Proposed Problems on the Standard Model

A. Pich (TAE 2010, Barcelona)

Problem 1

The scalar sector of the Standard Model Lagrangian has the form

$$\mathcal{L}_S \,=\, (D_\mu\phi)^\dagger\, D^\mu\phi \,-\, V(\phi)\,, \qquad\qquad D^\mu\phi \,=\, \left[\partial^\mu + i\,g\,\frac{\vec{\sigma}}{2}\,\vec{W}^\mu + i\,g'\,y_\phi\,B^\mu\right]\,\phi\,,$$

$$V(\phi) \,=\, \mu^2\phi^\dagger\phi + h\left(\phi^\dagger\phi\right)^2 \qquad\qquad (h>0\,,\; \mu^2<0)\,,$$

where $\phi(x)$ is an $SU(2)_L$ doublet of complex scalar fields. The potential, which only depends on the modulus of the scalar doublet, has its minimum at $|\phi|_{\rm min} = \sqrt{\frac{-\mu^2}{2h}} \equiv \frac{v}{\sqrt{2}}$. The scalar doublet can be then parametrized in the form

$$\phi(x) \,\equiv\, \left(\begin{array}{c}\phi_a(x)\\ \phi_b(x)\end{array}\right) \,=\, \exp\left\{i\,\frac{\sigma_i}{2}\,\theta^i(x)\right\}\,\frac{1}{\sqrt{2}}\left(\begin{array}{c}0\\ v+H(x)\end{array}\right)\,.$$

The (arbitrary) choice of vacuum configuration, $\phi_0^T = \frac{1}{\sqrt{2}}\,(0,\,v)$, breaks the $SU(2)_L\otimes U(1)_Y$ gauge symmetry 'spontaneously', leaving one generator unbroken:

$$Q\equiv (T_3+Y)\equiv \frac{1}{2}\,(\sigma_3+I_2)\,=\,\left(\begin{array}{cc}1&0\\0&0\end{array}\right)\,, \qquad\qquad Q\,\phi_0\,=\,0\,, \qquad\qquad {\rm e}^{iQ\gamma}\,\phi_0\,=\,\phi_0\,.$$

We would like to identify Q with the electric charge and the associated unbroken symmetry with $U(1)_{\rm em}$. Thus, $\phi_b(x)$ and $H(x)$ are neutral fields, $\phi_a(x)$ has $Q=+1$ and $y_\phi = \frac{1}{2}$.

a) Under a local $U(1)_{\rm em}$ transformation, $\phi'(x) = {\rm e}^{iQ\gamma(x)}\,\phi(x)$, the electromagnetic field should transform as $A'_\mu(x) = A_\mu(x) - \frac{1}{e}\,\partial_\mu\gamma(x)$, while the Z field should remain invariant. Show that this requirement implies $e = g\,\sin\theta_W = g'\,\cos\theta_W$, where θ_W is the electroweak mixing angle,

$$\left(\begin{array}{c}W^3_\mu\\ B_\mu\end{array}\right) \,\equiv\, \left(\begin{array}{cc}\cos\theta_W & \sin\theta_W\\ -\sin\theta_W & \cos\theta_W\end{array}\right)\left(\begin{array}{c}Z_\mu\\ A_\mu\end{array}\right)\,.$$

b) Find the explicit expression of the Lagrangian $\mathcal{L}_S$ in the 'unitary gauge' $\vec{\theta}(x) = \vec{0}$. Show that $M_Z\cos\theta_W = M_W = \frac{1}{2}\,g\,v$.

c) Working in a general gauge where charged scalars are present, show that one gets the correct $U(1)_{\rm em}$ covariant derivative $D^\mu = \partial^\mu + ieQA^\mu$.

Problem 2

Consider the free Lagrangian of a complex massless spin–1 field $W^\mu(x)$,

$$\mathcal{L}_0^W \;=\; -\frac{1}{2}\,(\partial_\mu W_\nu^\dagger - \partial_\nu W_\mu^\dagger)(\partial^\mu W^\nu - \partial^\nu W^\mu)\,.$$

a) Imposing the minimal coupling prescription $\partial^\mu \longrightarrow D^\mu = \partial^\mu + ieQA^\mu$, obtain the corresponding electromagnetic couplings. Check that, taking the charge of W^μ to be $Q=-1$, one correctly reproduces the Standard Model $W^\dagger W A$ and $W^\dagger W A^2$ vertices.

b) The Standard Model Lagrangian contains one additional $W_\mu^\dagger W_\nu F^{\mu\nu}$ term. Derive its explicit form.

c) The Standard Model does not contain any trilinear coupling of three neutral gauge bosons $[Z^3,\, \gamma Z^2,\, \gamma^2 Z\,, \gamma^3]$. Explain this fact.

Problem 3

The three polarizations of a massive spin–1 particle with momentum k^μ are described through a basis of three four-vectors $\varepsilon_r^\mu(\vec{k})$, satisfying $\varepsilon_r^\mu(\vec{k}) k_\mu = 0$ and $\varepsilon_r^\mu(\vec{k})\, \varepsilon_\mu^s(\vec{k}) = -\delta_{rs}$. In the rest frame, $k^\mu = (M,\, \vec{0})$, they correspond to the three independent space unit vectors with a zero time component $[\varepsilon_1^\mu(\vec{k}) = (0,1,0,0),\, \varepsilon_2^\mu(\vec{k}) = (0,0,1,0),\, \varepsilon_3^\mu(\vec{k}) = (0,0,0,1)]$. In the boosted frame $k^\mu = (k^0, 0, 0, |\vec{k}|)$, the transverse polarization vectors $\varepsilon_{1,2}^\mu(\vec{k})$ remain the same, while the longitudinal polarization is given by $\varepsilon_3^\mu(\vec{k}) = \frac{1}{M}\,(|\vec{k}|, 0, 0, k^0)$. Note that this longitudinal state diverges when the momentum of the particle approaches infinity

$$\varepsilon_3^\mu(\vec{k}) \quad \overset{k\,\to\,\infty}{\longrightarrow} \quad \frac{k^\mu}{M} \,+\, O\left(\frac{1}{|\vec{k}|^2}\right) .$$

a) Consider the process $\nu_e \bar{\nu}_e \to W_L^- W_L^+$. In the Standard Model there are two amplitudes contributing to lowest-order: t–channel electron exchange and s–channel Z exchange. Show that the t–channel amplitude leads to a cross section which increases with energy, violating unitarity.

b) Show that the s–channel amplitude cancels exactly the bad high-energy behaviour.

c) The process $\nu_e \bar{\nu}_e \to Z_L Z_L$ does not receive any s–channel contribution (a Z^3 vertex does not exist in the Standard Model). Show that the t–channel contribution is well-behaved in this case. Discuss the result.

Solutions

Problem 1

a) Under a local $U(1)_{\rm em}$ transformation, $\phi'(x) = {\rm e}^{iQ\gamma(x)}\,\phi(x)$,

$$A'_\mu(x) = A_\mu(x) - \frac{1}{e}\,\partial_\mu\gamma(x)\,, \qquad\qquad Z'_\mu(x) = Z_\mu(x)\,,$$
$$W^3_\mu(x) = W^3_\mu(x) - \frac{1}{g}\,\partial_\mu\gamma(x)\,, \qquad\qquad B'_\mu(x) = B_\mu(x) - \frac{1}{g'}\,\partial_\mu\gamma(x)\,,$$

This requires

$$\frac{1}{e} = \frac{s_W}{g} + \frac{c_W}{g'}\,, \qquad\qquad 0 = \frac{c_W}{g} - \frac{s_W}{g'}\,,$$

where $s_W \equiv \sin\theta_W$, $c_W \equiv \cos\theta_W$. Therefore,

$$\tan\theta_W \; = \; \frac{g'}{g}\,, \qquad\qquad e \; = \; g\,\sin\theta_W \; = \; g'\,\cos\theta_W \; = \; \frac{gg'}{\sqrt{g^2+g'^2}}\,.$$

b) Taking $\vec{\theta}(x) = \vec{0}$,

$$D^\mu\phi(x) \; = \; \frac{1}{\sqrt{2}}\left[\begin{array}{c} i\,\frac{g}{\sqrt{2}}\,W^{\mu\dagger} \\ \partial^\mu - i\,\frac{g}{2}\,W_3^\mu + i\,\frac{g'}{2}\,B^\mu \end{array}\right](v+H)\,.$$

$$\begin{aligned}(D_\mu\phi)^\dagger\, D^\mu\phi \;&=\; \frac{1}{2}\,\partial_\mu H\partial^\mu H + (v+H)^2\,\frac{g^2}{4}\left\{W_\mu^\dagger\, W^\mu + \frac{1}{2}\left(W^3_\mu - \frac{g'}{g}\,B_\mu\right)\left(W_3^\mu - \frac{g'}{g}\,B^\mu\right)\right\} \\ &=\; \frac{1}{2}\,\partial_\mu H\partial^\mu H + \left(1+\frac{H}{v}\right)^2\,\frac{g^2v^2}{4}\,\left\{W_\mu^\dagger\, W^\mu + \frac{1}{2\cos^2\theta_W}\,Z_\mu Z^\mu\right\}\,.\end{aligned}$$

Therefore,

$$M_Z\,\cos\theta_W \; = \; M_W \; = \; \frac{1}{2}\,v\,g\,.$$

The scalar potential takes the form $[M_H = \sqrt{-2\mu^2} = \sqrt{2h}\,v]$

$$-\,V(\phi) \; = \; \frac{1}{4}\,hv^4 - \frac{1}{2}\,M_H^2\,H^2 - \frac{M_H^2}{2v}\,H^3 - \frac{M_H^2}{8v^2}\,H^4\,.$$

c) In a general gauge $\;[T_\pm \equiv \frac{1}{\sqrt{2}}\,(T_1 \pm i\,T_2) = \frac{1}{2\sqrt{2}}\,(\sigma_1 \pm i\,\sigma_2)]$

$$D^\mu = \partial^\mu + i\,eQA^\mu - i\,\frac{g}{2c_W}\,(1-2c_W^2 Q)\,Z^\mu + i\,g\,(W^{\mu\dagger}T_+ + W^\mu T_-)\,.$$

Problem 2

a)

$$
\begin{aligned}
\mathcal{L}^W &= -\frac{1}{2}\left[(\partial_\mu + ieA_\mu)W_\nu^\dagger - (\partial_\nu + ieA_\nu)W_\mu^\dagger)\right]\left[(\partial^\mu - ieA^\mu)W^\nu - (\partial^\nu - ieA^\nu)W^\mu)\right] \\
&= \mathcal{L}_0^W + ie\left\{(\partial^\mu W^\nu - \partial^\nu W^\mu)W_\mu^\dagger A_\nu - (\partial^\mu W^{\nu\dagger} - \partial^\nu W^{\mu\dagger})W_\mu A_\nu\right\} \\
&\quad - e^2\left\{W_\mu^\dagger W^\mu A_\nu A^\nu - W_\mu^\dagger A^\mu W_\nu A^\nu\right\} .
\end{aligned}
$$

b) The pure gauge sector of the Standard Model Lagrangian is given by the following expression $[W_1^\mu = \frac{1}{\sqrt{2}}(W^{\mu\dagger} + W^\mu)$, $W_2^\mu = \frac{i}{\sqrt{2}}(W^{\mu\dagger} - W^\mu)$, $\epsilon^{ijk}\epsilon_{imn} = \delta^j_m\delta^k_n - \delta^j_n\delta^k_m]$:

$$
\begin{aligned}
\mathcal{L}_{\rm Kin} &= -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}W^i_{\mu\nu}W_i^{\mu\nu} \\
&= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}(\partial^\mu Z^\nu - \partial^\nu Z^\mu)(\partial_\mu Z_\nu - \partial_\nu Z_\mu) - \frac{1}{2}(\partial_\mu W_\nu^\dagger - \partial_\nu W_\mu^\dagger)(\partial^\mu W^\nu - \partial^\nu W^\mu) \\
&\quad + \frac{g}{2}\epsilon^{ijk}(\partial^\mu W_i^\nu - \partial^\nu W_i^\mu)W_\mu^j W_\nu^k - \frac{g^2}{4}\epsilon^{ijk}\epsilon_{imn}W_j^\mu W_k^\nu W_\mu^m W_\nu^n \\
&= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}(\partial^\mu Z^\nu - \partial^\nu Z^\mu)(\partial_\mu Z_\nu - \partial_\nu Z_\mu) - \frac{1}{2}(\partial_\mu W_\nu^\dagger - \partial_\nu W_\mu^\dagger)(\partial^\mu W^\nu - \partial^\nu W^\mu) \\
&\quad + ie\cot\theta_W\left\{(\partial^\mu W^\nu - \partial^\nu W^\mu)W_\mu^\dagger Z_\nu - \left(\partial^\mu W^{\nu\dagger} - \partial^\nu W^{\mu\dagger}\right)W_\mu Z_\nu + W_\mu W_\nu^\dagger(\partial^\mu Z^\nu - \partial^\nu Z^\mu)\right\} \\
&\quad + ie\left\{(\partial^\mu W^\nu - \partial^\nu W^\mu)W_\mu^\dagger A_\nu - \left(\partial^\mu W^{\nu\dagger} - \partial^\nu W^{\mu\dagger}\right)W_\mu A_\nu + W_\mu W_\nu^\dagger(\partial^\mu A^\nu - \partial^\nu A^\mu)\right\} \\
&\quad - \frac{e^2}{2\sin^2\theta_W}\left\{\left(W_\mu^\dagger W^\mu\right)^2 - W_\mu^\dagger W^{\mu\dagger}W_\nu W^\nu\right\} - e^2\cot^2\theta_W\left\{W_\mu^\dagger W^\mu Z_\nu Z^\nu - W_\mu^\dagger Z^\mu W_\nu Z^\nu\right\} \\
&\quad - e^2\cot\theta_W\left\{2W_\mu^\dagger W^\mu Z_\nu A^\nu - W_\mu^\dagger Z^\mu W_\nu A^\nu - W_\mu^\dagger A^\mu W_\nu Z^\nu\right\} \\
&\quad - e^2\left\{W_\mu^\dagger W^\mu A_\nu A^\nu - W_\mu^\dagger A^\mu W_\nu A^\nu\right\}.
\end{aligned}
$$

The $W^\dagger W A$ and $W^\dagger W A^2$ terms agree with the Lagrangian derived in section a). The $W_\mu^\dagger W_\nu F^{\mu\nu}$ vertex is generated by the term $g\,(\partial^\mu W_3^\nu - \partial^\nu W_3^\mu)W_\mu^1 W_\nu^2$. The $U(1)_{\rm em}$ invariance is better understood writing the $SU(2)_L$ covariant derivative in the form

$$
\tilde{D}^\mu \equiv \partial^\mu + i\,g\,\frac{\vec{\sigma}}{2}\vec{W}^\mu = (\partial^\mu + i\,eA^\mu T_3) + i\,g\,(W^{\mu\dagger}T_+ + W^\mu T_- + c_W Z^\mu T_3)\,.
$$

Since $[T_3, T_i] = Q_i T_i$, $(Q_\pm = \pm 1,\ Q_3 = 0)$,

$$
\begin{aligned}
\widetilde{W}_{\mu\nu} \equiv -\frac{i}{g}\left[\tilde{D}_\mu\,,\,\tilde{D}_\nu\right] &= s_W F_{\mu\nu}T_3 + \{(\partial_\mu - ieA_\mu)W_\nu - (\partial_\nu - ieA_\nu)W_\mu\}\,T_- \\
&\quad + \left\{(\partial_\mu + ieA_\mu)W_\nu^\dagger - (\partial_\nu + ieA_\nu)W_\mu^\dagger\right\}T_+ + \cdots
\end{aligned}
$$

Remember that $\mathcal{L}_{\rm Kin}^{SU(2)} = -\frac{1}{2}\,{\rm Tr}\left[\widetilde{W}_{\mu\nu}\widetilde{W}^{\mu\nu}\right]$.

c) The $U(1)_{\rm em}$ invariance guarantees that the photon does not couple to neutral particles. Moreover the $SU(2)_L$ commutation relation [the antisymmetric ϵ^{ijk} factor] cannot generate terms with three or four W_3^μ fields.

Problem 3

a) At very high energies, the reduced t–channel amplitude contributing to the scattering process $\nu_e(p_1) + \bar\nu_e(p_2) \to W_L^+(k_+) + W_L^-(k_-)$ is $[\mathcal{P}_L\, u_\nu(p_1) \equiv \frac{1}{2}(1-\gamma_5)\, u_\nu(p_1) = u_\nu(p_1)$, $l^\mu \equiv (p_1 - k_+)^\mu = (k_- - p_2)^\mu$, $t \equiv l^2]$:

$$\begin{aligned} T_t &\approx \frac{g^2}{2t}\, \varepsilon_3^{\mu *}(k_+)\varepsilon_3^{\nu *}(k_-)\; [\bar v_{\bar\nu_e}(p_2)\gamma_\nu \not{l}\gamma_\mu \mathcal{P}_L u_{\nu_e}(p_1)] \;\approx\; \frac{g^2}{2M_W^2 t}\; [\bar v_{\bar\nu_e}(p_2)\not{k}_- \not{l}\not{k}_+ u_{\nu_e}(p_1)] \\ &= \frac{g^2}{2M_W^2 t}\; [\bar v_{\bar\nu_e}(p_2)(\not{k}_- - \not{p}_2)\not{l}\not{k}_+ u_{\nu_e}(p_1)] \;=\; \frac{g^2}{2M_W^2}\; [\bar v_{\bar\nu_e}(p_2)\not{k}_+ u_{\nu_e}(p_1)]\,. \end{aligned}$$

On dimensional grounds, this implies $\sigma \sim g^4 s/M_W^4$, with $s \equiv q^2 \equiv (p_1+p_2)^2 = (k_- + k_+)^2$.

b) The s–channel exchange of a neutral Z boson generates the additional amplitude:

$$\begin{aligned} T_s &\approx \frac{g^2}{2}\, \frac{-g^{\alpha\beta} + q^\alpha q^\beta/M_Z^2}{s - M_Z^2}\, \varepsilon_3^{\mu *}(k_+)\varepsilon_3^{\nu *}(k_-)\; [\bar v_{\bar\nu_e}(p_2)\gamma_\alpha \mathcal{P}_L u_{\nu_e}(p_1)] \\ &\quad \times \{k_{+\nu} g_{\mu\beta} - k_{+\beta} g_{\mu\nu} - k_{-\mu} g_{\nu\beta} + k_{-\beta} g_{\mu\nu} - q_\mu g_{\nu\beta} + q_\nu g_{\mu\beta}\} \\ &\approx \frac{g^2}{2s}\left(\frac{q_\alpha q_\beta}{M_Z^2} - g_{\alpha\beta}\right) [\bar v_{\bar\nu_e}(p_2)\gamma^\alpha u_{\nu_e}(p_1)] \left\{(k_- - k_+)^\beta (\varepsilon_{3+}^*\cdot\varepsilon_{3-}^*) + 2\varepsilon_{3+}^{*\beta}(k_+\cdot\varepsilon_{3-}^*) - 2\varepsilon_{3-}^{*\beta}(k_-\cdot\varepsilon_{3+}^*)\right\} \\ &\approx -\frac{g^2}{2M_W^2 s}\; [\bar v_{\bar\nu_e}(p_2)\gamma_\alpha u_{\nu_e}(p_1)]\; (k_+\cdot k_-)\; \left\{(k_- - k_+)^\alpha + 2k_+^\alpha - 2k_-^\alpha\right\} \\ &\approx -\frac{g^2}{4M_W^2}\; [\bar v_{\bar\nu_e}(p_2)(\not{k}_+ - \not{k}_-) u_{\nu_e}(p_1)] \;\approx\; -\frac{g^2}{2M_W^2}\; [\bar v_{\bar\nu_e}(p_2)\not{k}_+ u_{\nu_e}(p_1)]\,, \end{aligned}$$

which cancels the t–channel contribution.

c) The process $\nu_e(p_1) + \bar\nu_e(p_2) \to Z_L(k_1) + Z_L(k_2)$ has two t–channel amplitudes, corresponding to the permutation of the two identical Z_L bosons $[l^\mu \equiv (p_1 - k_1)^\mu = (k_2 - p_2)^\mu$, $t \equiv l^2$, $r^\mu \equiv (p_1 - k_2)^\mu = (k_1 - p_2)^\mu$, $u \equiv r^2]$:

$$\begin{aligned} T_t &\approx \frac{g^2}{4c_W^2}\left\{\frac{1}{t}\; [\bar v_{\bar\nu_e}(p_2)\not{\varepsilon}_2^*\not{l}\not{\varepsilon}_1^* u_{\nu_e}(p_1)] + \frac{1}{u}\; [\bar v_{\bar\nu_e}(p_2)\not{\varepsilon}_1^*\not{r}\not{\varepsilon}_2^* u_{\nu_e}(p_1)]\right\} \\ &\approx \frac{g^2}{4c_W^2 M_Z^2}\left\{\frac{1}{t}\; [\bar v_{\bar\nu_e}(p_2)\not{k}_2\not{l}\not{k}_1 u_{\nu_e}(p_1)] + \frac{1}{u}\; [\bar v_{\bar\nu_e}(p_2)\not{k}_1\not{r}\not{k}_2 u_{\nu_e}(p_1)]\right\} \\ &= \frac{g^2}{4c_W^2 M_Z^2}\; [\bar v_{\bar\nu_e}(p_2)(\not{k}_1 - \not{k}_1) u_{\nu_e}(p_1)] \;=\; 0\,. \end{aligned}$$

This is a consequence of Bose symmetry