*Open session on* GEOMETRIC AND FUNCTIONAL INEQUALITIES

# Bonnesen type inequalities in the anisotropic case

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*PDE's, optimal design and numerics*

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## Formulation of the problem:

1) Given a convex bounded set $E \subset \mathbb{R}^n$ set with finite measure (i.e. $|E| < +\infty$) find $f : \mathbb{R}^+ \to \mathbb{R}^+$ "optimal" such that it holds

$$\boxed{\lambda_K(E) \leq f(\delta(E))}$$

where

$$\lambda_K(E) = \min_{x_0 \in \mathbb{R}^n} \{ d_{\mathcal{K}}(E, x_0 + rK) : |K| r^n = |E| \} \text{ anisotropic deviation}$$

$$\delta(E) = \frac{\mathrm{Per}_K(E) - n|K|r^{n-1}}{n|K|r^{n-1}} \text{ anisotropic perimeter deficit}$$

Rem. all quantities suitably defined in order to be scaling invariant.

2) extend the estimate to a proper class of sets with finite measure (i.e. $|E| < +\infty$)

## Interpretation and main motivation

Recall that

$$d_K(E,F) = \max\{\max_{x\in E} \mathrm{dist_K}(x,F), \max_{y\in F} \mathrm{dist}(y,E)\}$$

$\lambda_{\mathcal{H}}(E)$ $\approx$ computes the best way to cover uniformly $E$ with a Wulff shape of same measure

$\delta(E)$ $\approx$ computes the oscillation in perimeter

Two-folded goal:

- applications to models in phase transitions
- estimate of higher order terms for set functions along stationary points in the anisotropic case
- applications to problems in Convex analysis

## the conjectured optimal form of $f$ depending on $n$:

Convex case:

$$f(t)=\begin{cases} c_2(K)t^{\frac{1}{2}} & \text{for } n=2\\ c_3(K)t^{\frac{1}{2}}\left(\log\frac{1}{t}\right)^{\frac{1}{2}} & \text{for } n=3\\ c_n(K)t^{\frac{2}{n-1}} & \text{for } n\geq 4.\end{cases} \tag{1}$$

General case:

$$f(t)=\begin{cases} c_2'(K)t^{\frac{1}{2}} & \text{for } n=2\\ c_3'(K)t^{\frac{1}{2}}\left(\log\frac{1}{t}\right)^{\frac{1}{2}} & \text{for } n=3\\ c_n'(K)t^{\frac{1}{n-1}} & \text{for } n\geq 4.\end{cases} \tag{2}$$

# Results avalaible in this setting

- Bonnesen, (1924) isop. ineq. n=2
- Osserman, (1979) Bonnesen type ineq. for convex sets
- Fuglede, (1989) nearly convex domains
- Fusco, Maggi, Pratelli, (2008) quantitative isop. ineq. in the euclidean case
- Figalli, Maggi, Pratelli, (2010) anisotropic case
- Figalli, Maggi (2010) application to liquid drops
- Fusco, G., Pisante (2010) (euclidean case with Hausdorff asymmetry)

# Features of the problem in the anisotropic case :

Drawbacks for the general case:

- validity expected in a proper subclass of sets with finite perimeter
- need of selection of good representative in both topological and measure sense
- counterexamples also under regularity hypotheses
- no sufficiency of connectedness or indecomposability hypotheses
- no easy continuity property inherited by $\lambda_K(\cdot)$

$\hookrightarrow$ look for a special class of sets for which the result holds

Aims for the convex case:

1. find the best exponent $\alpha(n)$ so that $f(t) = c(n)t^{\alpha(n)}$
2. estimate the asymptotic behaviour of $c(n)$

# Result in the euclidean case

Assume $K$ is the unitary ball and denote
$\lambda_{\mathcal{H}}(E)$ the *spherical deviation*
$D(E)$ the standard *perimeter deficit*
$C_R$ the class of sets with an *interior cone property of radius $R$*

**Theorem**[Fusco-G.-Pisante] For any $R > 0$ there exist $0 < \delta_R < 1$ and a constant $C = C(R, n)$ depending only on $R$ and $n$ such that for any $E \in \mathcal{C}_R$ with $D(E) < \delta_R$ it holds

$$\lambda_{\mathcal{H}}(E) \leq C \begin{cases} D(E)^{\frac{1}{2}} & \text{for } n = 2 \\ D(E)^{\frac{1}{2}} \left(\log \frac{1}{D(E)}\right)^{\frac{1}{2}} & \text{for } n = 3 \\ D(E)^{\frac{1}{n-1}} & \text{for } n \geq 4. \end{cases} \tag{3}$$

## *Selection of a special class of sets*

In order to avoid tiny connected comp. and thin tentacles we need strong structure properties:

quantitative and qualitative geometry

$\Rightarrow$ interior cone property

Analytic consequences of the interior cone property

1. boundedness of the diameter under a volume constraint ($\Rightarrow$ compactness in $BV$ and in $d_K$);
2. continuity of $\lambda_K(E)$ with respect to $\delta(E)$;
3. closeness in $d_K$ of the optimal Wulff shapes w.r.t. the $L^1$ and Hausdorff distance resp. ($\Rightarrow$ we can reduce to estimate $d_K(E, K_1)$).

## Possible strategy of the proof

Step 1. Establish a functional inequality

*a*) rephrase the desired estimate as a functional inequality in $W^{1,\infty}$;

*b*) prove it on bounded sets of $W^{1,\infty}$ taking advantage of an integral condition;

*c*) select at the same time the "optimal" $f$ in the estimate;

Remark. $f$ would depend on the bound on $K$ and $\|u\|_{W^{1,\infty}}$.

Step 2. Reduction to a special structure of the boundary

*a*) "deal" with interior holes;

*b*) use the cone property to rule out the presence of "tentacles";

*c*) deduce some graph property of the boundary+Lipschitz regularity;

*d*) use the cone property to infer a uniform bound on the $W^{1,\infty}$-norm of the graph.

# Related functional inequality in the euclidean case

Let $\Sigma$ denote the unit sphere in $\mathbb{R}^n$ equipped with the surface measure $\sigma$ suitably normalized and for $u : \Sigma \to (-1, +\infty)$ set

$$\Delta(u) = \int_\Sigma (1+u)^{n-1}\sqrt{1+(1+u)^{-2}|\nabla u|^2}\, d\sigma - 1$$

**Theorem 1**[Fusco-G.-Pisante] For any $M > 0$ there exist constants $c_1(M,n) > 0,\ C_2(M,n) > 0$ such that for any $u \in W^{1,\infty}(\Sigma)$ with

$$\int_\Sigma (1+u(z))^n\, d\sigma = 1 \qquad \|u\|_1 \leq c_1' \sqrt{\Delta(u)}$$
$$\|u\|_\infty \leq c_1(M,n) \qquad \|\nabla u\|_\infty \leq M$$

it holds

$$\int_\Sigma |\nabla u|^2 \leq C_2(M,n)\Delta(u).$$

**Theorem 2** For any $M > 0$ there exist constants $c_1(M, n) > 0$, $C_2(M, n) > 0$ such that for any $u \in W^{1,\infty}(\Sigma)$ with

$$\int_\Sigma (1 + u(z))^n \, d\sigma = 1 \qquad \|u\|_1 \leq c_1' \sqrt{\Delta(u)}$$
$$\|u\|_\infty \leq c_1(M, n) \qquad \|\nabla u\|_\infty \leq M$$

it holds

$$\|u\|_\infty^{n-1} \leq C_2(M, n) \begin{cases} \Delta(u)^{\frac{1}{2}} & \text{for } n = 2 \\ \Delta(u) \left( \log \frac{1}{\Delta(u)} \right) & \text{for } n = 3 \\ \Delta(u) & \text{for } n \geq 4. \end{cases}$$

Analogously for the convex case up to refine the exponent for $n \geq 4$.