The route to dissipativity

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The route to dissipativity

Consider an evolution sistem $S(t)$ in a phase space X, a Banach space (or even a complete metric space)

$$\text{(initial state) } u_0 \in X \longmapsto S(t)u_0 \in X \text{ (state at time } t\text{)}$$

Assume solutions are global:

$$u(t; u_0) = S(t)u_0, \quad u_0 \in X, \quad t \geq 0.$$

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Uniform asymptotic estimates

Assume orbits of bounded sets are bounded:

$$\{S(t)B,\ t \geq 0\} \quad \text{bounded in } X \text{ for any bounded set} \quad B \subset X$$

Furthermore assume

$$\limsup_{t\to\infty} \|u(t;u_0)\| \leq K \quad \text{(independent of } u_0\text{)}$$

i.e the ball $B_0 := B_X(0, K+1)$ is absorbing.

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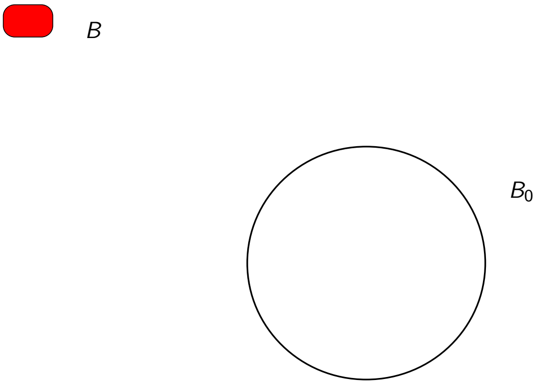

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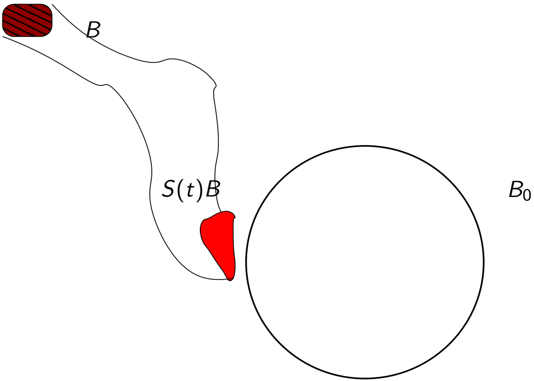

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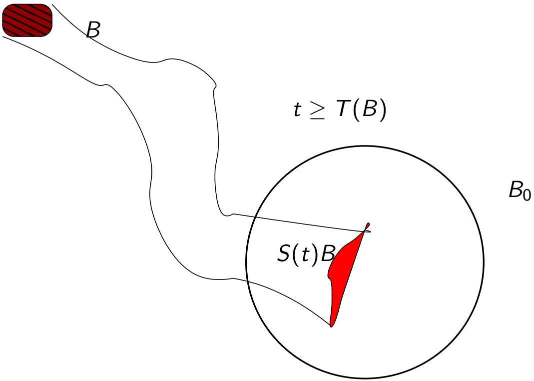

Asymptotic compactness

Finally, assume the asymptotic compactness:

for every bounded sequence $\{u_0^n\} \subset X$ and $t_n \to \infty$, the set $\{u(t_n; u_0^n)\}$ has a converging subsequence.

This is satisfied if:

- X is a finite dimensional space, or
- For a bounded set $B \subset X$, $\{S(t)B,\ t \geq t_B\}$ is bounded in Y and $Y \subset X$ is compact (Smoothing effect), or
- $S(t) = L(t) + K(t)$ with

$$\|L(t)B\| \to 0 \quad \text{as } t \to \infty, \text{ for any bounded } B \subset X$$

$$K(t) : X \to X \quad \text{is compact}$$

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Limit sets

- Given $u_0 \in X$ define

$$\omega(u_0) = \{z \in X,\ S(t_n)u_0 \to z,\ t_n \to \infty\}$$

Then

$$\omega(u_0) \text{ is compact, connected, invariant and}$$

$$\mathrm{dist}_X\big(S(t)u_0, \omega(u_0)\big) \to 0,\ \text{as } t \to \infty$$

- Given a bounded set $B \subset X$, define

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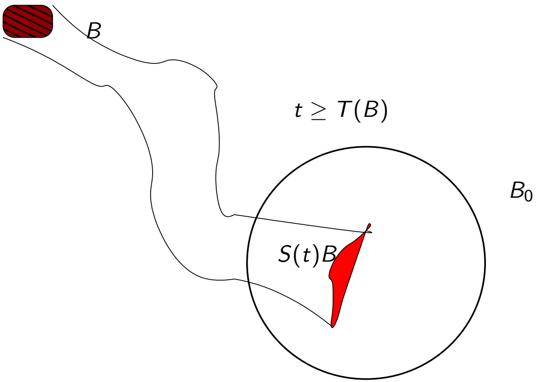

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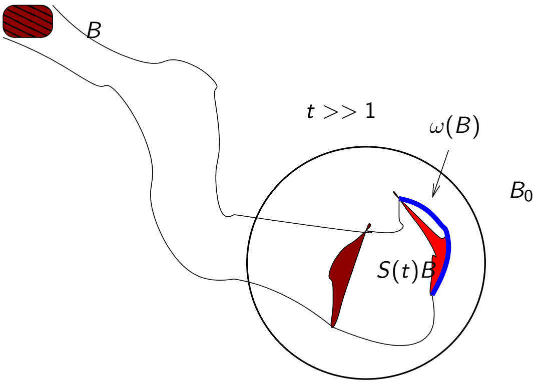

The global Attractor

Theorem

There exists a global attractor, $\mathcal{A} \subset X$, compact, connected, invariant

$$\operatorname{dist}_X\big(S(t)B, \mathcal{A}\big) \to 0, \quad \text{for all} \quad B \subset X, \quad \text{bounded.}$$

Proof Define

$$\mathcal{A} = \{z \in X,\ S(t_n)u_0^n \to z,\ t_n \to \infty,\ \{u_0^n\} \subset B_0\} = \omega(B_0)$$

i.e. all accumulation points of the orbit of the bounded absorbing set B_0.

Remark the set J is invariant if

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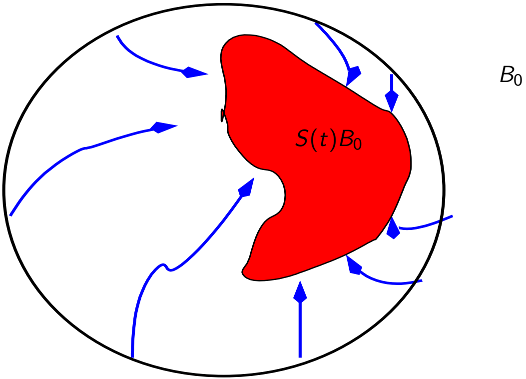

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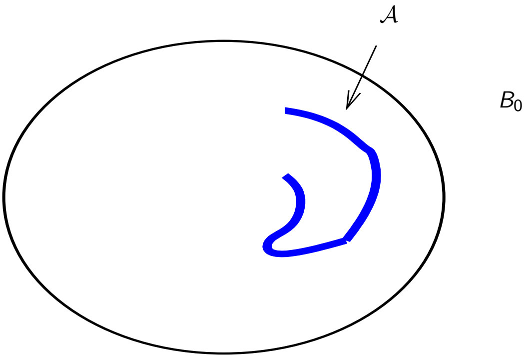

The global Attractor

The attractor contains:

- All equilibria: $S(t)u_0 = u_0$, for all $t \geq 0$.
- All periodic solutions: $u(t+T) = u(t)$, for all $t \geq 0$.
- All global and bounded solutions $\{u(t),\ t \in \mathbb{R}\}$.
- All bounded invariant sets J, $S(t)J = J$ for all $t \geq 0$. In particular, all ω-limit sets
- All unstable sets of an equilibria u_0

$$W^u(u_0) = \{z \in X,\ S(t)z \text{ is defined for } t \leq 0 \text{ and}$$

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Global versus fine dynamics

1) Global dynamics

• Show existence of attractors

• Determine mechanisms for dissipativity (e.g. competition between diffusion, reaction, convection, damping, etc). Each examples requieres its own analysis.

• Determine the structure of the global atractor. OPEN, only known in case of gradient flows.

2) Fine dynamics

• Analyze equilibria and linear stability

• Prove finite dimensionality of asymptotic dynamics

• Tools for analyzing dynamics inside the attractor

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