Open session on
“Geometrical and Functional Inequalities”
(some quantitative spectral inequalities)

Workshop on
“Partial differential equations, optimal design and numerics”

Aldo Pratelli (Pavia)

Benasque, September 1, 2011

Inequalities for the eigenvalues: I

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$$\lambda_1(E) = \frac{\int_E |Du|^2}{\int_E u^2} \geq \frac{\int_B |Du^*|^2}{\int_B {u^*}^2} \geq \lambda_1(B)\,.$$

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The proof is quite technical, and relies on Bessel functions.

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For the dimension $n = 2$, Bhattacharya showed that

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In other words, $\mathcal{A}_2(E) \leq c(n) KS(E)^{\frac{1}{2(N+1)}}$.

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$$\mathcal{A}_2(E) \lesssim \Big(\mathcal{A}(E^+) + \mathcal{A}(E^-) + \varepsilon\Big)^{\frac{2}{n+1}}$$
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- Is $\frac{2\kappa}{n+1}$ sharp?

It has been obtained putting together two sharp estimates, but this is not enough.

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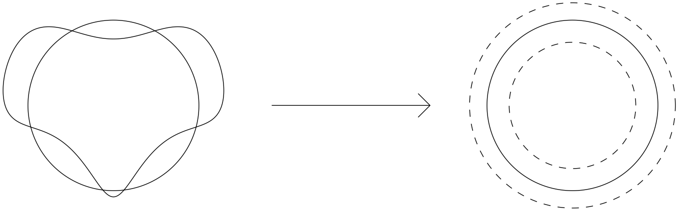

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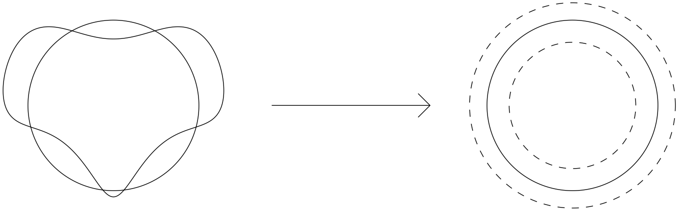

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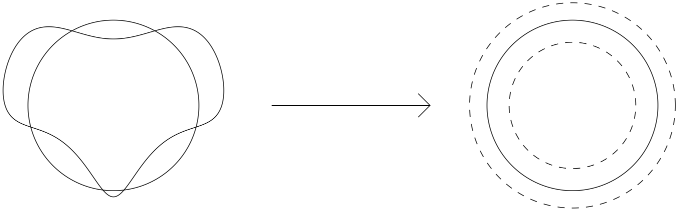

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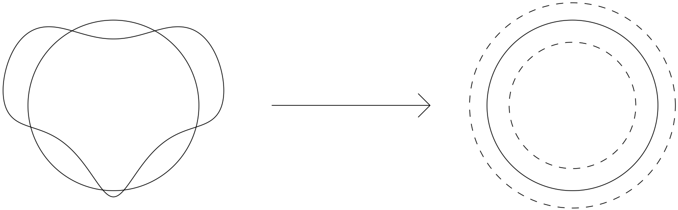

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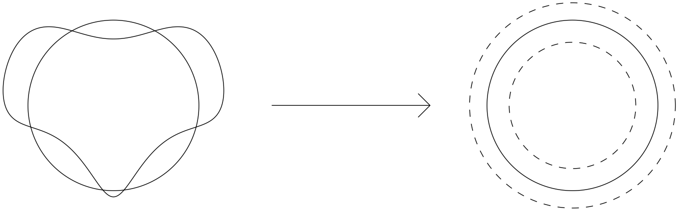

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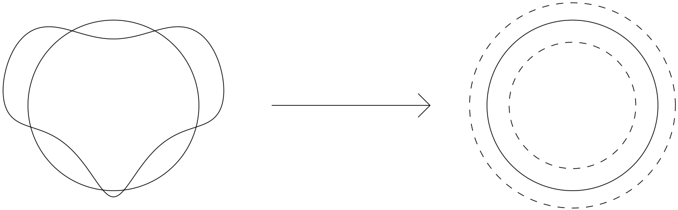

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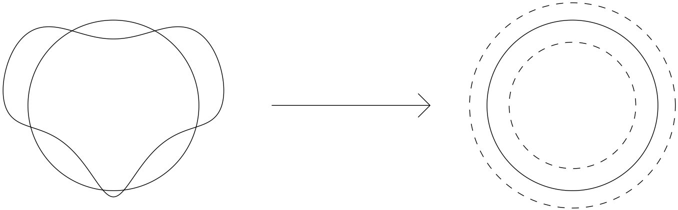

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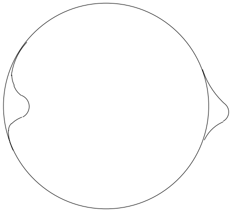

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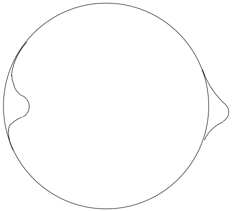

One needs to use an iterative procedure together with a spectral decomposition plus classical Calderon–Zygmund estimates, on an unknown eigenfunction!

Open problems

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- Show the sharp exponent κ $(= 2?)$ for the Faber–Krahn inequality for λ_1.

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- There are a lot of other open spectral inequalities!