1 Energy of a system of free complex scalars

Consider a free, massless, complex scalar field, with Lagrangian density

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi. \tag{1.1}$$

The general solution of the field equation

$$\partial^2 \phi + m^2 \phi = 0 \tag{1.2}$$

can be written as

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{\sqrt{2p^0}} \left[a(\vec{p}) e^{-ip\cdot x} + b^\dagger(\vec{p}) e^{ip\cdot x} \right] \tag{1.3}$$

where $p^2 = m^2$ (prove it as an exercise). Compute the total energy of the system

$$E_0 = \int d^3x \, \mathcal{H} \tag{1.4}$$

where

$$\mathcal{H} = \mathcal{T}^{00} = \sum_i \frac{\partial \mathcal{L}}{\partial \partial_0 \phi_i} \partial^0 \phi_i - g^{00} \mathcal{L} = \partial_0 \phi^\dagger \partial_0 \phi + \partial_i \phi^\dagger \partial_i \phi + m^2 \phi^\dagger \phi. \tag{1.5}$$

(the sum is extended to all fields in the theory, in this case ϕ and $\phi^\dagger$.)

Solution

Using the free-field solution eq. (1.3) we find

$$\begin{aligned}
E_0 &= \frac{1}{(2\pi)^3} \int d^3x \int \frac{d^3p}{\sqrt{2p^0}} \int \frac{d^3q}{\sqrt{2q^0}} (p_0 q_0 + p_i q_i) \left[a^\dagger(\vec{p}) e^{ip\cdot x} - b(\vec{p}) e^{-ip\cdot x} \right] \left[a(\vec{q}) e^{-iq\cdot x} - b^\dagger(\vec{q}) e^{iq\cdot x} \right] \\
&+ \frac{m^2}{(2\pi)^3} \int d^3x \int \frac{d^3p}{\sqrt{2p^0}} \int \frac{d^3q}{\sqrt{2q^0}} \left[a^\dagger(\vec{p}) e^{ip\cdot x} + b(\vec{p}) e^{-ip\cdot x} \right] \left[a(\vec{q}) e^{-iq\cdot x} + b^\dagger(\vec{q}) e^{iq\cdot x} \right] \\
&= \frac{1}{2} \int \frac{d^3p}{p^0} \Big\{ (p_0^2 + p_i p_i + m^2) \left[a^\dagger(\vec{p}) a(\vec{p}) + b(\vec{p}) b^\dagger(\vec{p}) \right] \\
&\qquad - (p_0^2 - p_i p_i - m^2) \left[a^\dagger(\vec{p}) b^\dagger(-\vec{p}) e^{2ip_0 t} + b(\vec{p}) a(-\vec{p}) e^{-2ip_0 t} \right] \Big\}
\end{aligned} \tag{1.6}$$

where we have used

$$\frac{1}{(2\pi)^3} \int d^3x \, e^{i(p\pm q)\cdot x} = \delta(\vec{p} \pm \vec{q}) \, e^{i(p^0 \pm p^0)t}. \tag{1.7}$$

(prove it as an exercise). Recalling that $p^2 = p_0^2 - p_i p_i = m^2$, we finally obtain

$$E_0 = \int d^3p \, p^0 \left[a^\dagger(\vec{p}) a(\vec{p}) + b(\vec{p}) b^\dagger(\vec{p}) \right]. \tag{1.8}$$

Comments

We see that a positive-definite expression for the total energy is achieved if we assume that annihilation and creation operators obey commutation relations:

$$\left[a(\vec{p}), a^{\dagger}(\vec{q})\right] = \left[b(\vec{p}), b^{\dagger}(\vec{q})\right] = \delta(\vec{p} - \vec{q}), \tag{1.9}$$

appropriate for Bose-Einstein statistics. We get

$$E_0 = \int d^3p\, p^0 \, \left[a^{\dagger}(\vec{p})a(\vec{p}) + b^{\dagger}(\vec{p})b(\vec{p})\right] \tag{1.10}$$

up to an infinite additive constant.

The scalar lagrangian (1.1) is invariant under the field transformation

$$\phi \to e^{i\alpha}\phi, \tag{1.11}$$

where α is a real constant. The corresponding conserved current is

$$j_\mu = i(\phi^{\dagger}\partial_\mu\phi - \phi\partial_\mu\phi^{\dagger}). \tag{1.12}$$

Proceeding as in the case of the total energy, we can obtain an expression for the total conserved charge:

$$Q = \int d^3x\, i(\phi^{\dagger}\partial_0\phi - \phi\partial_0\phi^{\dagger}) = \int d^3p \, \left[a^{\dagger}(\vec{p})a(\vec{p}) - b^{\dagger}(\vec{p})b(\vec{p})\right], \tag{1.13}$$

which shows that b-particles carry opposite charge with respect to a-particles; b-particles are usually called the antiparticles of a-particles.

2 Energy of a system of free massless Weyl spinors

Consider a free, massless, right-handed Weyl spinor field, with Lagrangian density

$$\mathcal{L}_R = i\xi_R^{\dagger}\,\bar{\sigma}_\mu\partial^\mu\,\xi_R. \tag{2.1}$$

The general solution of the field equation

$$\bar{\sigma}_\mu\partial^\mu\,\xi_R = 0 \tag{2.2}$$

can be written as

$$\xi_R(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{\sqrt{2p^0}} \left[u_R(\vec{p})a(\vec{p})e^{-ip\cdot x} + v_R(\vec{p})b^{\dagger}(\vec{p})e^{ip\cdot x}\right] \tag{2.3}$$

where $p^2 = 0$, and

$$\vec{\sigma}\cdot\vec{p}\,u_R(\vec{p}) = p^0\,u_R(\vec{p}) \tag{2.4}$$

$$\vec{\sigma}\cdot\vec{p}\,v_R(\vec{p}) = p^0\,v_R(\vec{p}). \tag{2.5}$$

Observe that eqs. (2.4,2.5) give

$$u_R^\dagger(\vec{p})v_R(-\vec{p}) = -\frac{p^i p^j}{p^{0\,2}} u_R^\dagger(\vec{p})\sigma_i\sigma_j v_R(-\vec{p}) = -\frac{p^i p^j}{p^{0\,2}} u_R^\dagger(\vec{p})\frac{1}{2}\left\{\sigma_i,\sigma_j\right\} v_R(-\vec{p}) = -u_R^\dagger(\vec{p})v_R(-\vec{p}) \tag{2.6}$$

and therefore

$$u_R^\dagger(\vec{p})v_R(-\vec{p}) = 0. \tag{2.7}$$

Compute the total energy of the system,

$$E_{1/2} = \int d^3x\, \mathcal{H} = -i\int d^3x\, \xi_R^\dagger\, \sigma_i\partial^i\, \xi_R. \tag{2.8}$$

Solution

Using the free-field solution eq. (2.3) we find

$$\begin{aligned}
E_{1/2} &= -\frac{i}{(2\pi)^3}\int d^3x \int \frac{d^3p}{\sqrt{2p^0}}\int \frac{d^3q}{\sqrt{2q^0}} \\
&\quad \left[u_R^\dagger(\vec{p})a^\dagger(\vec{p})e^{ip\cdot x} + v_R^\dagger(\vec{p})b(\vec{p})e^{-ip\cdot x}\right] i\sigma_i q^i \left[u_R(\vec{q})a(\vec{q})e^{-iq\cdot x} - v_R(\vec{q})b^\dagger(\vec{q})e^{iq\cdot x}\right] \\
&= \frac{1}{2}\int d^3p\, \Big[u_R^\dagger(\vec{p})u_R(\vec{p})a^\dagger(\vec{p})a(\vec{p}) - u_R^\dagger(\vec{p})v_R(-\vec{p})a^\dagger(\vec{p})b^\dagger(-\vec{p})e^{2iq^0t} \\
&\quad + v_R^\dagger(\vec{p})u_R(-\vec{p})b(\vec{p})a(-\vec{p})e^{-2iq^0t} - v_R^\dagger(\vec{p})v_R(\vec{p})b(\vec{p})b^\dagger(\vec{p})\Big] \\
&= \frac{1}{2}\int d^3p\, \left[u_R^\dagger(\vec{p})u_R(\vec{p})a^\dagger(\vec{p})a(\vec{p}) - v_R^\dagger(\vec{p})v_R(\vec{p})b(\vec{p})b^\dagger(\vec{p})\right]
\end{aligned} \tag{2.9}$$

where we have used eq. (1.7) and eq. (2.7).

Comments

We see that a positive-definite expression for the total energy can only be achieved if we assume that annihilation and creation operators obey anticommutation relations:

$$\left\{a(\vec{p}), a^\dagger(\vec{q})\right\} = \left\{b(\vec{p}), b^\dagger(\vec{q})\right\} = \delta(\vec{p}-\vec{q}), \tag{2.10}$$

appropriate for Fermi-Dirac statistics. Furthermore, the two-component spinors $u_R(\vec{p}), v_R(\vec{p})$ must be normalized as

$$u_R^\dagger(\vec{p})u_R(\vec{p}) = 2p^0 \tag{2.11}$$
$$v_R^\dagger(\vec{p})v_R(\vec{p}) = 2p^0, \tag{2.12}$$

so that

$$E_{1/2} = \int d^3p\, p^0\, \left[a^\dagger(\vec{p})a(\vec{p}) + b^\dagger(\vec{p})b(\vec{p})\right]. \tag{2.13}$$

The Weyl lagrangian (2.1) is invariant under the field transformation

$$\xi_R \rightarrow e^{i\alpha}\xi_R, \tag{2.14}$$

where α is a real constant. The corresponding conserved current is

$$j_\mu = \xi_R^\dagger \, \bar{\sigma}_\mu \, \xi_R. \tag{2.15}$$

Proceeding as in the case of the total energy, we can obtain an expression for the total conserved charge:

$$Q = \int d^3x \, \xi_R^\dagger \, \bar{\sigma}_0 \, \xi_R = \int d^3p \, \left[a^\dagger(\vec{p}) a(\vec{p}) - b^\dagger(\vec{p}) b(\vec{p}) \right], \tag{2.16}$$

which is consistent with the interpretation of b-particles as antiparticles of a-particles.