Material design and life-cycle optimization

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Joint work with Jaroslav Haslinger, Michal Kocvara, Michael Stingl, Peter Kogut

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ENGINEERING OF ADVANCED MATERIALS

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Part A: On Free Material Optimization and Inverse Homogenization

Günter Leugering and Michael Stingl
Michal Kocvara, Jaroslav Haslinger
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See also the lecture by Manuel Luna Laynez yesterday

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Free Material Optimization (FMO) setup

$$a_E(w,z) := \int_\Omega E(x)\varepsilon(w(x)) \cdot \varepsilon(z(x))\mathrm{d}x$$

$$\mathcal{E}_0 := \left\{ E \in \mathcal{L}^\infty(\Omega, \mathbb{S}^{\bar{N}}) \mid E \succcurlyeq 0 \text{ a.e. in } \Omega \right\}$$

$$\mathcal{E} := \{E \in \mathcal{E}_0 \mid \mathrm{Tr}(E) \leq \overline{\rho} \text{ a.e. in } \Omega,\ v(E) \leq \bar{v}\}$$

$$\inf_{E\in\mathcal{E}} c(E) \quad \text{s. t.}$$

$$u_E \in \mathcal{V}:\ u_E := \arg\inf_{u\in\mathcal{V}} \left\{ \frac{1}{2} a_E(u,u) - \int_\Gamma f \cdot u \,\mathrm{d}s \right\}$$

$$c(E) := \int_\Gamma f \cdot u_E \,\mathrm{d}s$$

Examples for additional state constraints

- *Quadratic or tracking type displacement constraints* of the form

$$\int_\Omega \left(u(x) - u_0(x)\right)^2 \,\mathrm{dx} \le C,$$

with here $u_0 \in \mathcal{V}$.

- *Integral stress constraints* of the form

$$\int_\omega \sigma(x)^\top M \sigma(x) \,\mathrm{dx} \le C,$$

where $\omega \subset \Omega$ and M is either the unit or the von Mises matrix.

Regularizations and additional constraints

We consider the regularization of the set $\mathcal{E}$:

$$\mathcal{E}^{\varepsilon} := \{E \in \mathcal{E} \mid E \succcurlyeq \varepsilon I_{\bar{N}} \quad \text{a.e. in } \Omega\}$$

$$\mathcal{E}^{\alpha,\beta} := \left\{E \in \mathcal{L}^{\infty}(\Omega, \mathbb{S}^{\bar{N}}) \mid \alpha I_{\bar{N}} \preccurlyeq E \preccurlyeq \beta I_{\bar{N}} \quad \text{a.e. in } \Omega\right\}$$

$$\mathcal{E}^{\varepsilon, g_I, g_{II}} := \{E \in \mathcal{E}^{\varepsilon} \mid g_I(u_E) \leq C_u,\ g_{II}(\sigma_E) \leq C_\sigma\}$$

$$\begin{aligned} &\inf_{E \in \mathcal{E}^{\varepsilon}} J(E, u) \text{ s.t.} \\ &\quad u = S(E), \\ &\quad g_I(u) \leq C_u,\ g_{II}(\sigma) \leq C_\sigma\,,\ \sigma = Ee(S(E)) \end{aligned}$$

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H-compactness (see also Haslinger et.al.1996, Allaire 2002)

Theorem: (H-compactness, Murat, Tartar '79)
For any sequence (E_n) in $\mathcal{E}^{\alpha,\beta}$ there exists a subsequence, still denoted by (E_n), and a ('homogenized') $E^* \in \mathcal{E}^{\alpha,\beta}$ such that (E_n) H-converges to E^*.

Lemma: (Haslinger, Stingl, Kocvara, G.L. '10) The sets

$$\mathcal{E}^{\epsilon} \quad := \quad \left\{ E \in \mathcal{L}^{\infty}(\Omega, S^N) \mid \epsilon I \preccurlyeq \underline{\rho} I \preccurlyeq E;\ \mathrm{tr}(E) \leq \overline{\rho}, \int_{\Omega} \mathrm{tr}(E)\, \mathrm{dx} \leq V \right\}$$

and $\mathcal{E}^{\epsilon,g_I,g_{II}}$ are H-compact.

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Existence result based on H-convergence

Consider cost functionals of the type

$$J : \mathcal{E}^{\alpha,\beta} \times \mathcal{V} \to \mathbf{R}$$

with the following property:

$$\left.\begin{array}{r} (E_n) \xrightarrow{H} E \text{ in } \mathcal{E}^{\varepsilon} \\ (v_n) \rightharpoonup v \text{ in } \mathcal{V} \end{array}\right\} \Rightarrow \lim \inf_{n\to\infty} J(E_n, v_n) \geq J(E, v).$$

Theorem: (HLKS '10) The regularized FMO problem

$$\min_{E\in\mathcal{E}^{\epsilon}} C(E) := J(E, u_E).$$

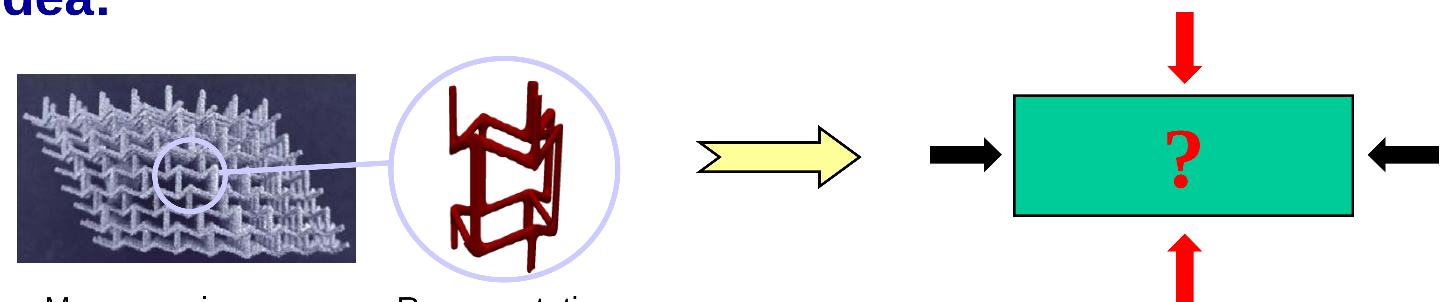
has at least one solution.

Material design

Bendsoe, Sigmund, Diaz, …

Goal: find a microstructure, which yields desired macroscopic material properties

Idea:



Macroscopic material

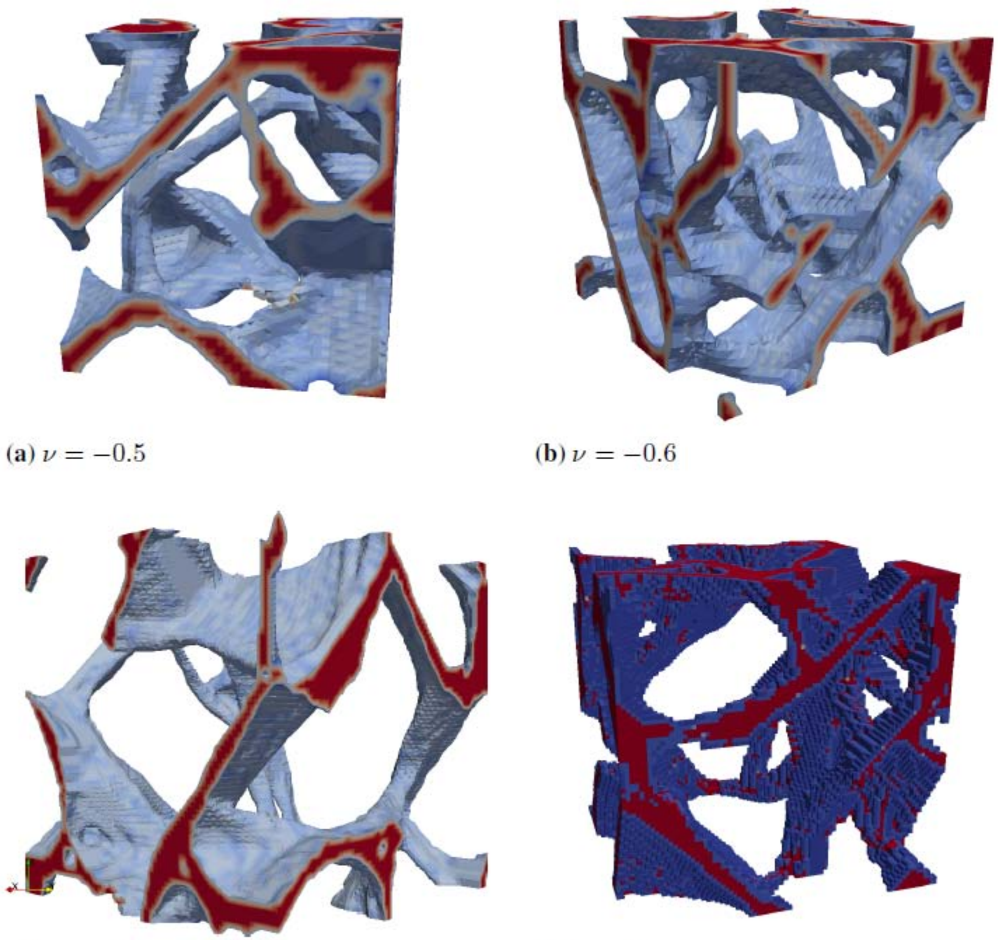
Representative base structure

$$S \xleftarrow{\text{material design}} E_H(S)$$

$$\boxed{\text{For given } E \text{ find } S \text{ s.t. } \|E - E_H(S)\| \to \min}$$

In particular, E may be given by the FMO-optimized matrix!

3-D auxetic elastic material

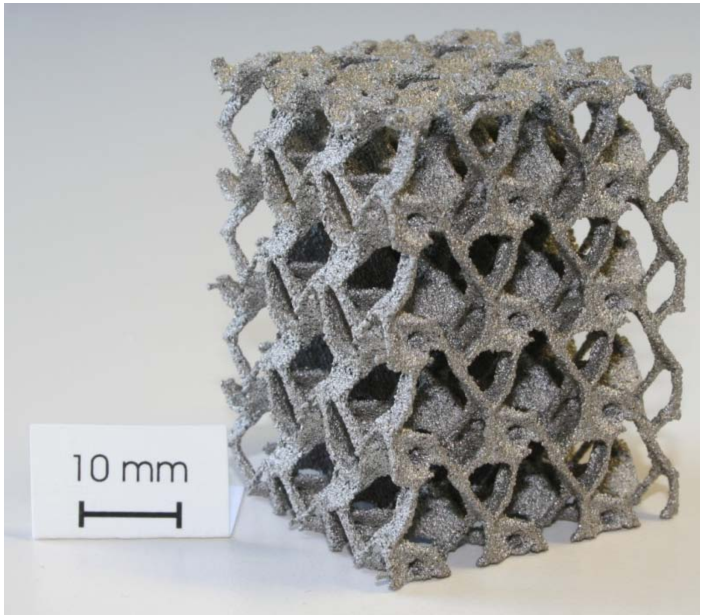


(a) $\nu = -0.5$ (b) $\nu = -0.6$

(c) $\nu = -0.3$ (d) $\nu = -0.4$

Optimization: auxetic materials

Isotropic auxetic structures in 3D



... still difficult to realize in practice ...

Optimization: other criteria

Optimization problem:

$$\max_{\rho} \; E^H_{1111}$$

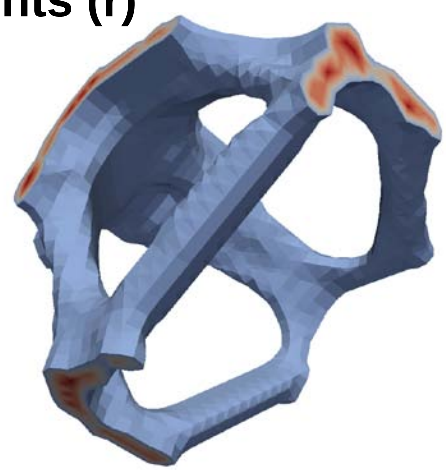
s. t. E^H is isotropic

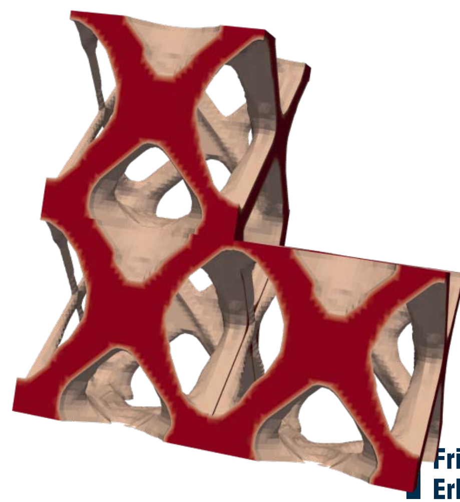
$$\frac{1}{|Y|}\sum_{e=1}^{N} \rho_e \le V_0$$

Goals:

- **light weight design**
- **given porosity**
- **maximize stiffness**
- **isotropy**

2 typical results: porosity 90%; a base structure (l); several elements (r)





Part B: Optimization of coeeficient matrices with vanishing eigenvalues

Günter Leugering
and
Peter I. Kogut
Submitted 2011

Strongly related to the work by
Giuseppe Buttazzo and Peter Kogut
Rev. Mat. Complutense 2010

Modeling

Let $\alpha \in \mathbb{R}$ be a fixed positive value and Ψ_* be a nonempty compact subset of $L^1(\Omega)$ such that $\zeta_* \in \Psi_*$ if and only if

$$0 < \zeta_*(x) \leq \alpha \text{ a.e. in } \Omega, \quad \zeta_*^{-1} \in L^1(\Omega),$$
$$\zeta_* : \Omega \to [0, \alpha] \text{ is smooth function along the boundary } \partial\Omega,$$
$$\zeta_* = \alpha \quad \text{on} \quad \partial\Omega.$$

By $\mathfrak{M}_\alpha^\beta(\Omega)$ we denote the set of all matrices $A(x) = [a_{i\,j}(x)] \in \mathbb{S}^N$ such that

$$A(x) \leq \beta(x) I \quad \text{a. e. in } \Omega,$$
$$\exists\, \zeta_* \in \Psi_* \text{ s.t. } \zeta_* I \leq A(x) \quad \text{a. e. in } \Omega.$$

Here $\beta \in L^1(\Omega)$ is a given function such that $\beta(x) > 0$ a.e. in Ω, I is the identity matrix in $\mathbb{R}^{N \times N}$.

Degeneration of eigenvalues

$$\text{if} \quad A \in L^1(\Omega; \mathbb{S}^N), \ \text{then} \quad \|A(x)\|_{L^1(\Omega;\mathbb{S}^N)} \leq \|\beta\|_{L^1(\Omega)} < +\infty,$$

$$\zeta_*(x)\|\xi\|^2_{\mathbb{R}^N} \leq (A(x)\xi, \xi)_{\mathbb{R}^N} \quad \text{a. e. in} \ \ \Omega, \ \forall \xi \in \mathbb{R}^N.$$

Remark Since every measurable matrix-valued function $A : \Omega \to \mathbb{S}^N$ can be associated with its eigenvalues $\{\lambda_1^A, \ldots, \lambda_N^A\}$, the second inequality means that eigenvalues of matrices $A \in \mathfrak{M}_\alpha^\beta(\Omega)$ may vanish on subdomains of Ω with zero Lebesgue measure.

Weighted spaces

Introduce:

$$W_A(\Omega;\Gamma_D) = W(\Omega;\Gamma_D; A\,dx) \quad \text{and} \quad H_A(\Omega;\Gamma_D) = H(\Omega;\Gamma_D; A\,dx),$$

where $W_A(\Omega;\Gamma_D)$ is the set of functions $y \in W^{1,1}(\Omega;\Gamma_D)$ whith finite norm

$$\|y\|_A = \left(\int_\Omega \left(y^2 + (\nabla y, A(x)\nabla y)_{\mathbb{R}^N} \right) dx \right)^{1/2}$$

and $H_A(\Omega;\Gamma_D)$ is the closure of $C_0^\infty(\Omega;\Gamma_D)$ in the $\|\cdot\|_A$-norm.

It is clear that $H_A(\Omega;\Gamma_D) \subset W_A(\Omega;\Gamma_D)$. If the eigenvalues $\{\lambda_1^A, \ldots, \lambda_N^A\}$ of $A : \Omega \to \mathbb{S}^N$ are bounded above and away from zero, then $W_A(\Omega;\Gamma_D) = H_A(\Omega;\Gamma_D)$.

Sequences in varying spaces

We assume that the measures $\vec{\mu}$ and $\{\vec{\mu}_k\}_{k\in\mathbb{N}}$ are defined by $d\vec{\mu}_k = A_k(x)\,dx$, $d\vec{\mu} = A(x)\,dx$, and $\vec{\mu}_k \overset{*}{\rightharpoonup} \vec{\mu}$ in $M(\Omega;\mathbb{S}^N)$. Further, we will use $L^2(\Omega, A\,dx)^N$ to denote the set of measurable vector-valued functions $\mathbf{f} \in \mathbb{R}^N$ on Ω such that

$$\|\mathbf{f}\|_{L^2(\Omega,A\,dx)^N} = \left(\int_\Omega (\mathbf{f}, A(x)\mathbf{f})_{\mathbb{R}^N}\, dx \right)^{1/2} < +\infty.$$

Any vector-valued function of $L^2(\Omega, A\,dx)^N$ is Lebesgue integrable on Ω.
We say that a sequence $\left\{\mathbf{v}_k \in L^2(\Omega, A_k\,dx)^N\right\}_{k\in\mathbb{N}}$ is bounded if

$$\limsup_{k\to\infty} \int_\Omega (\mathbf{v}_k, A_k(x)\mathbf{v}_k)_{\mathbb{R}^N}\, dx < +\infty.$$

Convergence in variable spaces

Definition 1. A bounded sequence $\left\{\mathbf{v}_k \in L^2(\Omega, A_k\, dx)^N\right\}_{k\in\mathbb{N}}$ is weakly convergent in the variable space $L^2(\Omega, A_k\, dx)^N$ to a function $\mathbf{v} \in L^2(\Omega, A\, dx)$ if

$$\lim_{k\to\infty} \int_\Omega (\vec{\varphi}, A_k(x)\mathbf{v}_k)_{\mathbb{R}^N}\ dx = \int_\Omega (\vec{\varphi}, A(x)\mathbf{v})_{\mathbb{R}^N}\ dx \quad \forall\, \vec{\varphi} \in C_0^\infty(\Omega)^N.$$

Proposition 1. If a sequence $\left\{\mathbf{v}_k \in L^2(\Omega, A_k\, dx)^N\right\}_{k\in\mathbb{N}}$ is bounded, then it is compact in the sense of weak convergence in $L^2(\Omega, A_k\, dx)^N$.

Lower semicontinuity

Proposition 2. If the sequence $\left\{\mathbf{v}_k \in L^2(\Omega, A_k\,dx)^N\right\}_{k\in\mathbb{N}}$ converges weakly to $\mathbf{v} \in L^2(\Omega, A\,dx)^N$, then

$$\liminf_{k\to\infty} \int_\Omega (\mathbf{v}_k, A_k(x)\mathbf{v}_k)_{\mathbb{R}^N}\ dx \geq \int_\Omega (\mathbf{v}, A(x)\mathbf{v})_{\mathbb{R}^N}\ dx.$$

Definition 2. A sequence $\left\{\mathbf{v}_k \in L^2(\Omega, A_k\,dx)^N\right\}_{k\in\mathbb{N}}$ is said to be strongly convergent to a function $\mathbf{v} \in L^2(\Omega, A\,dx)^N$, if

$$\lim_{k\to\infty} \int_\Omega (\mathbf{b}_k, A_k(x)\mathbf{v}_k)_{\mathbb{R}^N}\ dx = \int_\Omega (\mathbf{b}, A(x)\mathbf{v})_{\mathbb{R}^N}\ dx$$

whenever $\mathbf{b}_k \rightharpoonup \mathbf{b}$ in $L^2(\Omega, A_k\,dx)^N$ as $k \to \infty$.

Weak and strong convergence

Proposition 3. Weak convergence of $\left\{\mathbf{v}_k \in L^2(\Omega, A_k\, dx)^N\right\}_{k\in\mathbb{N}}$ to $\mathbf{v} \in L^2(\Omega, A\, dx)^N$ and

$$\lim_{k\to\infty} \int_\Omega (\mathbf{v}_k, A_k(x)\mathbf{v}_k)_{\mathbb{R}^N}\ dx = \int_\Omega (\mathbf{v}, A(x)\mathbf{v})_{\mathbb{R}^N}\ dx$$

are equivalent to strong convergence of $\{\mathbf{v}_k\}_{k\in\mathbb{N}}$ in $L^2(\Omega, A_k\, dx)^N$ to $\mathbf{v} \in L^2(\Omega, A\, dx)^N$.

Let $\{\zeta_{*,n}\}_{n\in\mathbb{N}}$ be any sequence in Ψ_*. Then there is an element $\zeta_* \in L^1(\Omega)$ such that, within a subsequence of $\{\zeta_{*,n}\}_{n\in\mathbb{N}}$, we have

$$\zeta_{*,n} \to \zeta_* \quad \text{in} \quad L^1(\Omega), \qquad \zeta_* \in \Psi_*,$$
$$\zeta_{*,n}^{-1} \to \zeta_*^{-1} \ \text{in} \ \ L^1(\Omega), \quad \text{and}$$
$$\zeta_{*,n}^{-1} \to \zeta_*^{-1} \quad \text{in variable space} \quad L^2(\Omega, \zeta_{*,n}\, dx).$$

w-Convergence

Definition 3. We say that a bounded sequence

$$\{(A_n, u_n) \in L^1(\Omega; \mathbb{S}^N) \times W_{A_n}(\Omega; \Gamma_D)\}_{n\in\mathbb{N}}$$

w-converges to $(A, u) \in L^1(\Omega; \mathbb{S}^N) \times W^{1,1}(\Omega; S))$ as $n \to \infty$, if

$$\begin{aligned} A_n \to A \quad &\text{in} \;\; L^1(\Omega; \mathbb{S}^N), \\ u_n \rightharpoonup u \quad &\text{in} \;\; L^2(\Omega), \\ \nabla u_n \rightharpoonup \nabla u \quad &\text{in the variable space} \;\; L^2(\Omega, A_n\, dx)^N, \end{aligned}$$

therefore,

$$\begin{aligned} \lim_{n\to\infty} \int_\Omega A_n \cdot \vec{\eta}\, dx &= \int_\Omega A \cdot \vec{\eta}\, dx \quad \forall\, \vec{\eta} \in L^\infty(\Omega; \mathbb{S}^N), \\ \lim_{n\to\infty} \int_\Omega u_n \lambda\, dx &= \int_\Omega u \lambda\, dx \quad \forall\, \lambda \in L^2(\Omega), \\ \lim_{n\to\infty} \int_\Omega \left(\vec{\xi}, A_n \nabla u_n\right)_{\mathbb{R}^N} dx &= \int_\Omega \left(\vec{\xi}, A\nabla u\right)_{\mathbb{R}^N} dx \quad \forall\, \vec{\xi} \in C_0^\infty(\Omega)^N. \end{aligned}$$

Compactness

Lemma Let $\{(A_n, u_n) \in L^1(\Omega;\mathbb{S}^N) \times W_{A_n}(\Omega;\Gamma_D)\}_{n\in\mathbb{N}}$ be a sequence such that

(i) the sequence $\{u_n \in W_{A_n}(\Omega;\Gamma_D)\}_{n\in\mathbb{N}}$is bounded, i.e.

$$\sup_{n\in\mathbb{N}} \int_\Omega \left(u_n^2 + (\nabla u_n, A_n \nabla u_n)\right) dx < +\infty;$$

(ii) $\{A_n\}_{n\in\mathbb{N}} \subset \mathfrak{M}_\alpha^\beta(\Omega)$ and there exists a matrix-valued function $A(x) \in \mathbb{S}^N$ such that

$$A_n \to A \quad \text{and} \quad A_n^{-1} \to A^{-1} \quad \text{in} \quad L^1(\Omega;\mathbb{S}^N) \quad \text{as} \quad n \to \infty.$$

Then, $A \in \mathfrak{M}_\alpha^\beta(\Omega) \cap L^1(\Omega;\mathbb{S}^N)$ and the original sequence is relatively compact with respect to w-convergence. Moreover, each w-limit pair (A, u) belongs to the space $L^1(\Omega;\mathbb{S}^N) \times W_A(\Omega;\Gamma_D)$.

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Assumptions

Let Q be a closed subdomain of Ω for which $\operatorname{dist}(\partial\Omega, \partial Q) \geq \delta > 0$, where δ is a prescribed value. Let $B \in L^\infty(Q; \mathbb{S}^N)$ be a given matrix-valued function such that

$$\alpha \|\xi\|^2_{\mathbb{R}^N} \leq (B(x)\xi, \xi)_{\mathbb{R}^N} \leq (\alpha + \sigma)\|\xi\|^2_{\mathbb{R}^N} \text{ a. e. in } Q \quad \forall \xi \in \mathbb{R}^N$$

with some $\sigma > 0$. Let $f \in L^2(\Omega)$ and $g \in L^2(\Gamma_N)$ be given functions.

Definition 4. We say that a matrix-valued function $A = A(x) \in \mathbb{S}^N$ is an admissible control to the boundary value problem (P) if

$$(AC) \begin{cases} A \in BV(\Omega \setminus Q; \mathbb{S}^N), & \displaystyle\int_{\Omega \setminus Q} A(x)\, dx = M, \\ A \in \mathfrak{M}^\beta_\alpha(\Omega \setminus Q), & A(x) = B(x) \text{ a.e. in } Q. \end{cases}$$

Weakly degenerate PDE

The main object of our consideration in this section is the following boundary value problem

$$(P)\begin{cases} -\operatorname{div}\big(A(x)\nabla y\big) = f \quad \text{in} \quad \Omega, \\ y = 0 \text{ on } \Gamma_D, \qquad \partial y/\partial \nu_A = g \text{ on } \Gamma_N. \end{cases}$$

Here

$$\frac{\partial y}{\partial \nu_A} = \sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial y}{\partial x_j}\cos(n, x_i),$$

$\cos(n, x_i)$ is i-th directing cosine of n, and n is the outward unit normal at Γ_N to Ω.

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Admissible controls and concept of solutions

Definition 5. We say that a function $y = y(A, f, g)$ is a weak solution to the boundary value problem (P) for a fixed control $A \in \mathfrak{A}_{ad}$ and given functions $f \in L^2(\Omega)$ and $g \in L^2(\Gamma_N)$, if

$$y \in W_A(\Omega; \Gamma_D)$$

and the integral identity

$$(WP) \qquad \int_\Omega (\nabla\varphi, A(x)\nabla y)_{\mathbb{R}^N}\, dx = \int_\Omega f\varphi\, dx + \int_{\Gamma_N} g\varphi\, d\mathcal{H}^{N-1}$$

holds for any $\varphi \in C_0^\infty(\mathbb{R}^N; \Gamma_D)$.

The set of (WP) solutions

$$\Xi_w = \left\{(A, y) \;\middle|\; A \in \mathfrak{A}_{ad},\ y \in W_A(\Omega; \Gamma_D)\,, (A, y) \text{ are related by (WP) } \right\}.$$

is sequentially closed wrt. w-convergence (if it is non-empty).

The optimal control problem

We are concerned with the following optimal control problem (OCP)

$$\text{Minimize } \Big\{ I(A,y) = \int_\Omega |y(x) - y_d(x)|^2 \, dx \\ + \int_\Omega \left(\nabla y(x), A(x)\nabla y(x)\right)_{\mathbb{R}^N} \, dx \\ + \sum_{i,j=1}^{N} \int_{\Omega \setminus Q} |D\, a_{ij}(x)| + \left\| \frac{\partial y}{\partial \nu_A} - y^* \right\|^2_{H^{-1/2}(\Gamma_D)} \Big\}$$

subject to the constraints (P), (AC).

Existence result

Hypothesis A. *The set of admissible solutions* Ξ_w *is nonempty, that is, the minimization problem* $\inf_{(A,y)\in\Xi_w} I(A,y)$ *is regular.*

We say that a pair $(A^0, y^0) \in L^1(\Omega;\mathbb{S}^N) \times W_A(\Omega;\Gamma_D)$ is weakly optimal for problem (OCP) if

$$(WO) \qquad (A^0, y^0) \in \Xi_w \quad \text{and} \quad I(A^0, y^0) = \inf_{(A,y)\in\Xi_w} I(A,y).$$

Theorem Let $f \in L^2(\Omega)$, $g \in L^2(\Gamma_N)$, $y_d \in L^2(\Omega)$, and $y^* \in L^2(\Gamma_D)$ be given functions. Assume that the Hypothesis A is valid. Then the optimal control problem (OCP) admits at least one solution

$$(A^0, y^0) \in L^1(\Omega;\mathbb{S}^N) \times W_{A^0}(\Omega;\Gamma_D).$$

Proofof the theorem ….

$I = I(A, y)$ is bounded below and $\Xi_w \neq \emptyset$, hence, there exists of a minimizing sequence $\{(A_n, y_n) \in \Xi_w\}_{n\in\mathbb{N}}$ to the problem (WO).

$$\inf_{(A,y)\in\Xi_w} I(A,y) = \lim_{n\to\infty} I(A_n, y_n) = \lim_{n\to\infty} \Big[\int_\Omega |y_n(x) - y_d(x)|^2 \, dx$$

$$+ \int_\Omega (\nabla y_n(x), A(x)\nabla y_n(x))_{\mathbb{R}^N} \, dx + \sum_{i,j=1}^{N} \int_{\Omega\setminus Q} |D\, a_{ij}^n(x)|$$

$$+ \left\| \frac{\partial y_n}{\partial \nu_{A_n}} - y^* \right\|^2_{H^{-1/2}(\Gamma_D)} \Big] < +\infty$$

implies the existence of a constant $C > 0$ such that

$$\sup_{n\in\mathbb{N}} \|\nabla y_n\|_{L^2(\Omega, A_n\, dx)^N} \leq C, \quad \|\frac{\partial y_n}{\partial \nu_{A_n}}\|_{H^{-1/2}(\Gamma_D)} \leq C,$$

$$\sup_{n\in\mathbb{N}} \|y_n\|_{L^2(\Omega)} \leq C, \quad \sup_{n\in\mathbb{N}} \|A_n\|_{BV(\Omega\setminus Q; \mathbb{S}^N)} \leq C.$$

Proof cont.

Hence, $\{(A_n, y_n) \in \Xi_w\}_{n\in\mathbb{N}}$ is bounded. and there exist functions $A^0 \in L^1(\Omega; \mathbb{S}^N)$ and $y^0 \in W_{A^0}(\Omega; \Gamma_D)$ s.t, up to a subsequence, $(A_n, y_n) \xrightarrow{w} (A^0, y^0)$. Ξ_w is sequentially closed w.r.t. w-convergence, hence, (A^0, y^0) is an admissible solution to the optimal control problem (OCP) (i.e. $(A^0, y^0) \in \Xi_w$). Moreover, by the estimates above and the Corollary to Proposition 4, we have:

$$\frac{\partial y_n}{\partial \nu_{A_n}} \rightharpoonup \frac{\partial y^0}{\partial \nu_{A^0}} \quad \text{in} \;\; H^{-1/2}(\Gamma_D).$$

This and $(A_n, u_n) \xrightarrow{w} (A^0, y^0)$ imply that the cost functional I is sequentially lower w-semicontinuous. Hence

$$I(A^0, y^0) \leq \liminf_{n\to\infty} I(A_n, y_n) = \inf_{(A,y)\in\Xi_w} I(A, y),$$

i.e. (A^0, y^0) is an optimal solution. The proof is complete.

Part C: Towards life-cycle optimization

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Submitted 2011

Related to the work of
Gregoire Allaire, Nicolas van Goethem….
and
Piermarco Cannarsa

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The evolution of damage

for a given body force $\mathbf{f} \in L^2\big(0,T; L^2(\Omega)^N\big)$, a surface traction $\mathbf{p} \in \mathcal{P}_{ad}$, the set Ψ_*, and an initial damage field $\zeta_0 \in L^2(\Omega)$ for which

$$\exists\, \zeta_*^0 \in \Psi_* \text{ such that } \; \zeta_*^0 \le \zeta_0 \le 1 \quad \text{a.e. in } \; \Omega,$$

a displacement field $\mathbf{u} : \Omega_T \to \mathbb{R}^N$, a stress field $\vec{\sigma} : \Omega_T \to \mathbb{S}^N$, and a damage field $\zeta : \Omega_T \to \mathbb{R}$ satisfy the relations

$$(P)\begin{cases} \rho u_{tt} \; - \operatorname{div} \vec{\sigma} = \mathbf{f} \quad \text{in} \quad \Omega_T, \\ \vec{\sigma} = \zeta A \mathbf{e}(\mathbf{u}) \quad \text{in} \;\; \Omega_T, \\ \mathbf{u} = 0 \quad \text{on} \quad (0,T) \times S, \\ \vec{\sigma}\vec{\nu} = \mathbf{p} \quad \text{on} \quad (0,T) \times \Gamma, \;\; \mathbf{p} \in \mathcal{P}_{ad}, \\ \zeta' - \kappa \Delta \zeta = \phi(\mathbf{e}(\mathbf{u}), \zeta) \quad \text{in} \quad \Omega_T, \\ \zeta(0,\cdot) = \zeta_0 \quad \text{in} \quad \Omega, \\ \zeta = 1 \quad \text{on } (0,T) \times \Gamma, \quad \partial\zeta/\partial n = 0 \quad \text{on } (0,T) \times S, \\ \exists\, \zeta_* \in \Psi_* \text{ such that } \; \zeta_* \le \zeta(t,x) \le 1 \quad \text{a.e. in} \quad \Omega_T. \end{cases}$$

Remarks and results in brief:

- For given data including the damage field, we derive a concept of weak solutions in weighted space
- We introduce a relaxed problem (with an extra $\epsilon\mathbf{u}$-term in the equation).
- We show existence of solutions (no uniqueness!)
- We introduce a concept of weak variational solutions to the full system
- We show existence for the relaxed problem

Life-cycle optimization

$$\text{Minimize} \quad \Bigg\{ I(\mathbf{p},\mathbf{u},\zeta) = \int_0^T \int_\Omega |\mathbf{u}-\mathbf{u}_d|^2_{\mathbb{R}^N}\,dxdt + \int_0^T \int_\Omega |\zeta - 1|\,dxdt + \int_0^T \int_\Omega \|\mathbf{e}(\mathbf{u})\|^2_{\mathbb{S}^N}\zeta\,dxdt \Bigg\}$$

subject to

$$\Xi := \{(\mathbf{p},\zeta,\mathbf{u}) \mid \mathbf{p}\in\mathcal{P}_{ad},\ \zeta\in\mathcal{Z},\ \mathbf{u}\in W_\zeta(\Omega\times(0,T);S),\ (\zeta,\mathbf{u}) \text{ is a weak variational solution to (P)}\}\,.$$

We show that the constraint set is sequentially τ-closed and that a minimum exists.

The extension to the fully dynamic case is under way.

ENGINEERING OF ADVANCED MATERIALS

Friedrich-Alexander-Universität Erlangen-Nürnberg

Typical picture….here for a spring