta(p_1, p_2)} \cdot \frac{u_1 - u_2 + i}{u_1 - u_2 - i}$$

$$e^{ip_2 L} = e^{2\pi i \beta L} \cdot e^{-2i\theta(p_1, p_2)} \cdot \frac{u_2 - u_1 + i}{u_2 - u_1 - i}$$

$$u_i = \frac{1}{2} \cot \frac{p_i}{2} \sqrt{1 + 16g^2 \sin^2 \frac{p_i}{2}}$$

$$E(p_i) = \sqrt{1 + 16g^2 \sin^2 \frac{p_i}{2}}$$

BAE cont'd

$$p_1 + p_2 = 4\pi\beta$$

$$p_1 = \tilde{p} + 2\pi\beta, \quad p_2 = -\tilde{p} + 2\pi\beta$$

$$e^{4i\tilde{p}} = e^{2i\theta(\tilde{p}+2\pi\beta, -\tilde{p}+2\pi\beta)} \left(\frac{u(\tilde{p}+2\pi\beta) - u(-\tilde{p}+2\pi\beta) + i}{u(\tilde{p}+2\pi\beta) - u(-\tilde{p}+2\pi\beta) - i} \right)$$

$$E_{\text{total}}(\beta) = E(\tilde{p} + 2\pi\beta) + E(-\tilde{p} + 2\pi\beta)$$

perturbative solutions of BAE

$$\tilde{p} = \tilde{p}_0 + g^2 \tilde{p}_1 + g^4 \tilde{p}_2 + g^6 \tilde{p}_3 + \dots$$

$$E_{\text{total}}(\beta) = 2 + g^2 E_1(\beta) + g^4 E_2(\beta) + g^6 E_3(\beta) + g^8 E_4(\beta) + \dots$$

zeroth order

$$e^{4i\tilde{p}_0} = \frac{\cot(\frac{\tilde{p}_0}{2} - \pi\beta) + \cot(\frac{\tilde{p}_0}{2} + \pi\beta) + 2i}{\cot(\frac{\tilde{p}_0}{2} - \pi\beta) + \cot(\frac{\tilde{p}_0}{2} + \pi\beta) - 2i}$$

$$\cos \tilde{p}_0 = \frac{1 - 3\Delta}{4 \cos(2\pi\beta)}$$

$$E_1(\beta) = 8 \sin^2 \left(\frac{\tilde{p}_0}{2} - \pi\beta \right) + 8 \sin^2 \left(\frac{\tilde{p}_0}{2} + \pi\beta \right) = 6(1 + \Delta)$$

higher order solutions

$$E_2 = -\frac{3}{\Delta} - 15 - 21\Delta - 9\Delta^2$$

$$E_3 = -\frac{3}{4\Delta^3} + \frac{153}{4\Delta} + 114 + \frac{495}{4}\Delta + 54\Delta^2 + \frac{27}{4}\Delta^3$$

$$\begin{aligned} E_4 &= \frac{3(1+\Delta)^4}{8\Delta^5(1+3\Delta)^2}(-1 - 2\Delta + 49\Delta^2 + 84\Delta^3 \\ &\quad - 1359\Delta^4 - 5562\Delta^5 - 2673\Delta^6 + 1944\Delta^7) \\ &\quad + \left(-\frac{9}{\Delta} + 27 + 54\Delta - 90\Delta^2 - 189\Delta^3 - 81\Delta^4\right)\zeta(3) \end{aligned}$$

wrapping order correction

$$\begin{aligned}\Delta E_{\text{wrapping}} = g^8 & \left[- 54(1 + \Delta)^3(-5 + 3\Delta)\zeta(3) - 360(1 + \Delta)^2\zeta(5) \right. \\ & \left. + \frac{81(1 - 3\Delta)^2(1 + \Delta)^4}{(1 + 3\Delta)^2} \right]\end{aligned}$$

Bajnok-Janik Lüscher formula for $su(2)$ Konishi operator

$$\begin{aligned} \Delta E = & - \sum_{\ell=1}^{\infty} \int_{-\infty}^{\infty} \frac{dq}{2\pi} \left(\frac{z^-}{z^+} \right)^L \\ & \times \sum_{j,j'} (-1)^{F(jj')} \left[\mathcal{S}^{(\ell f)}(z^\pm, x_1^\pm) \mathcal{S}^{(\ell f)}(z^\pm, x_2^\pm) \right]_{(jj')(11)}^{(jj')(11)} \end{aligned}$$

$$x_i^\pm = \frac{1}{4g} \left(\cot \frac{p_i}{2} \pm i \right) \left(1 + \sqrt{1 + 16g^2 \sin^2 \frac{p_i}{2}} \right)$$

$$z^\pm = \frac{q + i\ell}{4g} \left(\sqrt{1 + \frac{16g^2}{\ell^2 + q^2}} \pm 1 \right)$$

S-matrix

$$\mathcal{S}^{(\ell f)}(z^\pm, x_i^\pm) = S_{\text{scalar}}^{(\ell f)}(q, u_i) \left[\mathcal{S}_{\text{matrix}}^{(\ell f)}(q, u_i) \otimes \mathcal{S}_{\text{matrix}}^{(\ell f)}(q, u_i) \right]$$

$$\begin{aligned} & \sum_{j,j'} (-1)^{F_{(jj')}} \left[\mathcal{S}^{(\ell f)}(q, u_1) \mathcal{S}^{(\ell f)}(q, u_2) \right]_{(jj')(11)}^{(jj')(11)} \\ &= S_{\text{scalar}}^{(\ell f)}(q, u_1) S_{\text{scalar}}^{(\ell f)}(q, u_2) \\ & \quad \times \left(\sum_j (-1)^{F_j} \left[\mathcal{S}_{\text{matrix}}^{(\ell f)}(q, u_1) \mathcal{S}_{\text{matrix}}^{(\ell f)}(q, u_2) \right]_{j1}^{j1} \right)^2 \end{aligned}$$

S-matrix cont'd

- ℓ : antisymmetric rep. (ℓ bound-states in the mirror space)
- Related to S -matrix in physical space

$$\mathcal{S}^{(\ell f)}(z^\pm, x^\pm) = S^{(f\ell)}(x^\pm, z^\pm)[1 \leftrightarrow 3, 2 \leftrightarrow 4]$$

deformed S -matrix elements

- Bosonic bound-states:

$$\mathcal{S}^{(\ell\text{f})j1}_{j1} = \mathcal{S}_0^{(\ell\text{f})j1}, \quad j = 1, \dots, 2\ell$$

- Fermionic bound-states:

$$\begin{aligned}\mathcal{S}^{(\ell\text{f})j1}_{j1} &= e^{i\pi\beta} \mathcal{S}_0^{(\ell\text{f})j1}, \quad j = 2\ell + 1, \dots, 3\ell \\ &= e^{-i\pi\beta} \mathcal{S}_0^{(\ell\text{f})j1}, \quad j = 3\ell + 1, \dots, 4\ell\end{aligned}$$

Lüscher corrections

- Kinematic factor:

$$\left(\frac{z^-}{z^+}\right)^4 = \frac{256g^8}{(q^2 + \ell^2)^4} + \dots$$

- S -matrix part: only leading order is enough
- Residue calculation due to a pole at $q = i\ell$

$$\Delta E_{\text{Luscher}} = \sum_{\ell=1}^{\infty} \left[\frac{f_1}{\ell^5} + \frac{f_2}{\ell^3} + f_3(\ell) \right]$$

$$f_1 = -\frac{2560(1+2u_1^2+2u_2^2)^2}{(4u_1^2+1)^2(4u_2^2+1)^2}$$

$$f_2 = \frac{\text{num}}{(4u_1^2+1)^4(4u_2^2+1)^4}$$

$$\begin{aligned} \text{num} = & 2048 (-1 + 5u_1^2 + 48u_1^4 + 96u_1^6 - 2u_1u_2 - 16u_1^3u_2 - 32u_1^5u_2 + 5u_2^2 + 224u_1^2u_2^2 \\ & + 1024u_1^4u_2^2 + 1536u_1^6u_2^2 + 768u_1^8u_2^2 - 16u_1u_2^3 - 128u_1^3u_2^3 + 64u_1^8 \\ & - 256u_1^5u_2^3 + 48u_2^4 + 1024u_1^2u_2^4 + 3200u_1^4u_2^4 + 2560u_1^6u_2^4 - 32u_1u_2^5 \\ & - 256u_1^3u_2^5 - 512u_1^5u_2^5 + 96u_2^6 + 1536u_1^2u_2^6 + 2560u_1^4u_2^6 + 64u_2^8 + 768u_1^2u_2^8) \end{aligned}$$

$$f_3 = 1775 \text{ terms}$$

with leading-order Bethe roots

$$u_{1,2} = \frac{(1 - 3\Delta)^2}{2 \sqrt{-1 + 9\Delta^2} \left(3 \sqrt{1 - \Delta^2} \pm 2 \sqrt{\frac{3+3\Delta}{1+3\Delta}} \right)}$$

$$\begin{aligned}
& 3355443200 \mathbf{i} Q^2 u1^5 u2^{15} - 43486543872 \mathbf{i} Q^4 u1^5 u2^{15} + 130996502528 \mathbf{i} Q^6 u1^5 u2^{15} - 111400714240 \mathbf{i} Q^8 u1^5 u2^{15} - \\
& 838860800 Q u1^6 u2^{15} + 31608274944 Q^3 u1^6 u2^{15} - 162135015424 Q^5 u1^6 u2^{15} + 187636383744 Q^7 u1^6 u2^{15} + 13421772800 \mathbf{i} Q^2 u1^7 u2^{15} - \\
& 139586437120 \mathbf{i} Q^4 u1^7 u2^{15} + 240518168576 \mathbf{i} Q^6 u1^7 u2^{15} - 2516582400 Q u1^8 u2^{15} + 79591112704 Q^3 u1^8 u2^{15} - \\
& 233001975808 Q^5 u1^8 u2^{15} + 26843545600 \mathbf{i} Q^2 u1^9 u2^{15} - 163208757248 \mathbf{i} Q^4 u1^9 u2^{15} - 4026531840 Q u1^{10} u2^{15} + \\
& 76772540416 Q^3 u1^{10} u2^{15} + 21474836480 \mathbf{i} Q^2 u1^{11} u2^{15} - 2684354560 Q u1^{12} u2^{15} - 81920 \mathbf{i} u2^{16} + 1376256 \mathbf{i} Q^2 u2^{16} - \\
& 8912896 \mathbf{i} Q^4 u2^{16} + 27787264 \mathbf{i} Q^6 u2^{16} - 29360128 \mathbf{i} Q^8 u2^{16} + 8388608 \mathbf{i} Q^{10} u2^{16} - 2621440 Q u1 u2^{16} + 33554432 Q^3 u1 u2^{16} - \\
& 150994944 Q^5 u1 u2^{16} + 232783872 Q^7 u1 u2^{16} - 134217728 Q^9 u1 u2^{16} + 33554432 Q^{11} u1 u2^{16} - 1966080 \mathbf{i} u1^2 u2^{16} + \\
& 59506688 \mathbf{i} Q^2 u1^2 u2^{16} - 427819008 \mathbf{i} Q^4 u1^2 u2^{16} + 1050673152 \mathbf{i} Q^6 u1^2 u2^{16} - 973078528 \mathbf{i} Q^8 u1^2 u2^{16} + 369098752 \mathbf{i} Q^{10} u1^2 u2^{16} - \\
& 52428800 Q u1^3 u2^{16} + 721420288 Q^3 u1^3 u2^{16} - 2818572288 Q^5 u1^3 u2^{16} + 3741319168 Q^7 u1^3 u2^{16} - 1879048192 Q^9 u1^3 u2^{16} - \\
& 19660800 \mathbf{i} u1^4 u2^{16} + 731906048 \mathbf{i} Q^2 u1^4 u2^{16} - 4815060992 \mathbf{i} Q^4 u1^4 u2^{16} + 9353297920 \mathbf{i} Q^6 u1^4 u2^{16} - 6106906624 \mathbf{i} Q^8 u1^4 u2^{16} - \\
& 419430400 Q u1^5 u2^{16} + 5435817984 Q^3 u1^5 u2^{16} - 16374562816 Q^5 u1^5 u2^{16} + 13925089280 Q^7 u1^5 u2^{16} - 104857600 \mathbf{i} u1^6 u2^{16} + \\
& 3951034368 \mathbf{i} Q^2 u1^6 u2^{16} - 20266876928 \mathbf{i} Q^4 u1^6 u2^{16} + 23454547968 \mathbf{i} Q^6 u1^6 u2^{16} - 1677721600 Q u1^7 u2^{16} + 17448304640 Q^3 u1^7 u2^{16} - \\
& 30064771072 Q^5 u1^7 u2^{16} - 314572800 \mathbf{i} u1^8 u2^{16} + 9948889088 \mathbf{i} Q^2 u1^8 u2^{16} - 29125246976 \mathbf{i} Q^4 u1^8 u2^{16} - 3355443200 Q u1^9 u2^{16} + \\
& 20401094656 Q^3 u1^9 u2^{16} - 503316480 \mathbf{i} u1^{10} u2^{16} + 9596567552 \mathbf{i} Q^2 u1^{10} u2^{16} - 2684354560 Q u1^{11} u2^{16} - 335544320 \mathbf{i} u1^{12} u2^{16}) / \\
& (Q^5 (-1 + 2 Q + 2 \mathbf{i} u1)^4 (1 + 2 Q + 2 \mathbf{i} u1)^4 (-\mathbf{i} + 2 u1)^4 (\mathbf{i} + 2 u1)^4 (-1 + 2 Q + 2 \mathbf{i} u2)^4 (1 + 2 Q + 2 \mathbf{i} u2)^4 (-\mathbf{i} + 2 u2)^4 (\mathbf{i} + 2 u2)^4)
\end{aligned}$$

summation over ℓ

$$\begin{aligned}\Delta E_{\text{Luscher}} &= g^8 \left[- 54(1 + \Delta)^3(-5 + 3\Delta)\zeta(3) - 360(1 + \Delta)^2\zeta(5) \right. \\ &\quad \left. + \frac{81(1 - 3\Delta)^2(1 + \Delta)^4}{(1 + 3\Delta)^2} \right] \\ &\equiv \Delta E_{\text{wrapping}}\end{aligned}$$

Other approach

- Y -system [Gromov-Vieira-Kazakov]
- can be extended to the β -deformed SYM [Gromov-Maslyuk]
- one-impurity Lüscher correction is correctly derived

Conclusion

- the β -deformed SYM is probably all-loop integrable
- may need some nontrivial boundary condition
- some relevant S -matrix amplitudes are known
- many open questions:
 - exact S -matrix
 - derivation of asymptotic BAE

- TBA
- strong coupling Lüscher correction

