Asymptotic behavior of degenerate logistic equations

Aníbal Rodríguez-Bernal

Departamento de Matemática Aplicada
Universidad Complutense de Madrid

Instituto de Ciencias Matemáticas (ICMAT)
CSIC-UAM-UC3M-UCM

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Joint work with J.M. Arrieta and R. Pardo, UCM Madrid (in progress)

The problem: a logistic equation

$$\begin{cases} u_t - \Delta u = \lambda u - n(x)u^\rho & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \\ u(0) = u_0 \geq 0 \end{cases}$$

- $\Omega \subset \mathbb{R}^N$ bounded domain.
- $n(x) \geq 0$ in Ω is a continuous function not identically zero.
- $\rho > 1$, $\lambda \in \mathbb{R}$.
- $0 \leq u_0 \in L^1(\Omega)$ and the solution, which will be denoted $u(t; u_0)$, becomes classical for $t > 0$.

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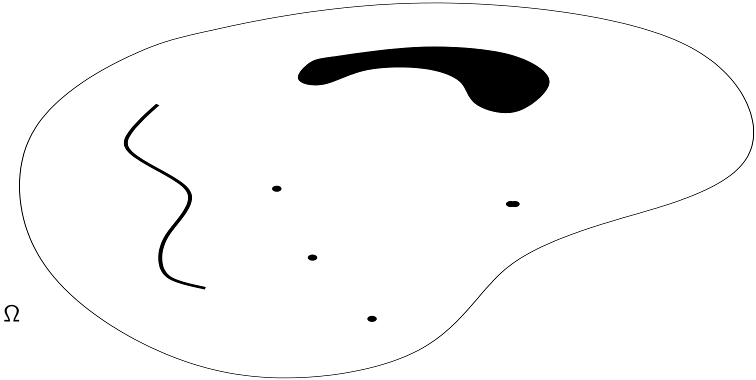
which is a compact set in Ω (does not touch the boundary).



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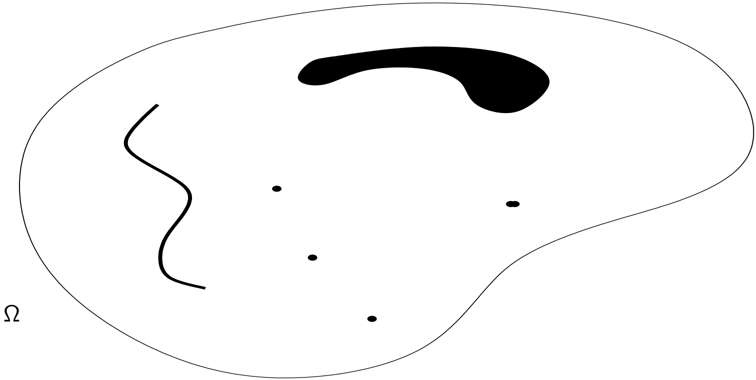

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Previous results

Proposition (ARB and A. Vidal-López, *JDE*, 244:2983–3030, 2008

i) Suppose either $n(x) \geq \gamma > 0$ in $\overline{\Omega}$ or $1/n \in L^s(\Omega)$, $s > N/2\rho$. Then for any $\lambda \in I\!R$ there exists a unique globally asymptotically stable nonegative equilibria φ: for every $u_0 \geq 0$ and nonzero in Ω,

$$\lim_{t\to\infty} u(t,x;u_0) = \varphi(x)$$

uniformly in $x \in \overline{\Omega}$. Moreover if $\lambda \leq \lambda_1(\Omega)$ then $\varphi = 0$, while if $\lambda > \lambda_1(\Omega)$ then $\varphi(x) > 0$ in Ω.

ii) Let $K_0 = \{x \in \Omega : \ n(x) = 0\}$ and Ω_δ be a neighborhood of K_0 such that $n(x) \geq \delta > 0$ for all $x \in \Omega \setminus \overline{\Omega}_\delta$. Denote by $\lambda_1(\Omega_\delta)$ first eigenvalue of $-\Delta$ with Dirichlet boundary conditions in Ω_δ. Then, if $\lambda < \lambda_1(\Omega_\delta)$, the above applies.

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$$\lambda_1(\Omega) < \lambda_0(K_0) = \lim_{\delta\to 0} \lambda_1(\Omega_\delta) \leq \infty$$

Hence we get the following

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With the notations above, for any

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- What happens to equilibria φ_λ as $\lambda \to \lambda_0(K_0)$?
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Thick and thin sets

If $K \subset I\!R^N$ is a compact set, consider a decreasing family of smooth open sets Ω_δ such that

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Then for each $\delta > 0$ consider

$$\lambda_1(\Omega_\delta) = \inf_{u \in H_0^1(\Omega_\delta)} \frac{\int_{\Omega_\delta} |\nabla u|^2}{\int_{\Omega_\delta} u^2} > 0,$$

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Then we have the following

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A compact set $K \subset \mathbb{R}^N$ is **thick** iff $\lambda_1(\Omega_\delta)$ is bounded in δ, or equivalently iff

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Otherwise, $\lambda_0(K) = \infty$ and K is said **thin**.

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iii) If $K = K_1 \cup K_2$ are separated compact sets, $K_1 \cap K_2 = \emptyset$, then

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Corollary. Decomposition of compact sets

For any compact set $K \subset \mathbb{R}^N$, there exist a (non necessarily unique) decomposition on pairwise separated connected components

$$K = K_1 \cup \ldots \cup K_n \cup K_{n+1} \cup \ldots \cup K_m$$

such that

$$K_{n+1}, \ldots, K_m \quad \text{are thin}$$

and

$$K_1, \ldots, K_n \quad \text{are thick}$$

in decreasing thickness, that is,

$$\lambda_0(K_1) \leq \cdots \leq \lambda_0(K_n).$$

Then $\lambda_0(K) = \min\{\lambda_0(K_1), \ldots, \lambda_0(K_n)\} = \lambda_0(K_1)$.

Proposition

i) Any superset of a thick set is thick.

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Proposition

i) K is thick iff

$$H_0^1(K) := \{\xi \in H^1(\mathbb{R}^N), \quad \xi(x) = 0 \quad \text{a.e. } x \in \mathbb{R}^N \setminus K\}$$

is a nontrivial (closed) linear subspace of $H^1(\mathbb{R}^N)$.

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Positive equilibria

Recall

$$K_0 = \{x \in \Omega : \; n(x) = 0\} \subset \Omega$$

Lemma

i) Assume that the logistic equation has a nonnegative stationary solution in $L^1(\Omega)$. Then

$$\lambda < \lambda_0(K_0).$$

ii) For $\lambda_1(\Omega) < \lambda < \lambda_0(K_0)$ the positive equilibria φ_λ is a smooth and increasing function of λ.
Even more, as $\lambda \to \lambda_0(K_0)$, we have

$$\|\varphi_\lambda\|_{L^\infty(\Omega)} \to \infty, \quad \textit{even} \quad \|\varphi_\lambda\|_{L^1(\Omega)} \to \infty$$

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Proof If u is such stationary solution, then $u \in L^s(\Omega)$ with $s > \frac{N}{2}(\rho - 1)$ and then

$$\lambda = \mu_1(-\Delta + n(x)u^{\rho-1}, \Omega),$$

(the first eigenvalue of the operator $-\Delta + n(x)u^{\rho-1}$ in Ω, with Dirichlet BC).

Take a decreasing family Ω_δ with $n(x) \leq \delta$ in Ω_δ. Then, for some $p > N/2$, $\|nu^{\rho-1}\|_{L^p(\Omega_\delta)} \leq \delta \|u\|_{L^s(\Omega_\delta)} \to 0$.

This and the monotonicity with respect to the domain of this eigenvalue gives

$$\lambda < \mu_1(-\Delta + n(x)u^{\rho-1}, \Omega_\delta) \to \lambda_0(K_0).$$

Monotonicity in λ follows from the equation satisfied by $\frac{\partial \varphi_\lambda}{\partial \lambda}$. $\square$

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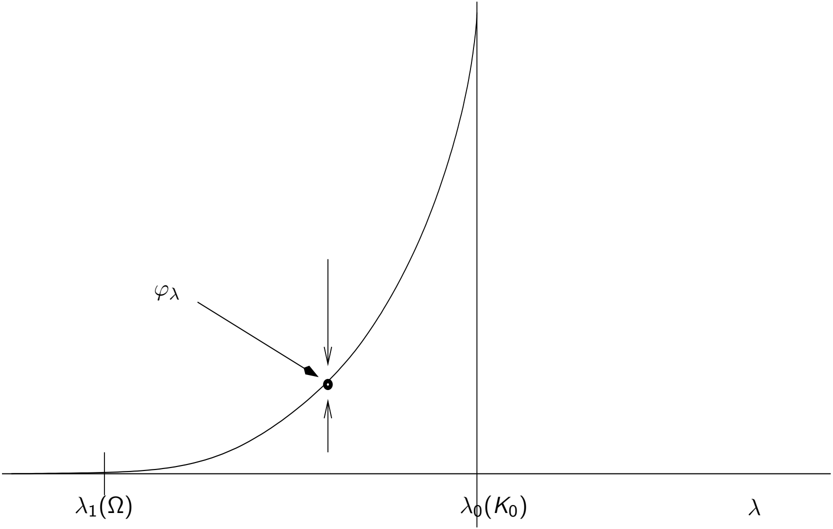

φ_λ
$\lambda_1(\Omega)$
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λ

Solutions for $\lambda > \lambda_0(K_0)$

- For $\lambda > \lambda_1(\Omega)$, $u = 0$ is unstable.
- If $\lambda > \lambda_0(K_0)$ and $u_0 \geq 0$ then $u(t; u_0)$ is globally defined but can not be bounded in Ω (if it was, using compactness, there would exist a bounded stationary solution).

Therefore

$$\limsup_{t\to\infty} \|u(t; u_0)\|_{L^\infty(\Omega)} = \infty$$

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Where and how solutions become unbounded?

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Upper universal bounds

Using Garcia-Melian et al. ARMA, 145, 261-289 (1998),

Lemma

Assume $\rho > 1$ and $\lambda, \beta > 0$ and consider

$$\begin{cases} -\Delta z = \lambda z - \beta z^{\rho} & \text{in } B(0,a) \\ z = \infty & \text{on } \partial B(0,a). \end{cases}$$

Then
i) There exists a unique positive radial solution, $z_a(x)$.
ii) The solution satisfies

$$\left(\frac{\lambda}{\beta}\right)^{\frac{1}{\rho-1}} \leq z_a(0) = \inf_{B(0,a)} z_a(x) \leq \left(\frac{\lambda(\rho+1)}{2\beta} + \frac{B}{\beta a^2}\right)^{\frac{1}{\rho-1}}$$

for some constant $B > 0$.

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Solutions remain bounded out of K_0

As a consequence

Proposition

Let $x_0 \in \Omega \setminus K_0$ and let $u_0 \geq 0$ be a bounded initial data. Then for any given $\lambda \geq \lambda_0(K_0)$ there exists $b > 0$ and $M > 0$ such that

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Solutions become unbounded first on the thicker parts of K_0.

We can assume

$$K_0 = K_1 \cup \ldots \cup K_n \cup K_{n+1} \cup \ldots \cup K_m$$

a decomposition in pairwise separated components, and such that

$$K_{n+1}, \ldots, K_m \quad \text{are thin}$$

and

$$K_1, \ldots, K_n \quad \text{are thick}$$

in decreasing thickness, that is,

$$\lambda_0(K_1) \leq \cdots \leq \lambda_0(K_n).$$

Solutions become unbounded first on the thicker parts of K_0.

Theorem

i) For any $\lambda \in I\!R$ all solution are bounded in $K_{n+1} \cup \ldots \cup K_m$.

ii) If for some $j = 1, \ldots, n-1$

$$\lambda_0(K_j) \leq \lambda < \lambda_0(K_{j+1})$$

then all solution are bounded in $K_{j+1} \cup \ldots \cup K_n$.

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Proof Let K be any component of K_0 in the decomposition above. Take a decreasing family Ω_δ for K. For δ small enough $u(x,t)$ remains bounded on $\partial\Omega_\delta$ by M. Then the the solution of

$$\begin{cases} z_t - \Delta z = \lambda z & \text{in} \quad \Omega_\delta \\ z = M & \text{on} \quad \partial\Omega_\delta \\ z(0) = u_0 \geq 0 & \text{in} \quad \Omega_\delta \end{cases}$$

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Let K be any of the thick components $K_1, \dots, K_n$ and assume a solution, $u(x,t)$, remains bounded on a neighborhood of K for all time.

Take a decreasing family Ω_δ for K such that $0 \leq n(x) \leq \delta$ in Ω_δ, since $n(x)$ is continuous. Then, given $\varepsilon > 0$, for δ small enough, we have

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in Ω_δ and for all times.

Then the the solution of

$$\left\{ \begin{array}{rcll} z_t - \Delta z &=& (\lambda - \varepsilon) z & \text{in} \quad \Omega_\delta \\ z &=& 0 & \text{on} \quad \partial\Omega_\delta \\ z(0) &=& z_0 \geq 0 & \end{array} \right.$$

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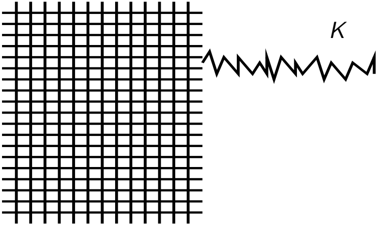
Where in a thick component is a solution bounded?



A thick component of K_0.

Improved universal bound

Lemma

For any solution there exists a constant $A = A(u_0, \lambda)$ such that

$$0 \leq u(t, x; u_0) \leq h(x) = \left(\frac{A}{\varepsilon^2(x) \inf_{B(x,\varepsilon(x))} n} \right)^{\frac{1}{\rho - 1}}, \quad x \in \Omega$$

with $\varepsilon(x) = C\mathrm{dist}(x, K_0)$, with $0 < C < 1$.

Note that $h(x) \to \infty$ as $x \to K_0$.

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Estimates on transversal sections

Definition. Transversal section

Let S be a bounded closed regular piece of a hyperplane in $\mathbb{R}^N$. That is, $S = \overline{S_0}$ with S_0 a bounded open set in the hyperplane.

We say S is **transversal** to the compact set K, if $K \not\subset S$, $K_S = K \cap S \neq \emptyset$ and for $x \in S$

$$dist_S(x, K_S) \sim dist(x, K)$$

where $dist_S$ denotes the $N-1$ dimensional distance on the hyperplane containing S.

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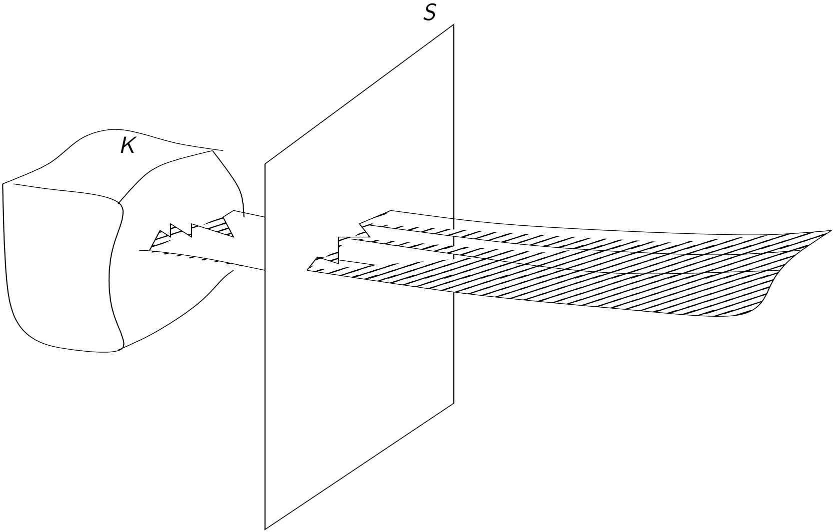

S
K

Proposition. Estimate on transversal sections

Assume S is transversal to K and for $x \in S$, close enough to K, and for $\varepsilon(x) = C\,\text{dist}(x, K)$, with $0 < C < 1$, we have

$$\inf_{B(x,\varepsilon(x))} n \ge n^*(dist(x, K))$$

with n^*, continuous and $n^*(s) > 0$ if $s > 0$, $n^*(0) = 0$.

Furthermore we assume $j(s) = s^2 n^*(s)$ is increasing in $s \ge 0$.

Finally assume that the $N-1$ fractal dimension of $K_S = K \cap S$ is $0 \le d^* < N-1$; that is, the fractal dimension of K_S as a subset of $\mathbb{R}^{N-1}$.

Note that $B(x, \varepsilon(x))$ is N–dimensional ball. This condition gives information in the way $n(x)$ vanishes, as $x \in S$ approaches K.

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then for any solution there exists $h \in L^r(S)$ such that

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In particular, if

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Remark

The result also holds without assuming that S is a piece of a hyperplane. It is enough that $S = F(S^*)$ where F is a bi–Lipschitz diffeomorphism in $\mathbb{R}^{N-1}$ as long as

$$dist_S(x, K_S) \sim dist(x, K)$$

Note that now $dist_S$ denotes the geodesic distance on S.
Also, we require that $K_S = F(K^*)$ where $K^* \subset \mathbb{R}^{N-1}$ has fractal dimension $0 \leq d^* < N - 1$.

A sufficient condition for boundedness in a part of a thick component

Assume K is a thick component of K_0, and $K = K_1 \cup K_2$ where $K_1 \cap K_2 \neq \emptyset$.

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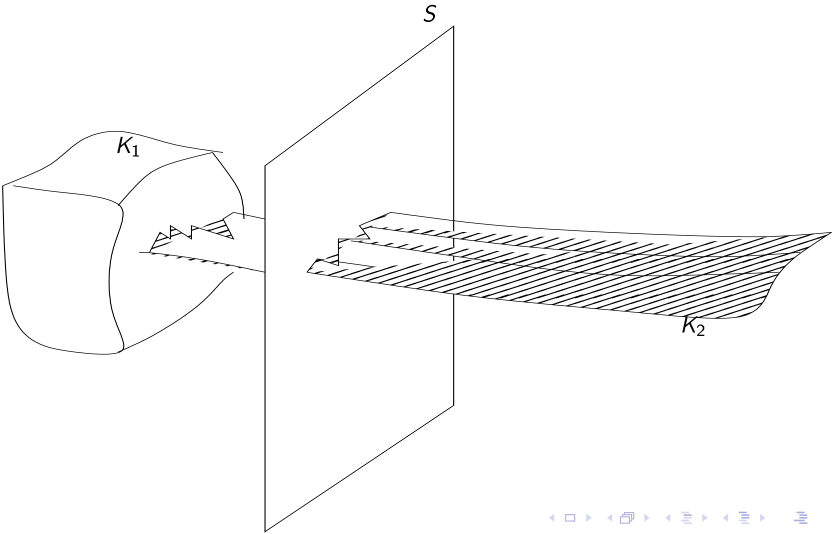
We will assume estimates on a suitable transversal section



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ii) Assume B is an "isolation box", for K_2, that is, an open bounded set B such that $\overline{B} \supset K_2$, $K_1 \cap B = \emptyset$ and $K_1 \cap \overline{B} = K_1 \cap K_2$.

iii) Moreover, assume one of the "faces" of its boundary, say $S \subset \partial B$, is transversal to K.

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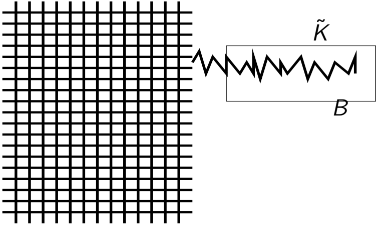
An isolation box



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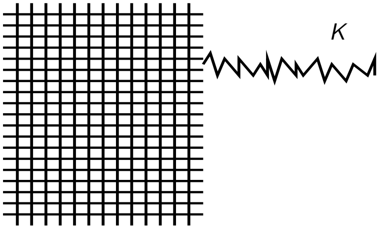

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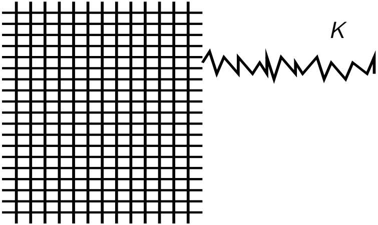

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All solutions of the logistic equation become unbounded at the same rate and at the same points when $\lambda \geq \lambda_0(K_0)$.

Proof Fix any $C^1(\overline{\Omega})$, $u_0 > 0$ with strictly negative normal derivative at the boundary of Ω. Then for any other initial data v_0 we can assume there exists $\alpha < 1$ and $\beta > 1$ such that

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In particular we get

Corollary

All solutions of the logistic equation become unbounded at the same rate and at the same points when $\lambda \geq \lambda_0(K_0)$.

Proof Fix any $C^1(\overline{\Omega})$, $u_0 > 0$ with strictly negative normal derivative at the boundary of Ω. Then for any other initial data v_0 we can assume there exists $\alpha < 1$ and $\beta > 1$ such that

$$\alpha u_0 \leq v_0 \leq \beta u_0.$$

Otherwise replace v_0 by $u(\varepsilon; v_0)$ for any $\varepsilon > 0$.

Then

$$\alpha u(t,x;u_0) \leq u(t,x;\alpha u_0) \leq u(t,x;v_0) \leq u(t,x;\beta u_0) \leq \beta u(t,x;u_0)$$

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$\square$