The spectrum of anomalous dimensions in $\mathcal{N} = 4$ Yang-Mills and BFKL

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Outline

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Introduction

The AdS/CFT correspondence

The large N limit of $\mathcal{N} = 4$ Yang-Mills is dual to type IIB string theory on $AdS_5 \times S^5$ $\Rightarrow$ **Spectra of both theories should agree**

$\longrightarrow$ Difficult to test, because

$$\frac{R^2}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi}$$

with $\lambda \equiv g_{\mathrm{YM}}^2 N$ the 't Hooft coupling,

and thus **strongly-coupled gauge theory** corresponds to **large radius of curvature** (classical string regime), and viceversa

$\longrightarrow$ **The correspondence is a strong/weak-coupling duality**

A complete formulation of the AdS/CFT correspondence $\Rightarrow$ Precise identification of **string states** with local **gauge invariant operators**

$$\Rightarrow E\sqrt{\alpha'} = \Delta$$

$E \equiv$ **String energy** in the global time coordinate of AdS

$\Delta \equiv$ **Scaling dimension** of gauge operators

Difficulties:

$\Rightarrow$ **String quantization** in $AdS_5 \times S^5$

$\Rightarrow$ Solving the **complete gauge spectrum**

Integrability and asymptotic anomalous dimensions

→ The **one-loop planar dilatation operator** of $\mathcal{N}=4$ Yang-Mills is the hamiltonian of an **integrable spin chain**

[Minahan,Zarembo] [Beisert,Staudacher]

$\Downarrow$

Given a local gauge-invariant operator

$$\mathcal{O} = \text{tr}\left(\phi_1\phi_2(D_1D_2\phi_2)D_1\psi_2\ldots\right)$$

scaling dimensions $\Delta(\lambda)$, obtained from two-point functions

$$\langle\mathcal{O}(x)\mathcal{O}(y)\rangle = C|x-y|^{-2\Delta(\lambda)}$$

amount to **energies in an integrable system**

Single trace operators can be mapped to states in a closed spin chain
$\Rightarrow$ BMN impurities: magnon excitations

$$\mathrm{tr}(XXXYYX\ldots) \leftrightarrow |\uparrow\uparrow\uparrow\downarrow\downarrow\uparrow\ldots\rangle$$

$$\Downarrow$$

The Bethe ansatz

$\rightarrow$ The rapidities u_j parameterizing the momenta of the magnons satisfy a set of one-loop **Bethe equations**

$$e^{ip_j J} \equiv \left(\frac{u_j + i/2}{u_j - i/2}\right)^J = \prod_{k\neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i} \equiv \prod_{k\neq j}^{M} S(u_j, u_k)$$

$$\text{Energy} \quad \longrightarrow \quad E = -\sum_{j=1}^{M} \frac{d}{du_j} p(u_j) \quad \Rightarrow \quad \gamma = \frac{\lambda}{8\pi^2} E$$

→ There is strong evidence in favor of **higher-loop integrability**

[Beisert,Kristjansen,Staudacher] [Beisert] [Zwiebel]

→ **Assuming integrability** an **asymptotic long-range Bethe ansatz** has been proposed [Beisert,Dippel,Staudacher]

$$\left(\frac{x_j^+}{x_j^-}\right)^J = \prod_{k\neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i} = \prod_{k\neq j}^{M} \frac{x_j^+ - x_k^-}{x_j^- - x_k^+} \frac{1 - \lambda/16\pi^2 x_j^+ x_k^-}{1 - \lambda/16\pi^2 x_j^- x_k^+}$$

where $x_j^\pm$ are generalized rapidities

$$x_j^\pm \equiv x(u_j \pm i/2)\ , \quad x(u) \equiv \frac{u}{2} + \frac{u}{2}\sqrt{1 - 2\frac{\lambda}{8\pi^2}\frac{1}{u^2}}$$

→ The spectrum of length $L \to \infty$ operators is ruled by the **asymptotic Bethe ansatz equations**

The long-range Bethe ansatz is constructed to fit the spectrum of anomalous dimensions, or the $\mathcal{N} = 4$ magnon dispersion relation

→ The (one-loop) **Heisenberg chain** has dispersion relation

$$E = 4\sin^2\left(\frac{p}{2}\right)$$

→ The Bethe ansatz can be **deformed** to include the magnon dispersion relation for planar $\mathcal{N} = 4$ Yang-Mills,

$$E^2 = 1 + \frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{2}\right)$$

The extension/deformation is the long-range Bethe ansatz

[Beisert,Dippel,Staudacher]

Integrability in $AdS_5 \times S^5$

$\longrightarrow$ **Integrable structures are also present on the string side** of the correspondence: there exists a family of flat connections

[Mandal,Suryanarayana,Wadia] [Bena,Polchinski,Roiban]

$\longrightarrow$ **Classical integrability** of coset σ-models [Lüscher,Pohlmeyer] also holds for the $AdS_5 \times S^5$ string ($\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$ coset)

$\rightarrow AdS_4 \times CP^3$ and $\mathcal{N} = 6$ Chern-Simons (AdS_4/CFT_3)

[Minahan,Zarembo] [Arutyunov,Frolov] [Stefanski] [Grignani,Harmark,Orselli] [Ahn,Nepomechie]

$\rightarrow AdS_3 \times S^3 \times T^4$, $AdS_3 \times S^3 \times S^3 \times S^1$ and $\mathcal{N} = 4$ symmetric orbifold CFTs (AdS_3/CFT_2)

[Babichenko,Stefanski,Zarembo]

are also integrable

$\rightarrow$ The **spectrum of classical strings** is governed by a set of **Bethe ansatz equations**, quite similar to the ones on the gauge theory side

[Arutyunov,Frolov,Staudacher]

Asymptotically long operators $\longrightarrow$ Strings with large quantum numbers

Symmetries and the S-matrix

The **all-loop asymptotic S-matrix is fixed** completely by the $SU(2|2) \otimes SU(2|2)$ symmetry **up to a scalar dressing factor**

[Beisert]

$\Downarrow$

$$S(p_j, p_k) = \sigma(p_j, p_k) S_{SU(2|2)}(p_j, p_k) S_{SU(2|2)'}(p_j, p_k)$$

$\rightarrow$ The all-loop S-matrix describes succesfully the **asymptotic** spectrum of states in the AdS/CFT correspondence

The **dressing phase** $\sigma(p_j, p_k; \lambda)$
is responsible for the interpolation

→ The **leading term** in the $1/\sqrt{\lambda}$ expansion is found discretizing the classical finite-gap equations [Arutyunov,Frolov,Staudacher]

→ The **one-loop corrections** to the energies of rotating strings provide the subleading term [Beisert,Tseytlin] [RH,López] [Freyhult,Kristjansen]

→ Solving the crossing symmetry conditions of [Janik] a **strong-coupling expansion** was suggested [Beisert,RH,López], and tested up to two-loops [Roiban,Tirziu,Tseytlin] [Klose,McLoughlin,Minahan,Zarembo]

→ The proposal for a **weak-coupling expansion** [Beisert,Eden,Staudacher] agrees with four-loop computations [Bern,Czakon,Dixon,Kosower,Smirnov]

The **asymptotic Bethe ansatz** relies on the $\mathcal{N} = 4$ **dispersion relation**

$\rightarrow$ The **symmetry algebra fixes the dispersion relation** [Beisert]

$$E(p) = \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2\left(\frac{p}{2}\right)}$$

Symmetry $\Rightarrow$ **Dispersion relation**

$\Downarrow$

Asymptotic spectrum of states

Finite-size operators and wrapping corrections

(See the talks by Balog, Bozhilov, Ahn, Aniceto, Kim and Bajnok)

→ The all-loop S-matrix describes the **asymptotic** spectrum of states in the AdS/CFT correspondence

→ For **finite-size** operators/chains **wrapping interactions** arise and the long-range dressed S-matrix is no longer valid

Wrapping effects appear when the length of the spin chain reaches the perturbative order

$\Downarrow$

Wrapping → Operator length $L \leq$ Number of loops

→ At **four-loops** agreement between perturbative and Bethe ansatz computations breaks down for the **length-four** Konishi operator

[Kotikov,Lipatov,Rej,Staudacher,Velizhanin] [Fiamberti,Santambrogio,Sieg,Zanon] [Bajnok,Janik]

→ The **Konishi operator** ($\mathrm{Tr}([Z,Y]^2)$ in the $SU(2)$ sector, or $\mathrm{Tr}(ZD^2Z)$ in the $SL(2)$ sector), with length L=4, is the simplest example where **wrapping** effects appear, already at four-loops

[Fiamberti,Santambrogio,Sieg,Zanon] [Bajnok,Janik] [Velizhanin]

$$\Delta(\lambda) = \frac{3}{4\pi^2}\lambda - \frac{3}{16\pi^4}\lambda^2 + \frac{21}{256\pi^6}\lambda^3 - \frac{2584 - 384\zeta(3) + 1440\zeta(5)}{65536\pi^8}\lambda^4 + \cdots$$

(Five-loop contribution by [Bajnok,Hegedüs,Janik.Lukowski])

Dimensions of short operators → Energies of quantum string states

- → The spectrum of states with **large quantum numbers** is obtained from solutions to the **Asymptotic Bethe Ansatz equations** ⟶ Solving string theory on a plane $\mathbb{R}^{1,1}$ leads to the Asymptotic Bethe Ansatz for the spectrum
- → The generalization to **short states, with any quantum number**, amounts to **solving string theory** on $\mathbb{R} \times S^1$ and provides a set of **Thermodynamic Bethe Ansatz equations** [Arutyunov,Frolov]
- → A set of discrete **Y-system equations** arising from the Thermodynamic Bethe Ansatz is expected to encode the spectrum of finite-size operators, to any order in the coupling [Gromov,Kazakov,Vieira] [Bombardelli,Fiorvanti,Tateo] (See the talk by D. Bombardelli)

Anomalous dimensions for twist-two operators

Twist-two operators in the $SL(2)$ sector of $\mathcal{N}=4$ Yang-Mills

$$\mathrm{Tr}(\mathcal{D}^{s_1} Z \mathcal{D}^{s_2} Z)$$

$s_1 + s_2 = N \rightarrow$ **Total spin**

Number of $Z = \phi_5 + i\phi_6$ fields $\rightarrow$ **Twist**

The anomalous dimension can be obtained from the Bethe ansatz for the long-range (at **one-loop** agrees with the $XXX_{s=-1/2}$ Heisenberg) chain,

$$\left(\frac{u_k + i/2}{u_k - i/2}\right)^L = \prod_{j\neq k} \frac{x_k^- - x_j^+}{x_k^+ - x_j^-} \frac{1 - \lambda/16\pi^2 x_k^+ x_j^-}{1 - \lambda/16\pi^2 x_k^- x_j^+}$$

In the **asymptotic case**, for infinitely long operators, the **anomalous dimension** is obtained from the $\mathcal{N}=4$ dispersion relation

$$\gamma_2(N) = \sum_{i=1}^{N} E(p_i) = \sum_{i=1}^{N} \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2\left(\frac{p_i}{2}\right)}$$

$N \longrightarrow$ **Number of magnons**

The result from the Bethe ansatz **agrees** with the perturbative evaluation of the twist-two anomalous dimension, up to three-loops

[Moch,Vermaseren,Vogt] [Kotikov,Lipatov,Onischenko,Velizhanin]

$$\begin{aligned}\gamma_{2,1}(N) &= 4S_1 \,, \\ \gamma_{2,2}(N) &= -4(S_3 + S_{-3} - 2S_{-2,1} + 2S_1(S_2 + S_{-2}))\,, \\ \gamma_{2,3}(N) &= -8(2S_{-3}S_2 - S_5 - 2S_{2,3} - \cdots + 16S_{-2,1,1})\,,\end{aligned}$$

where $S_{\bar{a}} \equiv S_{\bar{a}}(N)$ are harmonic sums,

$$\begin{aligned}S_a(N) &= \sum_{j=1}^{N} \frac{(\mathrm{sgn}(a))^j}{j^a}\,, \\ S_{a_1,\ldots,a_n}(N) &= \sum_{j=1}^{N} \frac{(\mathrm{sgn}(a_1))^j}{j^{a_1}} S_{a_2,\ldots,a_n}(j)\end{aligned}$$

(truncations of Riemann's zeta function $\longrightarrow S_a(\infty) = \zeta(a)$)

Several limits of the twist-two anomalous dimension can be considered

- $\longrightarrow$ The **pomeron** singularity, upon continuation to $N=-1+\omega$
- $\longrightarrow$ The **cusp** anomalous dimension, $N \gg 1$
- $\longrightarrow$ The **spin-one** regime, $N=1$

BFKL pomeron

A **four-loop** term can be obtained
trusting the dressed asymptotic Bethe ansatz

[Kotikov,Lipatov,Rej,Staudacher,Velizhanin]

$$\gamma_{2,4}(N) = 16(4S_{-7} + 6S_7 + \cdots - \zeta(3)S_1(S_3 - S_{-3} + 2S_{-2,1}))$$

$\longrightarrow$ It is **asymptotic**: **Wrapping** contributions are **excluded**, and it fails to reproduce the four-loop prediction from the BFKL pomeron ($N = -1 + \omega$, $\omega \to 0$ configuration)

$\Downarrow$

Breakdown of the (asymptotic) Bethe ansatz for **finite-size operators**

Cusp anomalous dimension

In the **high spin-N limit**, $N \gg 1$,

$$\Delta(\lambda) \sim \Delta_{\text{cusp}}(\lambda) \log N$$

with $\Delta_{\text{cusp}}(\lambda)$ the cusp anomalous dimension

For **twist-two** operators

$$\Delta_{\text{cusp},2}(\lambda) = \frac{1}{2}\frac{\lambda}{\pi^2} - \frac{1}{96}\frac{\lambda^2}{\pi^2} + \frac{11}{23040}\frac{\lambda^3}{\pi^2} - \left(\frac{73}{2580480} + \frac{\zeta(3)^2}{1024\pi^6}\right)\frac{\lambda^4}{\pi^2} + \cdots$$

In agreement with the prediction from integrability: the cusp dimension can be computed from the BES [Beisert,Eden,Staudacher] equation

→ The **strong-coupling** limit of the BES integral equation provides a strong-coupling expansion for the cusp anomalous dimension

[Kostov,Serban,Dvolin] [Basso,Korchemsky,Kotanski]

$$\Delta_{\text{cusp},2}(\lambda) = \frac{\sqrt{\lambda}}{\pi} - \frac{3\log 2}{\pi} - \frac{K}{\pi}\frac{1}{\sqrt{\lambda}} + \cdots$$

$$\Downarrow$$

Agrees with the string result: **Twist-two operators** correspond to a **folded string** with spin N along AdS_5, and thus the $N \to \infty$ limit is reachable from semiclassical methods

[Gubser,Klebanov,Polyakov] [Roiban,Tirziu,Tseytlin] [Casteill,Kristjansen]

Spin-one regime

$\longrightarrow$ Integrability had shown before in a four-dimensional gauge theory: the **reggeized limit of scattering amplitudes in QCD** [Lipatov]
(also [Belitsky] [Belitsky,Braun,Gorsky,Korchemsky])

$\rightarrow$ It is thus natural to search for a common origin, or at least some relic, of the integrable structure in $\mathcal{N}=4$ Yang-Mills still present in the Regge limit of scattering amplitudes in QCD

Consider now a **spin-one twist-two** operator $\longrightarrow$ $\boxed{N = 1}$

[Gómez,Gunnesson,RH]

$\rightarrow$ A non-vanishing result is obtained
when evaluating the harmonic sums at $N = 1$,

$$\gamma_2(1) = 4g^2 - 8g^4 + 32g^6 - 160g^8 + \mathcal{O}(g^{10})$$

$$\Downarrow$$

Agrees with the $\mathcal{N} = 4$ dispersion relation

$$\gamma_2(N = 1) = \sqrt{1 + \lambda/\pi^2 \sin^2\left(\frac{p}{2}\right)} - 1$$

for a single magnon with "non-physical" momentum $p = \pi$

$$\Downarrow$$

Probably holds to all orders, and thus

$$\boxed{\gamma_2(N = 1) = E(p = \pi)}$$

Furthermore, the expansion of $\gamma_2(N=1) = E(p=\pi)$ is related to the **analytic extension of the DGLAP anomalous dimension** $\gamma_2(N)$ to $N = -r + \omega$, with $\omega \to 0$

$$\gamma_2(-r+\omega) = \sum_{i=1} \frac{a_{i,r}}{\omega^{2i-1}} g^{2i} + \cdots$$

$$\Downarrow$$

$$\boxed{a_{i,r} = (-1)^i e_i}\ , \quad \text{with } \gamma_2(1) = \textstyle\sum_{i=1} e_i g^{2i}$$

$\rightarrow$ This analytic extension agrees with the **resummation of non-collinear double logarithms in QCD**

BFKL and double logarithms in QCD

The BFKL integral equation, arising in the Regge limit of QCD, allows to compute the leading log contributions to two-particle → two-particle amplitudes, and predicts the leading poles at $N = -r + \omega\ ,\ \ r = 1, 2, \dots$

→ The harmonic sums can be analytically continued to the whole complex plane [Basso,Korchemsky,Kotanski]

$$S_1(N) = \sum_{i=1}^{N} \frac{1}{i} \longrightarrow \psi(N+1) - \psi(1)$$

→ The eigenfunctions of the BFKL kernel are obtained through a Mellin transformation ...

Parton distribution functions are governed by the Bethe-Salpeter equation. In the **Regge limit** of QCD and in **deep inelastic scattering**

$$f(x,Q^2) = f_0(x,Q^2) + 2g^2 \int_x^1 \frac{dz}{z} \int_{Q'^2}^{Q^2} \frac{dk^2}{Q^2} f\left(\frac{x}{z}, k^2\right)$$

with $Q^2 \longrightarrow$ Momentum of the photon
$Q'^2 \longrightarrow$ Transversal scale of the hadron
$s \longrightarrow$ Center of mass energy
$x \equiv Q^2/s$

The Bethe-Salpeter equation performs a perturbative resummation of the **double logarithms in ladder diagrams**

$$\left(g^2 \log\left(\frac{1}{x}\right) \log\left(\frac{Q^2}{Q'^2}\right)\right)^n$$

After Mellin transformation ($s \to \omega$)
the solution to the Bethe-Salpeter equation is

$$f(\omega,\gamma) = \frac{\omega\, f_0(\omega,\gamma)}{\omega + 4g^2\frac{1}{\gamma}}$$

In the **leading logarithm approximation in BFKL**
the pole in the solution for the parton distribution function is corrected by

$$\frac{1}{\omega - 2g^2 \chi_{\mathsf{LLA}}}$$

where χ_{LLA} is the **BFKL kernel**,

$$\chi_{\mathsf{LLA}}(\omega, \gamma) = 2\psi(1) - \psi\left(-\frac{\gamma}{2}\right) - \psi\left(1 + \frac{\gamma}{2} + \omega\right)$$

However, it is not only collinear double logarithms that should be considered, but also **non-collinear double logarithms**

$$\log s \log s \quad \rightarrow \quad \log s \log Q^2$$

$$\Downarrow$$

This resummation can be performed with a **change in the kinematic region of integration** in the Bethe-Salpeter equation

→ The pole in the solution of the Bethe-Salpeter equation should be

$$\omega = 2g^2 \left(-\frac{2}{\gamma} + \frac{1}{\omega + \gamma/2} \right)$$

$$\Downarrow$$

$$\gamma = \omega \sqrt{1 - \frac{8g^2}{\omega^2}} - \omega \quad \longrightarrow \quad \textbf{Modifies the BFKL kernel}$$

Now, a closer look at the anomalous dimension
after resummation of non-collinear double logarithms,

$$\boxed{\gamma = \omega\sqrt{1 - \frac{8g^2}{\omega^2}} - \omega}$$

shows that it is precisely

$$\boxed{\gamma_2(N = 1) = E_{\mathcal{N}=4}(p = \pi)}$$

Single-magnon dispersion relation for $\mathcal{N} = 4$ Yang-Mills!!

$$\Downarrow$$

Resummation of non-collinear double logarithms in the Regge limit of QCD!!

Conclusions

- Integrable structures have provided extremely precise tests of the AdS/CFT correspondence
  - The Asymptotic Bethe ansatz probes semiclassical strings
  - Finite-size operators start being systematically analyzed
- The **dispersion relation in planar $\mathcal{N} = 4$ Yang-Mills** can be related to **non-collinear double logarithms in QCD**
  - The dispersion relation follows from the anomalous dimension for spin-one twist-two operators

Open questions

- Relate integrable structures in the AdS/CFT correspondence and in QCD: symmetries and structure of the central extension in $\mathcal{N} = 4$ allow to determine the dispersion relation $\longrightarrow$ Hidden symmetries in the Regge limit of QCD
- **Symmetry pattern organizing the correspondence**
    - (Quantum) Deformation of a gauge theory into a string theory